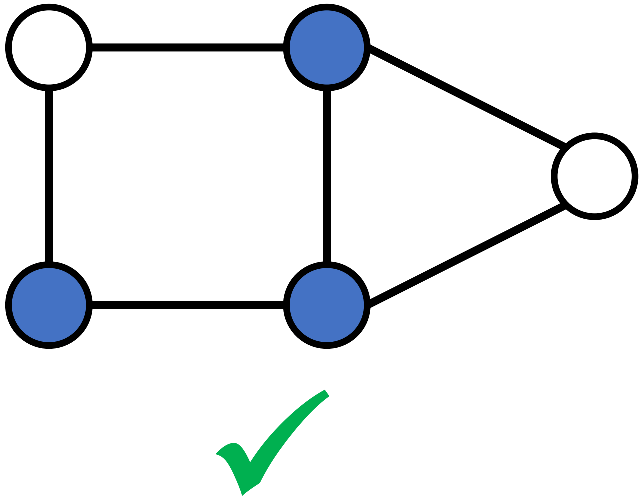
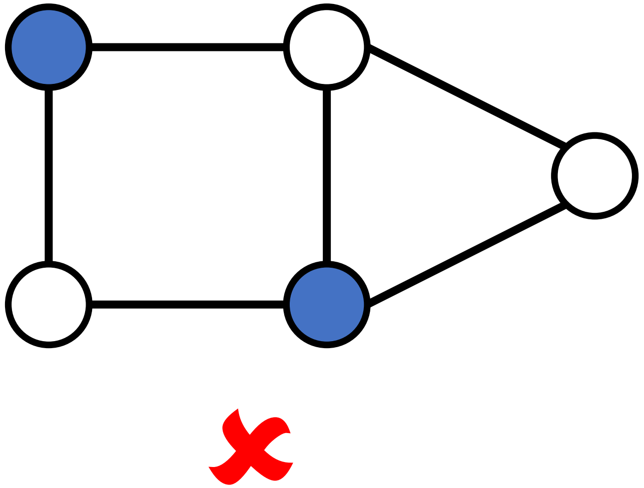
Vertex Cover
CSCI 338

Vertex Cover (VC)

Vertex Cover: Given graph $G = (V, E)$ and integer $k \leq |V|$, is there $V' \subseteq V$, with $|V'| \leq k$, such that each edge in E contains an end point in V'?





Vertex Cover (VC)

Claim: VC $\in NP$-Complete

Proof:

1. VC $\in NP$

2. $CLIQUE \leq_P$ VC

Vertex Cover (VC)

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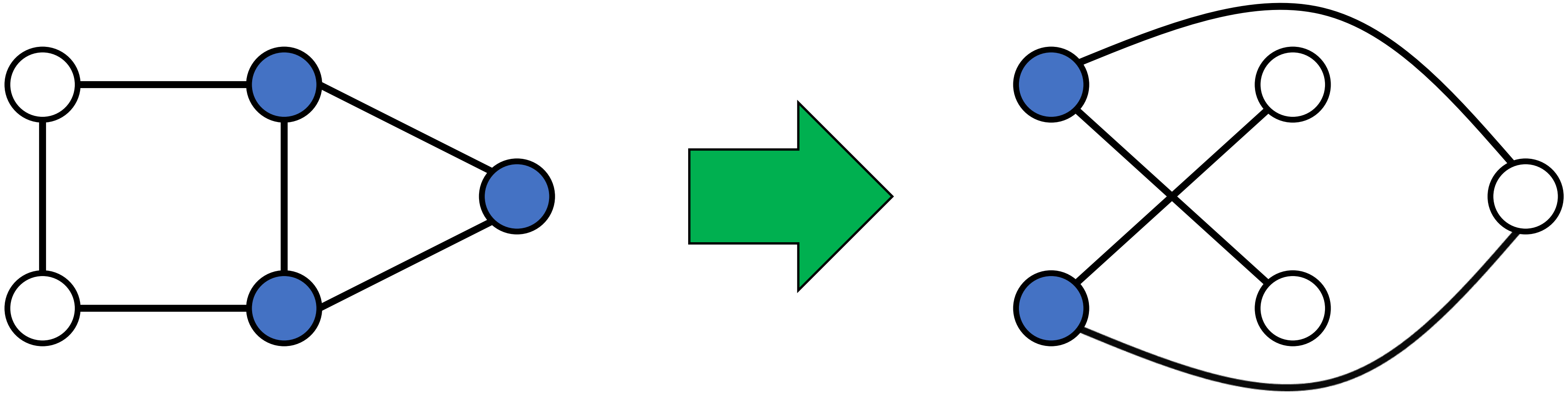

