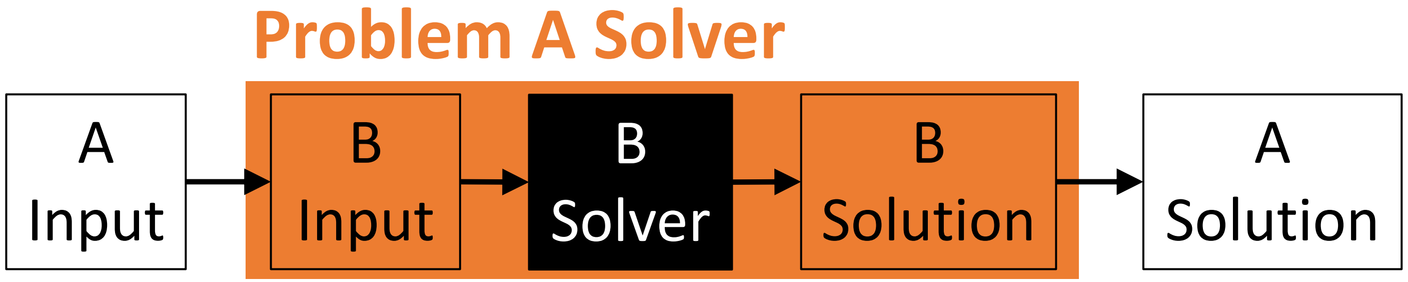
NP-Complete

CSCI 338

NP-Complete

How to show something (B) is in NP-Complete?

1. Show it is in NP.
2. Pick some known NP-Complete problem A.
3. Show that a solver for B can solve A in polynomial extra time.



To show A reduces to B:

- Show ***every*** instance of A can be translated to ***some*** instance of B.
- The solution to B can be translated back to a solution to A.

B is in NP-Complete if it satisfies two conditions:

1. $B \in NP$.
2. For some $A \in NP$-C, $A \leq_P B$.

$3SAT$

Claim: $3SAT$ is in NP-Complete.

Proof:

$$3SAT = \{\langle\phi\rangle\text{: } \phi \text{ is a satisfiable formula with 3 variables per clause}\}$$

$$\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor x_2)$$

$$(F \lor F \lor T) \qquad (T \lor F \lor F) \qquad (T \lor T \lor T)$$

$x_1 = F$ $\quad T \qquad T \qquad T$

$x_2 = T$

$$T$$

$3SAT$

Claim: $3SAT$ is in NP-Complete.

Proof:

1. $3SAT$ is in NP. ✓

2. $SAT \leq_P 3SAT$

B is in NP-Complete if it satisfies two conditions:

1. $B \in NP$.
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$SAT \leq_P 3SAT$

Claim: $SAT \leq_P 3SAT$

Proof:

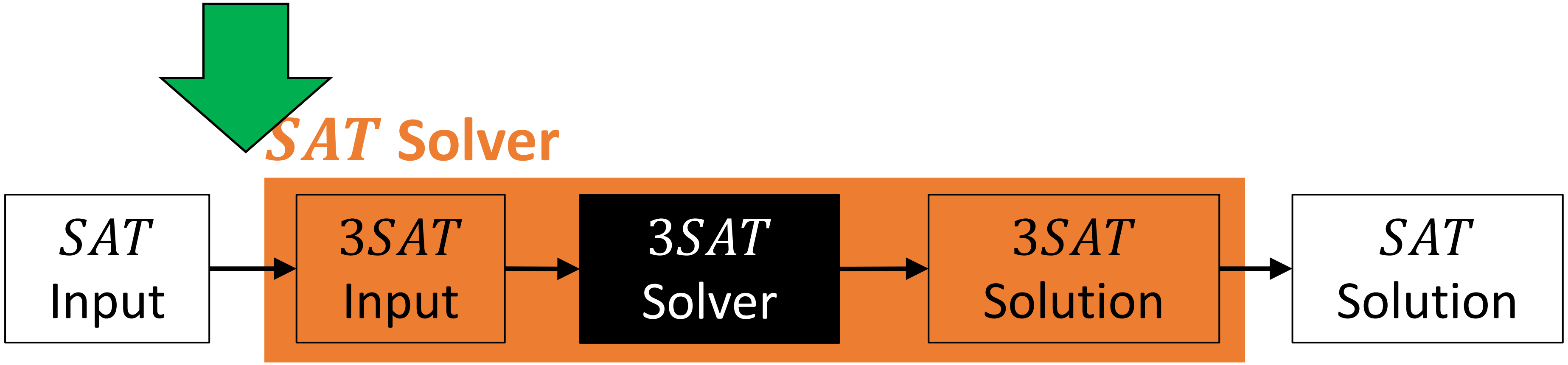
We need to turn instances of SAT into instances of $3SAT$.

If a clause has one literal? $(x_1) \rightarrow (x_1 \vee x_1 \vee x_1)$

If a clause has two literals? $(x_1 \vee x_2) \rightarrow (x_1 \vee x_1 \vee x_2)$

If a clause had three literals? No change.

If a clause has more than three literals?



