Announcement:

• CIA representatives speaking to ESOF 322 for first 10-20 minutes.

• REID 103 @ 12:00.

• You are invited.
How to show problem $B$ is in $NP$-Complete:

1. Show $B$ is in $NP$.
2. Pick some known $NP$-Complete problem $A$.
3. Show how generic instances of $A$ can be translated into instances of $B$.
4. Show that the translation process runs in polynomial time.
5. Show that if the answer to $A$’s instance is ‘yes’, the answer to $B$’s instance is also ‘yes’.
6. Show that if the answer to $B$’s instance is ‘yes’, the answer to $A$’s instance is also ‘yes’.
**CLIQUE**

Clique: a subgraph where every pair of nodes share an edge (i.e. a complete subgraph).

$k$-Clique: A clique that contains $k$ vertices.

\[
\text{CLIQUE} = \{ \langle G, k \rangle : G \text{ is an undirected graph with a } k\text{-clique} \}\]
**NP-Complete**

How to show problem $B$ is in NP-Complete:

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6. Show that if the answer to $B$’s instance is ‘yes’, the answer to $A$’s instance is also ‘yes’.
Claim: $3SAT \leq_P CLIQUE$

Proof: Let $\phi$ be a formula with $k$ clauses. Generate an undirected graph $G$:

For each clause in $\phi$, make a node for each literal. Make an edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.

Need to show: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

\[
\phi = \left( x_1 \lor x_1 \lor x_2 \right) \land \left( \overline{x_1} \lor \overline{x_2} \lor \overline{x_2} \right) \land \left( \overline{x_1} \lor x_2 \lor x_2 \right) \land \left( x_1 \lor \overline{x_2} \lor \overline{x_2} \right)
\]
Claim: $3SAT \leq_p CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:
1. Nodes in the same clause
2. Nodes that are negations of each other.
Claim: $3SAT \leq_p CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then...

$\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor x_2)$

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
Claim: 3SAT \leq_p CLIQUE

Proof: \( \phi \) is satisfiable \iff \( G \) has a \( k \)-clique.

\( \Rightarrow \) Suppose \( \phi \) is satisfiable. Then at least one literal is true in each clause.

\[
\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor x_2)
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For each clause in \( \phi \), make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
Claim: $3SAT \leq_p CLIQUE$

Proof: $\phi$ is satisfiable $\iff$ $G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal is true in each clause. For each clause, select a node in $G$ for one of the true literals.

$$\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor x_2 \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor x_2)$$

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
Claim: $3SAT \leq_p CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal is true in each clause. For each clause, select a node in $G$ for one of the true literals. This forms a $k$-clique, since...

$$
\phi = (x_1 \lor x_1 \lor x_2) \land \\
\neg(x_1 \lor x_2 \lor x_2) \land \\
\neg(x_1 \lor x_2 \lor x_2)
$$

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.

![Graph example](image)
Claim: 3SAT $\leq_P$ CLIQUE

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal is true in each clause. For each clause, select a node in $G$ for one of the true literals. This forms a $k$-clique, since $k$ nodes are selected and each is joined by an edge.

$$\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor x_2 \lor x_2) \land (\overline{x_1} \lor x_2 \lor x_2)$$

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:
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\[
\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x}_1 \lor \overline{x}_2 \lor \overline{x}_2) \land (\overline{x}_1 \lor x_2 \lor x_2)
\]

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
NP-Complete

How to show problem $B$ is in NP-Complete:

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Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal is true in each clause. For each clause, select a node in $G$ for one of the true literals. This forms a $k$-clique, since $k$ nodes are selected and each is joined by an edge.

$\Leftarrow$ Suppose $G$ has a $k$-clique.

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
**CLIQUE**

Claim: $3SAT \leq_p CLIQUE$

Proof: $\phi$ is satisfiable $\iff$ $G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal is true in each clause. For each clause, select a node in $G$ for one of the true literals. This forms a $k$-clique, since $k$ nodes are selected and each is joined by an edge.

$\Leftarrow$ Suppose $G$ has a $k$-clique. Then there is a non-contradictory node from the $k$-clique in each clause.