$\exists\ k$ - Clique $\iff$ $\exists\ (n-k)$ - Vertex Cover

Vertex Cover (VC)

Claim: $CLIQUE \leq_P$ VC

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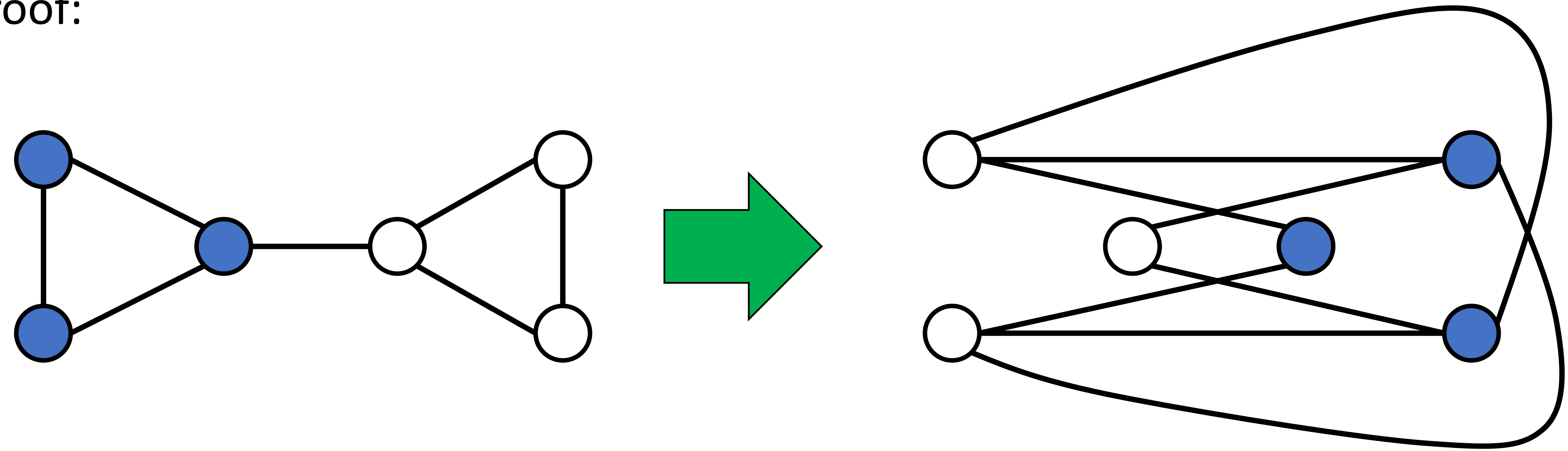

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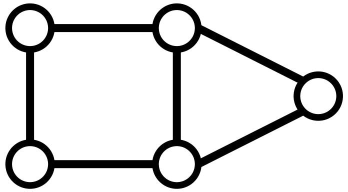

Claim: $CLIQUE \leq_P$VC

Proof: Let $G = (V, E)$, k be input to the clique problem, where $|V| = n$.

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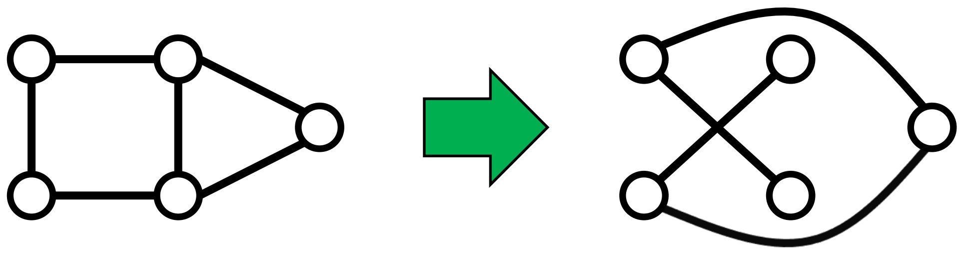

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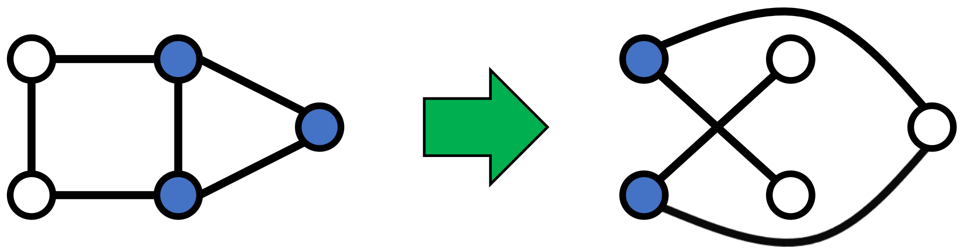

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G has a k-clique $\Leftrightarrow$ $\bar{G}$ has an $(n - k)$-VC.