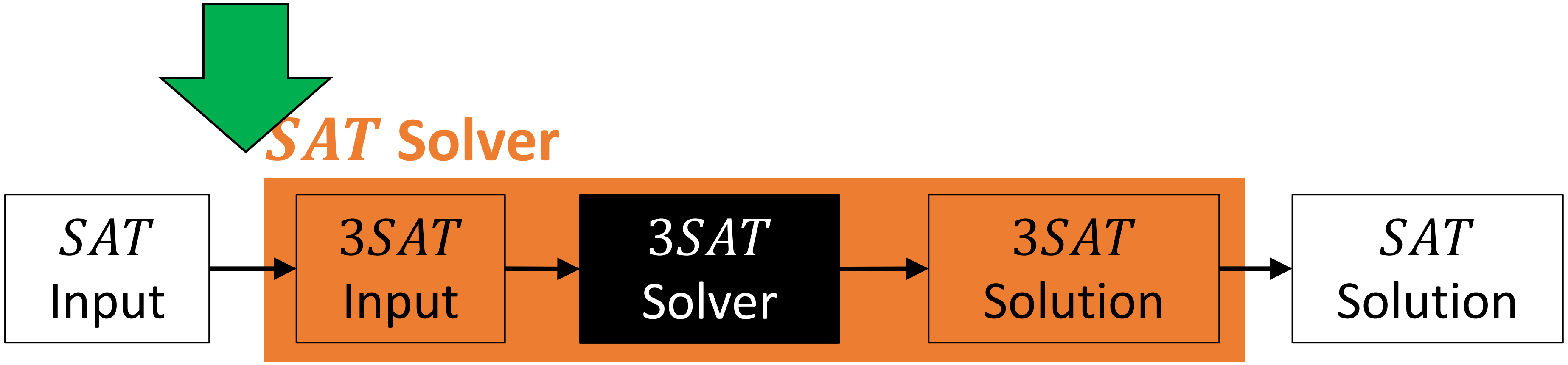
$SAT \leq_P 3SAT$

Claim: $SAT \leq_P 3SAT$

Proof: Convert SAT clauses with >3 literals into $3SAT$ clauses.

$$\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)$$

$$\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor \boldsymbol{z_1}) \land (\overline{\boldsymbol{z_1}} \lor x_3 \lor \boldsymbol{z_2}) \land \cdots \land (\overline{\boldsymbol{z_{k-3}}} \lor x_{k-1} \lor x_k)$$



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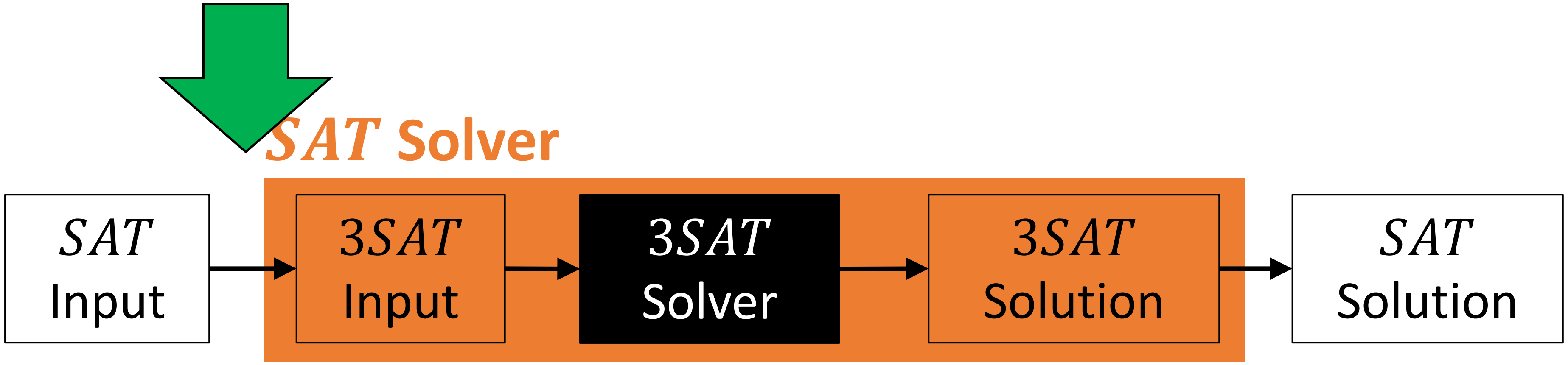
Need to show: ϕ_{SAT} can be true $\Leftrightarrow$ ϕ_{3SAT} can be true.



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Suppose ϕ_{SAT} can be true.

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Suppose ϕ_{SAT} can be true. Then some x_m is true. Let x_m be true in ϕ_{3SAT}. Let all z_i's before x_m be true…

$$\ldots \wedge (\overline{z_{i-1}} \vee x_{m-1} \vee z_i) \wedge (\overline{z_i} \vee x_m \vee z_{i+1}) \wedge (\overline{z_{i+1}} \vee x_{m+1} \vee z_{i+2}) \wedge \ldots$$

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Suppose ϕ_{SAT} can be true. Then some x_m is true. Let x_m be true in ϕ_{3SAT}. Let all z_i's before x_m be true and all z_i's after be false.
$\Rightarrow$ Every clause has a variable set to true.

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$\Rightarrow$ Every clause has a variable set to true. $\therefore \phi_{3SAT} = T$.

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$\therefore SAT \leq_P 3SAT$

$3SAT$

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1. $B \in NP$.
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Claim: $3SAT$ is in NP-Complete.