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
Claim: 3SAT \leq_p CLIQUE

Proof: \phi is satisfiable \iff G has a \(k\)-clique.

\[ \Rightarrow \text{Suppose } \phi \text{ is satisfiable. Then at least one literal is true in each clause. For each clause, select a node in } G \text{ for one of the true literals. This forms a } k\text{-clique, since } k \text{ nodes are selected and each is joined by an edge.} \]

\[ \Leftarrow \text{Suppose } G \text{ has a } k\text{-clique. Then there is a non-contradictory node from the } k\text{-clique in each clause. (nodes in the same clause can’t share an edge!)} \]

For each clause in \(\phi\), make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
**CLIQUE**

Claim: $3SAT \leq_p CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal in $\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor x_2)$ is true. For each clause, select a node in $G$ for one of the true literals. This forms a $k$-clique, since $k$ nodes are selected and each is joined by an edge.

$\Leftarrow$ Suppose $G$ has a $k$-clique. Then there is a non-contradictory node from the $k$-clique in each clause.

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.

$$\phi = x_1 \lor \overline{x_1} \lor x_2 \lor \overline{x_2} \land \overline{x_1} \lor x_2 \lor x_2$$
Claim: $3SAT \leq_p CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal is true in each clause. For each clause $\phi = (x_1 \lor \overline{x}_1 \lor x_2) \land (\overline{x}_1 \lor \overline{x}_2 \lor \overline{x}_2) \land (\overline{x}_1 \lor x_2 \lor \overline{x}_2)$. Since $k$ nodes are selected and each is joined by an edge.

$\Leftarrow$ Suppose $G$ has a $k$-clique. Then there is a non-contradictory node from the $k$-clique in each clause. Making each node in the $k$-clique true results in...

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.

$\overline{x}_1 \lor \overline{x}_2 \lor \overline{x}_2$
**CLIQUE**

Claim: $3SAT \leq_p CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal is true in each clause of $\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor x_2)$. For each clause, select a node in $G$ for one of the true literals. Since $k$ nodes are selected and each is joined by an edge, this forms a $k$-clique.

$\Leftarrow$ Suppose $G$ has a $k$-clique. Then there is a non-contradictory node from the $k$-clique in each clause. Making each node in the $k$-clique true results in $\phi$ being true.

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:
1. Nodes in the same clause
2. Nodes that are negations of each other.
**NP-Complete**

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Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

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$\Leftarrow$ Suppose $G$ has a $k$-clique. Then there is a node from the $k$-clique in each clause. Making each node in the $k$-clique true results in $\phi$ being true.

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
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CLAIM

Claim: CLIQUE ∈ NP-Complete
Proof:

1. CLIQUE ∈ NP ✓
   Given a graph $G = (V, E)$, where $|V| = n$, and a subset $S \subseteq V$, where $|S| \geq k$, check if all pairs of vertices in $S$ are in $E$. Running time: $O(n^2)$.

2. 3SAT $\leq_p$ CLIQUE ✓

$\therefore$ CLIQUE ∈ NP − C
$NP - C$

- All of $NP$
  - $SAT$
    - $3SAT$
      - $CLIQUE$

- "Can be solved by"
Vertex Cover (VC)

Vertex Cover: Given graph $G = (V, E)$ and integer $k \leq |V|$, is there $V' \subseteq V$, with $|V'| \leq k$, such that each edge in $E$ contains an end point in $V'$?
Vertex Cover (VC)

Claim: \( VC \in NP \)-Complete

Proof:

1. \( VC \in NP \)

2. ??? \( \leq_p VC \)
Vertex Cover (VC)

Claim: $VC \in NP$-Complete

Proof:
1. $VC \in NP$
2. $CLIQUE \leq_p VC$
Vertex Cover (VC)

Claim: $CLIQUE \leq_p VC$

Proof:

Clique $\Rightarrow$ Vertex Cover
Claim: $CLIQUE \leq_p VC$

Proof:
Claim: $CLIQUE \leq_p VC$

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Claim: $CLIQUE \leq_p VC$

Proof:

Clique \[ \rightarrow \] Vertex Cover
Claim: $CLIQUE \leq_P VC$

Proof:

$\exists k - Clique \iff \exists (n - k) - Vertex\ Cover$