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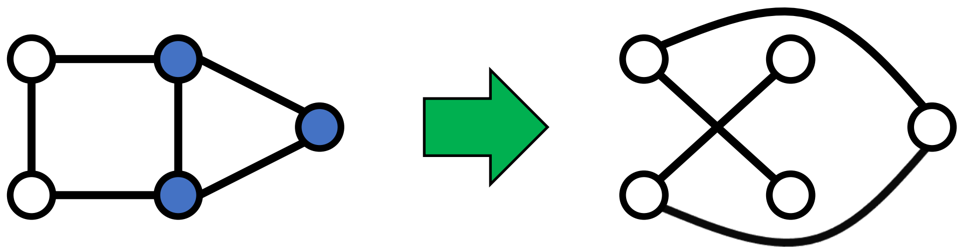

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$\Rightarrow$ Suppose G has a ***k*-clique *Q***.

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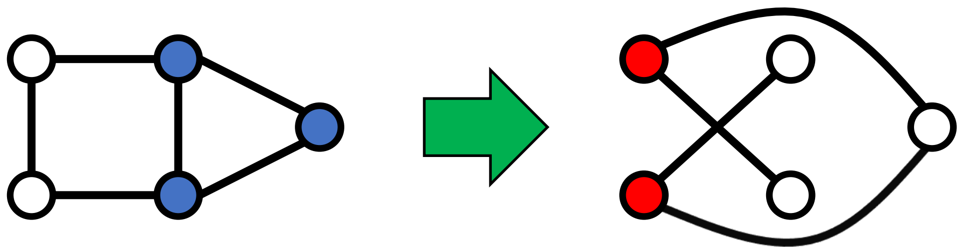

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$$G \text{ has a } k\text{-clique} \Leftrightarrow \bar{G} \text{ has an } (n-k)\text{-VC}.$$

$\Rightarrow$ Suppose G has a **k-clique Q**. Consider **$C = V \setminus Q$**.

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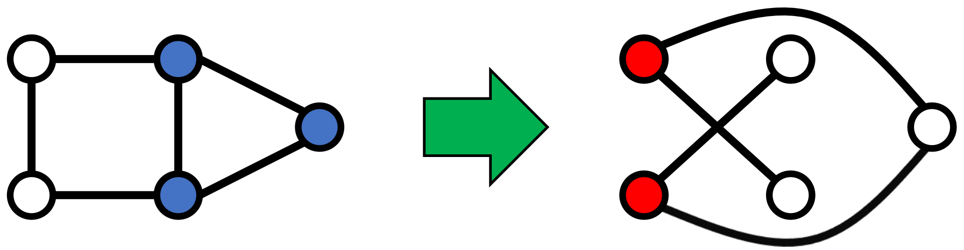

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G has a k-clique $\Leftrightarrow$ $\bar{G}$ has an $(n-k)$-VC.

$\Rightarrow$ Suppose G has a ***k*-clique *Q***. Consider ***$C = V \setminus Q$***. Show C is an $(n-k)$-VC.

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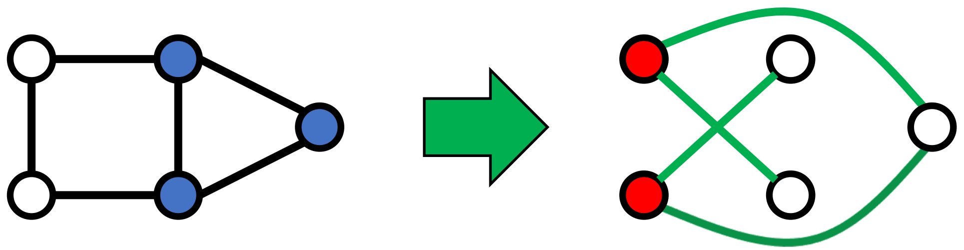