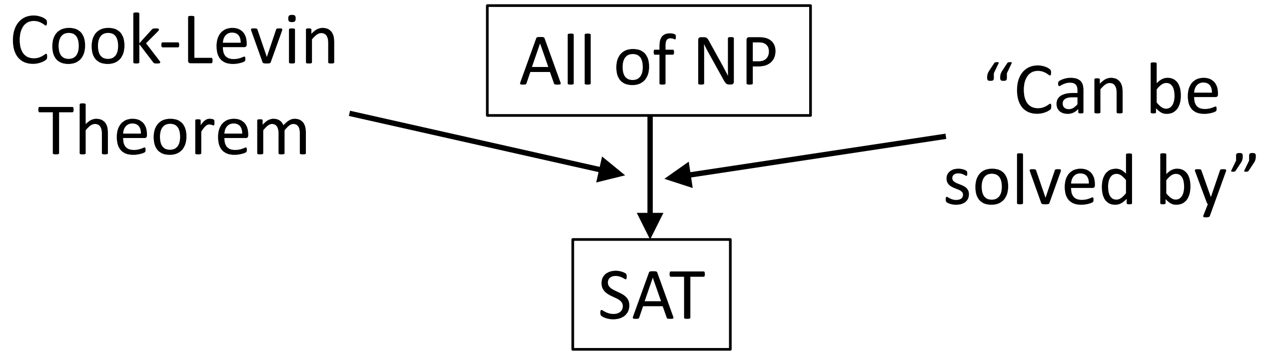
Proof:

1. $3SAT$ is in NP. ✓

2. $SAT \leq_P 3SAT$ ✓

Therefore, $3SAT$ is in NP-Complete.

$NP - C$



Cook-Levin
Theorem
All of NP
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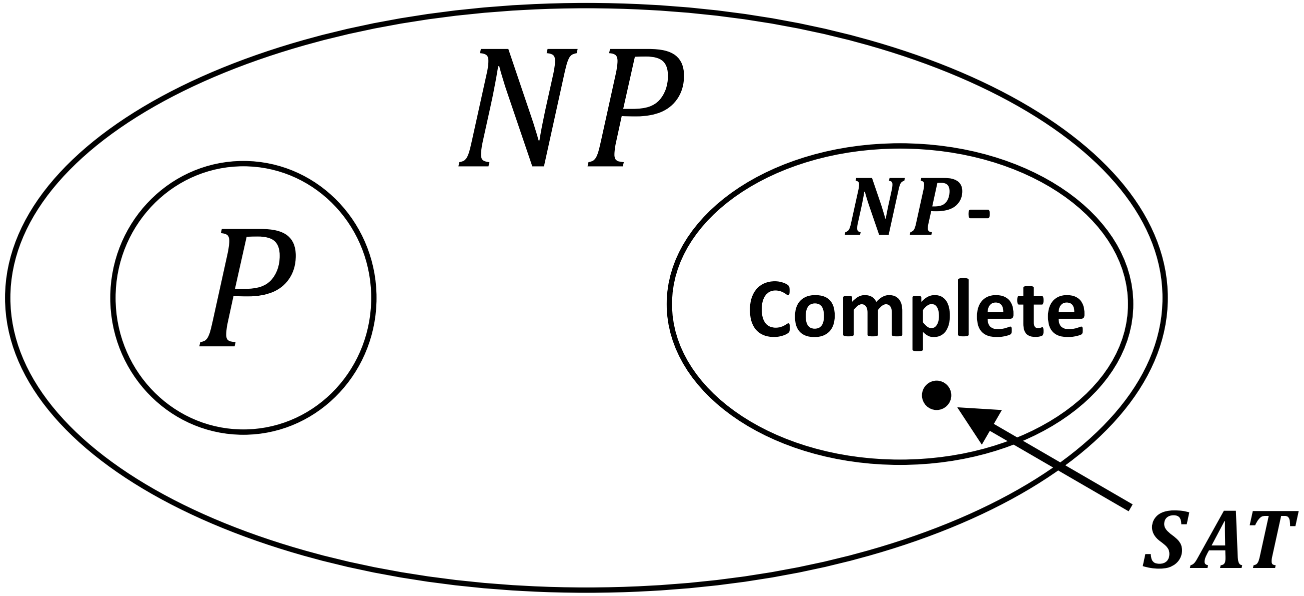

NP
P
NP-
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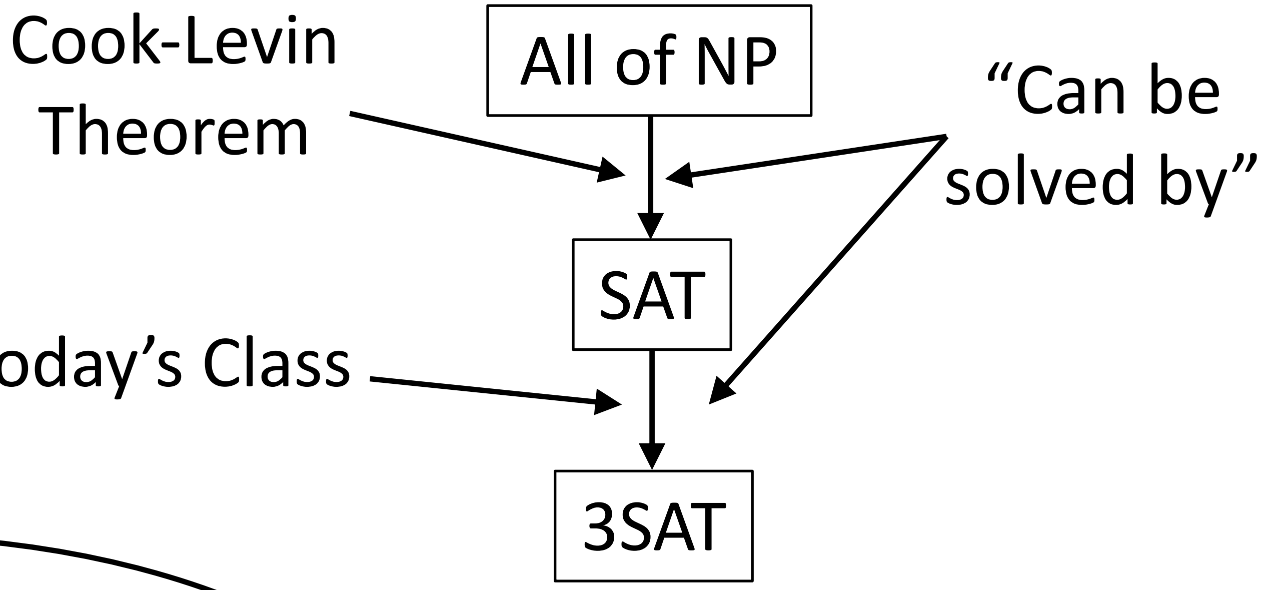
$NP - C$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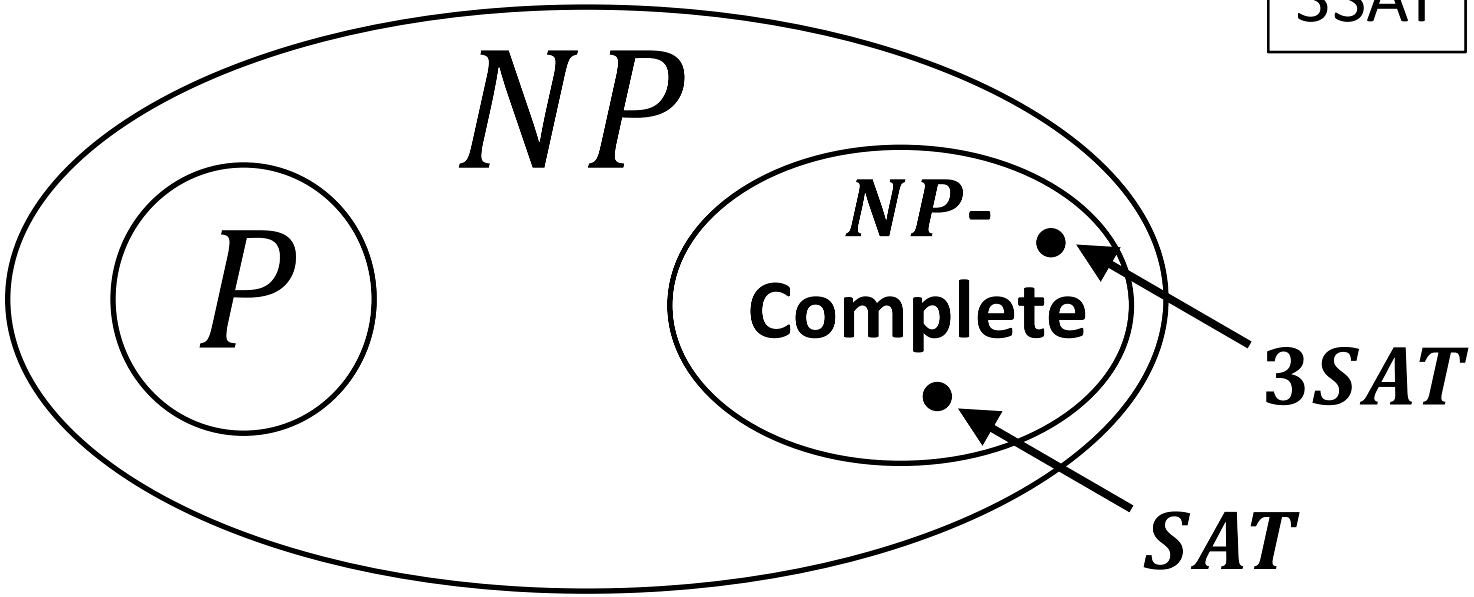
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Accuracy, speed.

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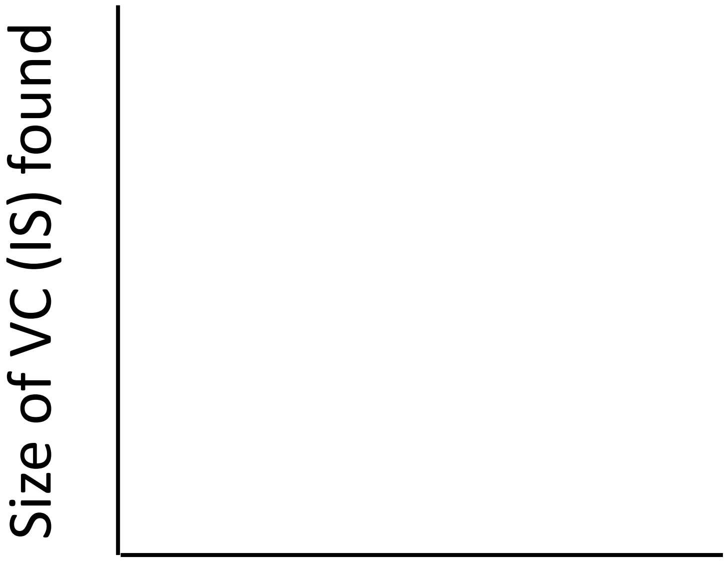