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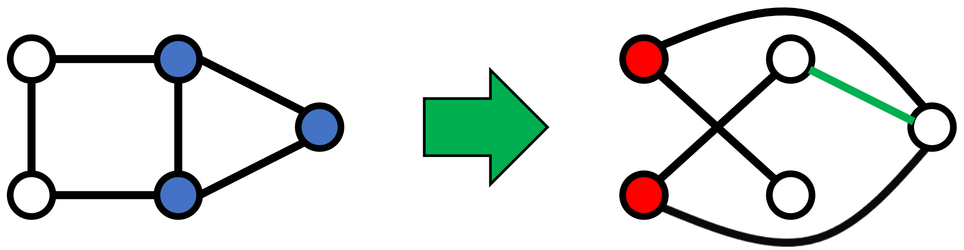

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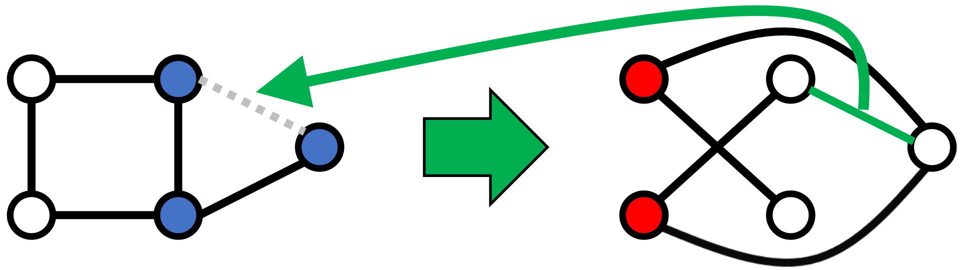

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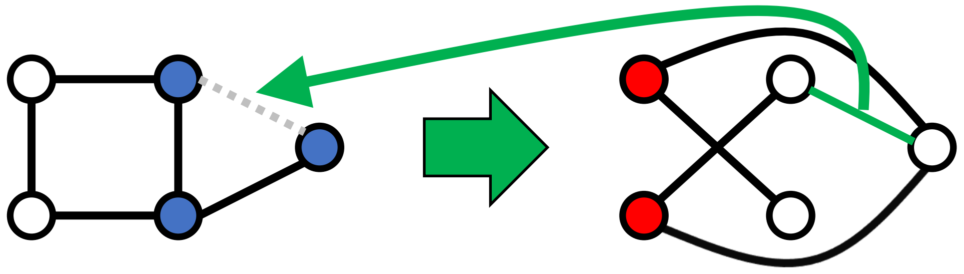

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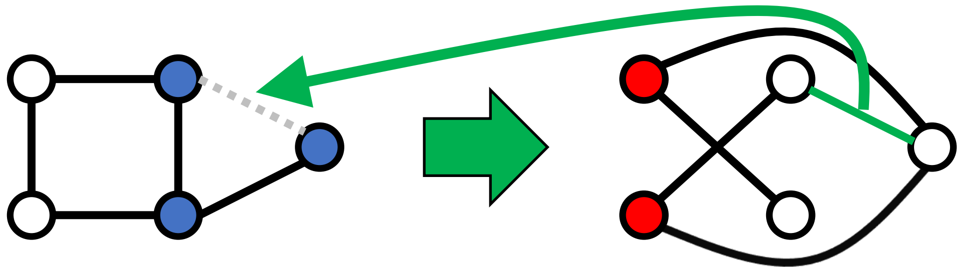

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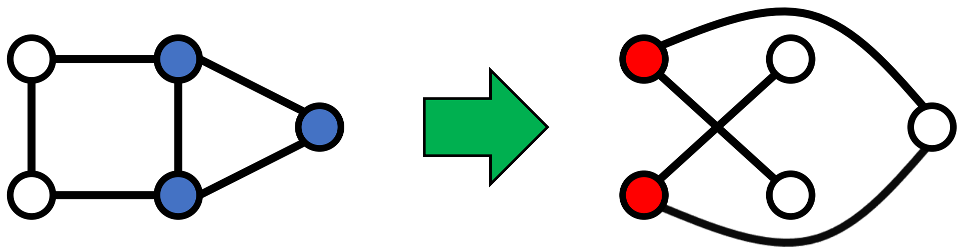