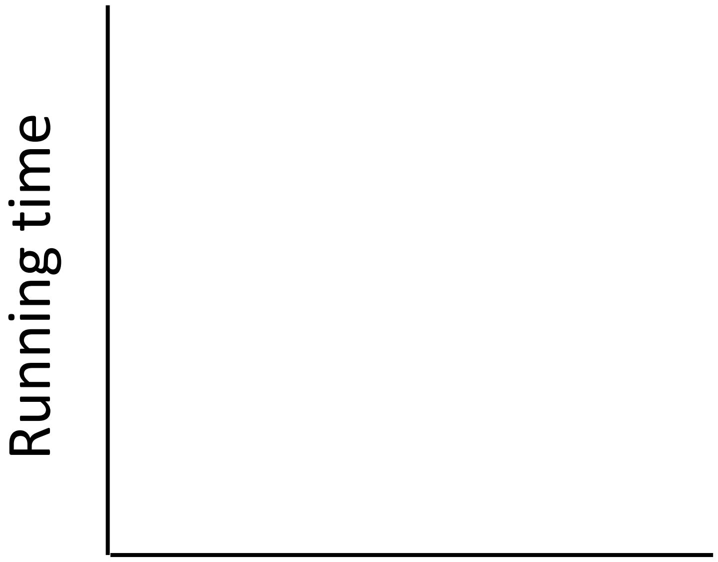

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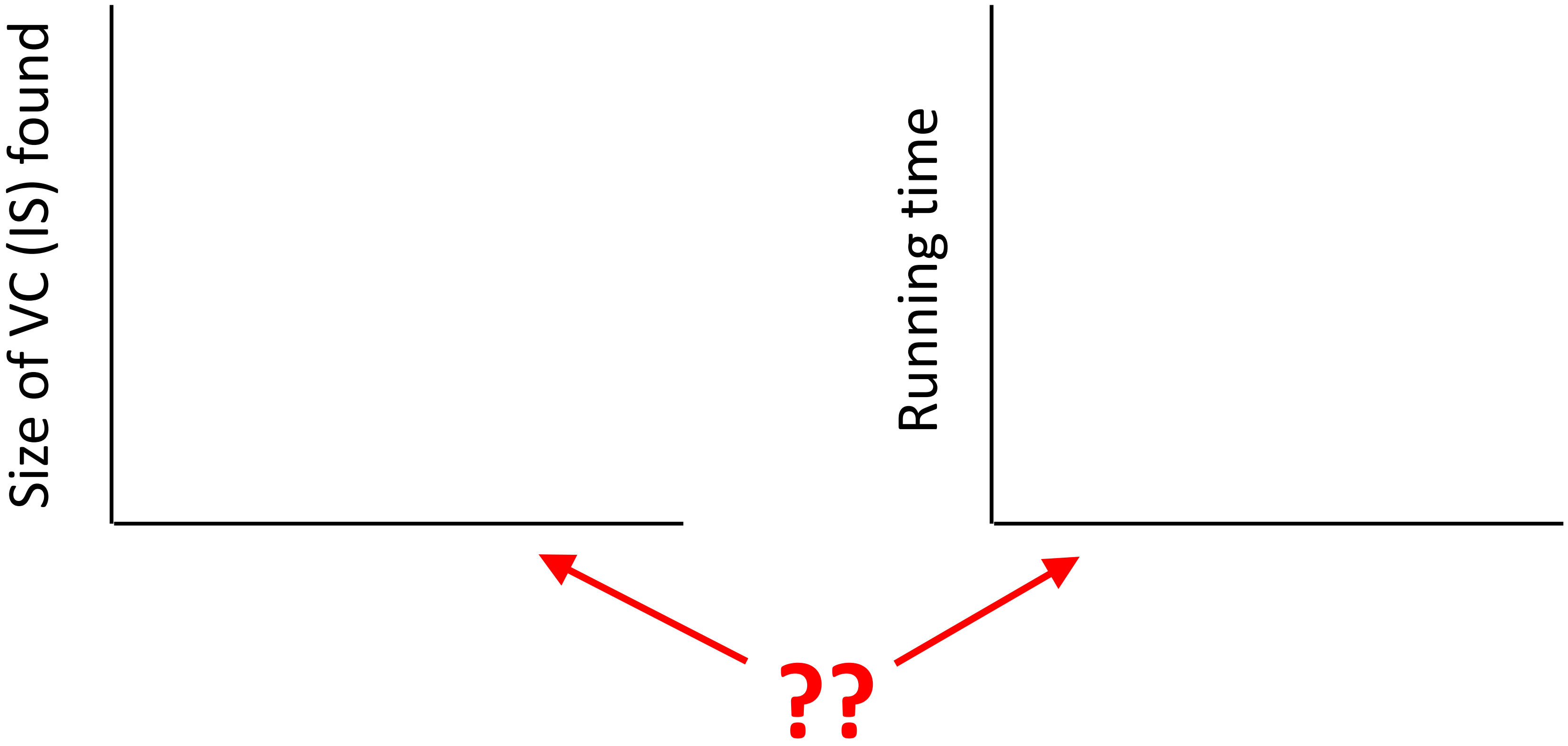

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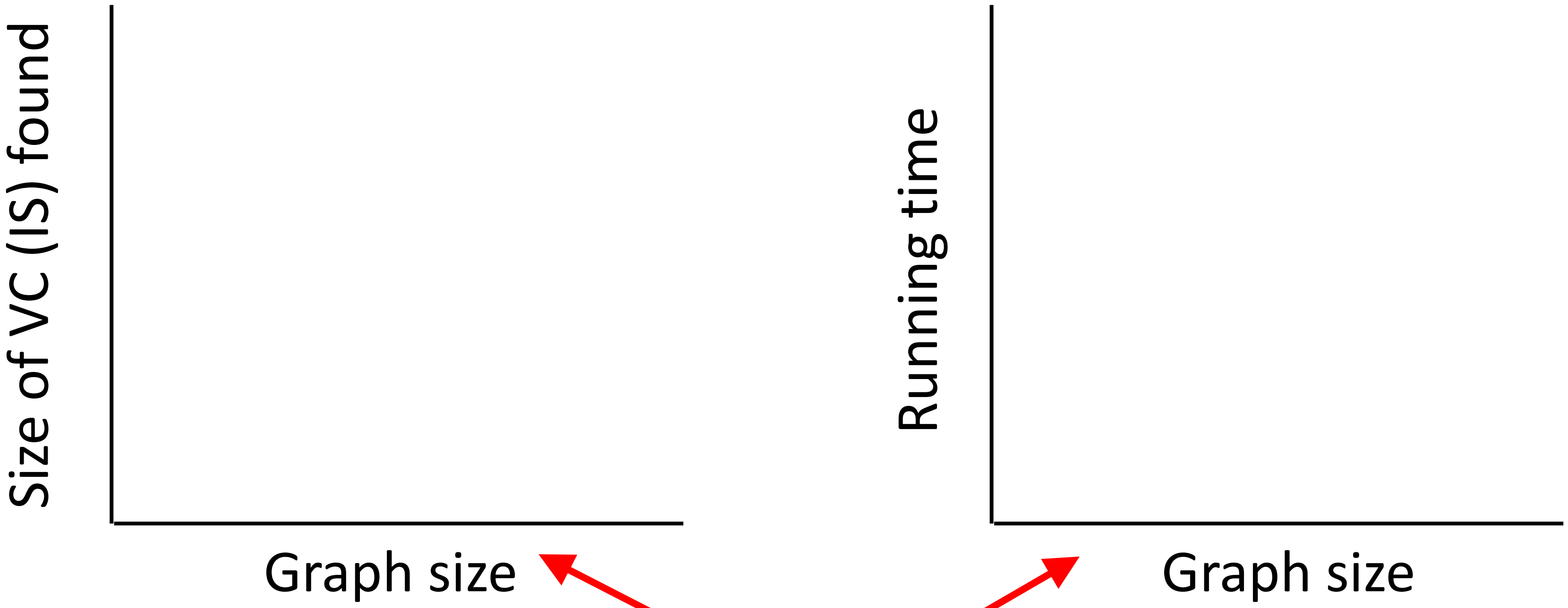

"Size" of graph (# vertices, # edges, connectivity)

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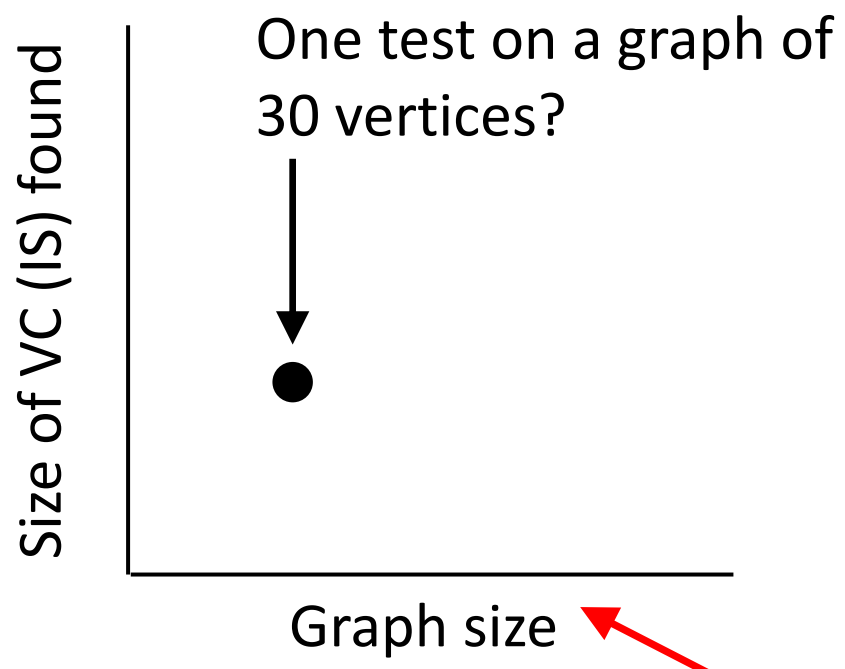

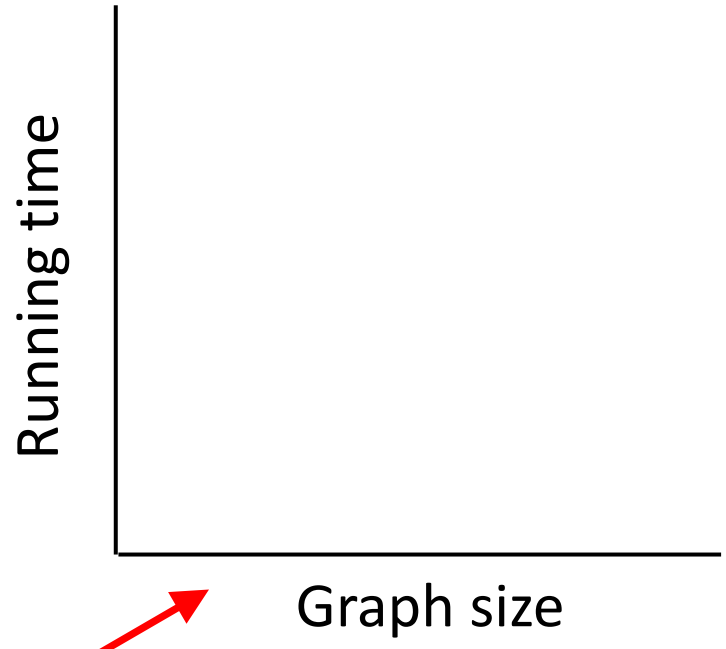

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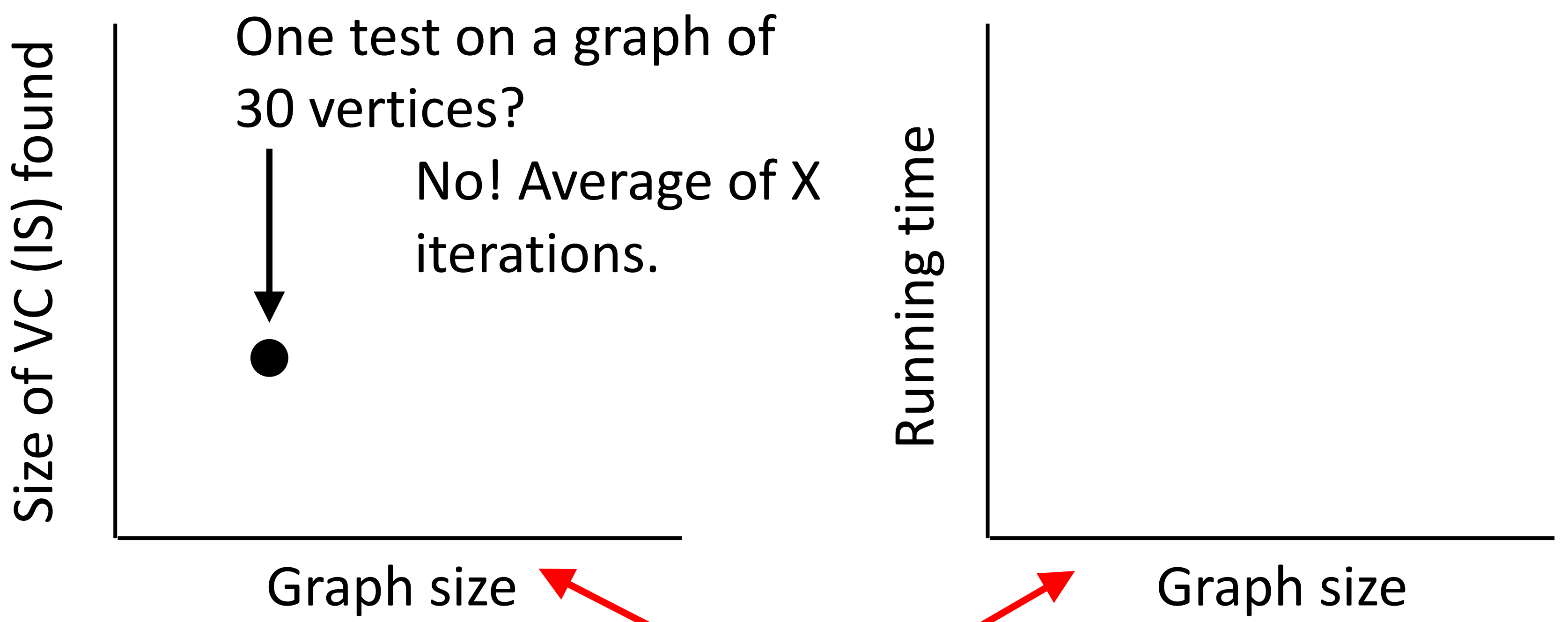

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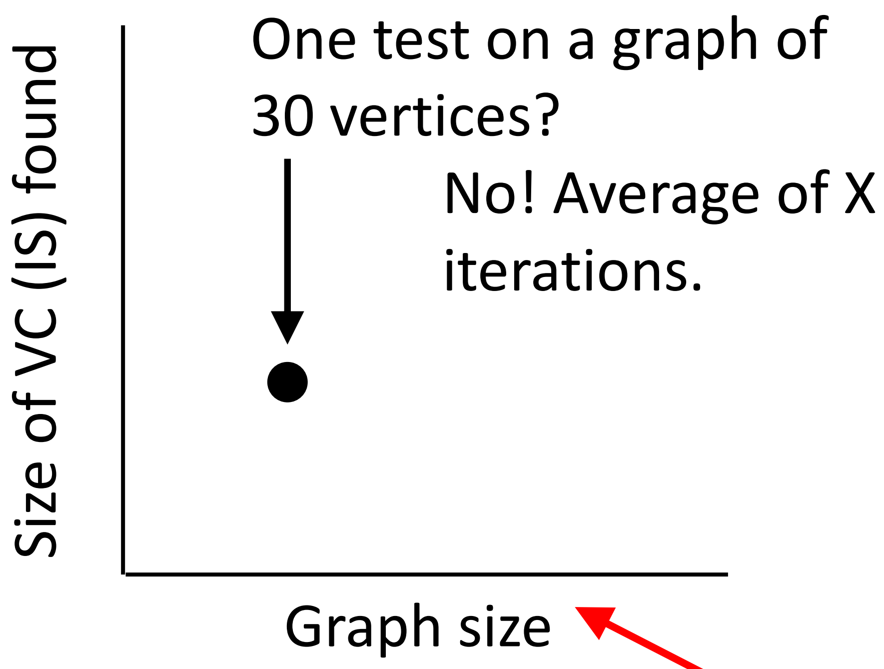

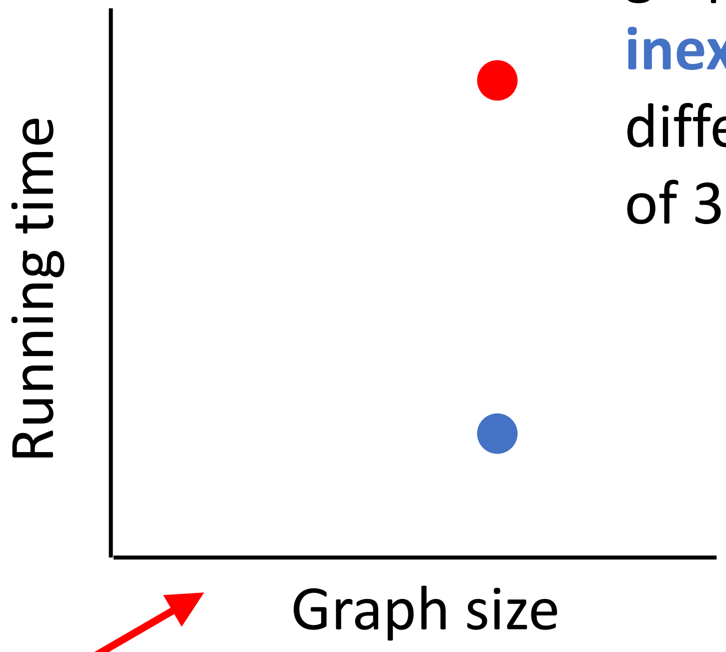

Exact result on random graph of 30 vertices and **inexact result** on different random graph of 30 vertices?

"Size" of graph (# vertices, # edges, connectivity)

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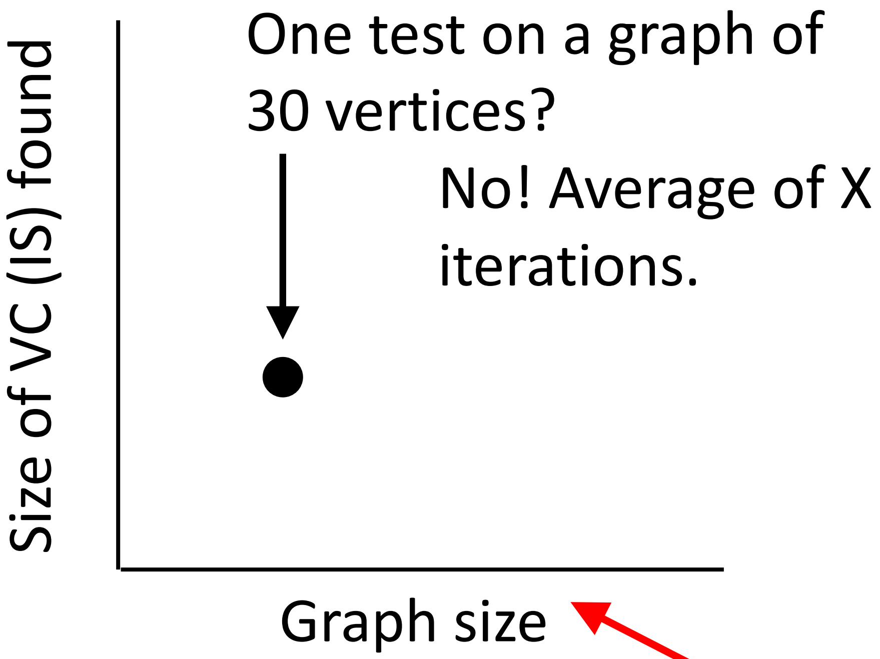

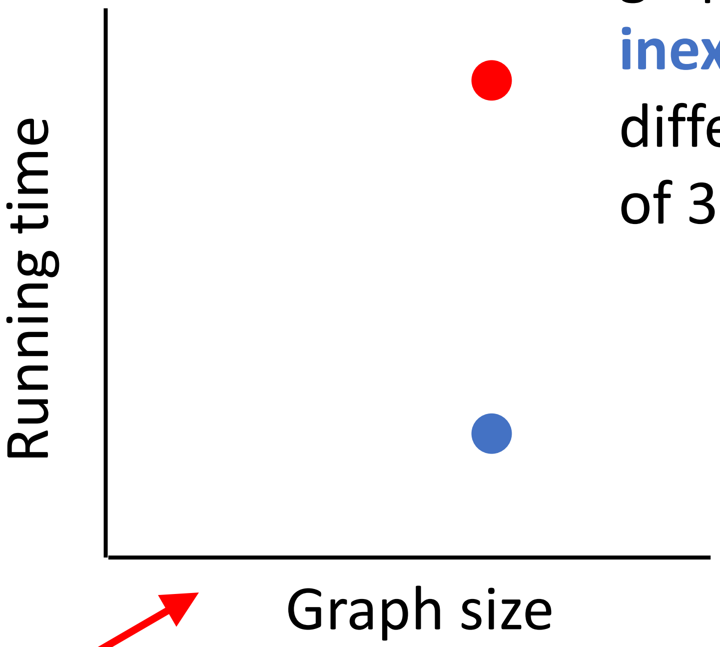

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No! results on same random graph.

“Size” of graph (# vertices, # edges, connectivity)