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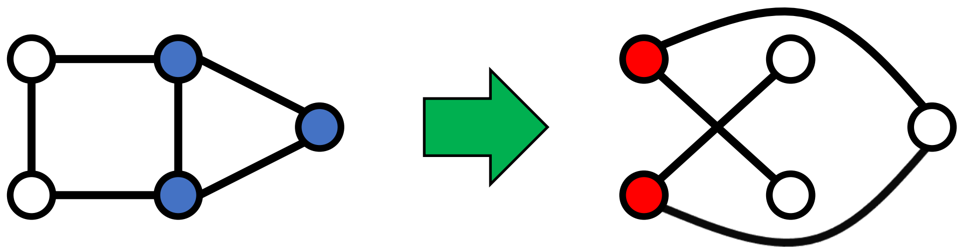

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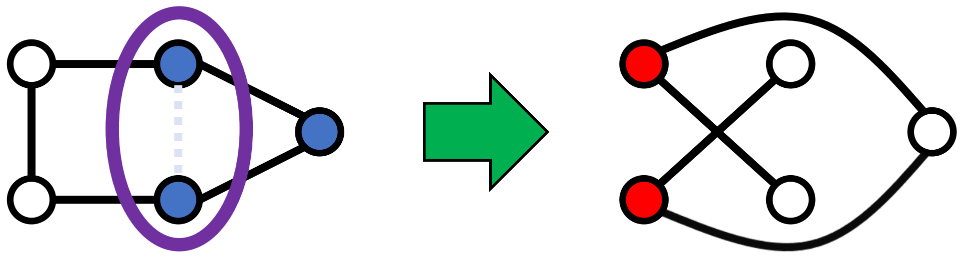

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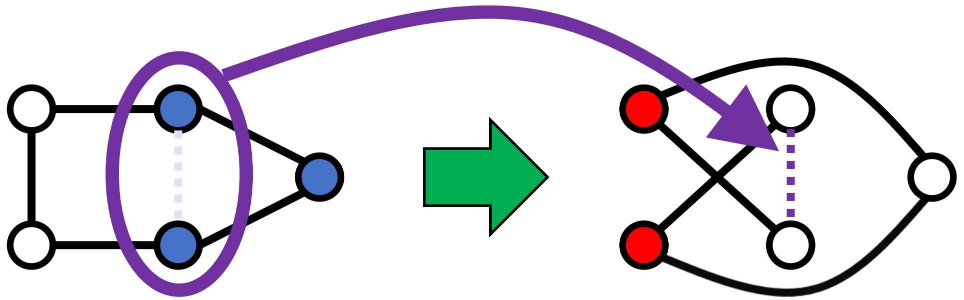

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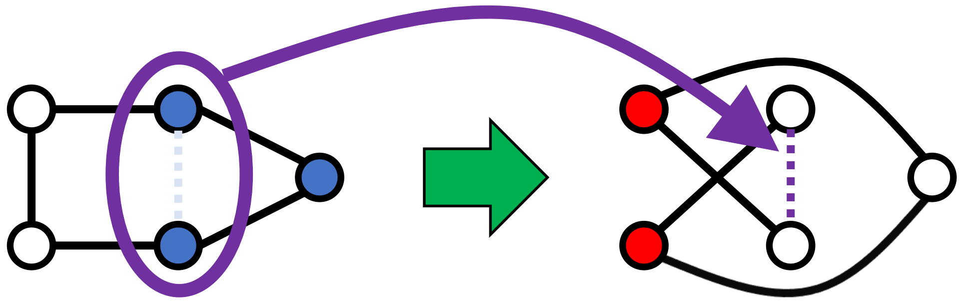

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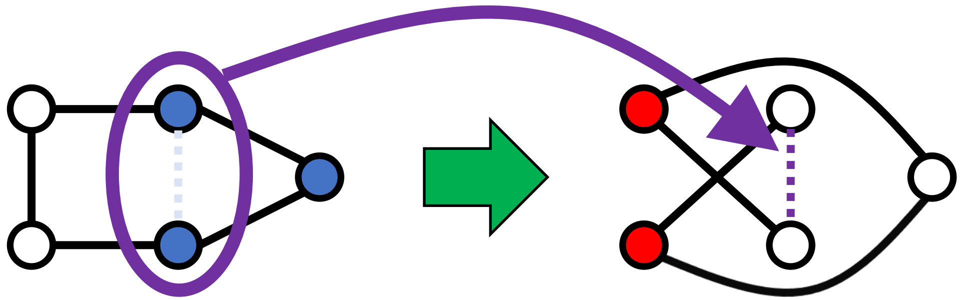

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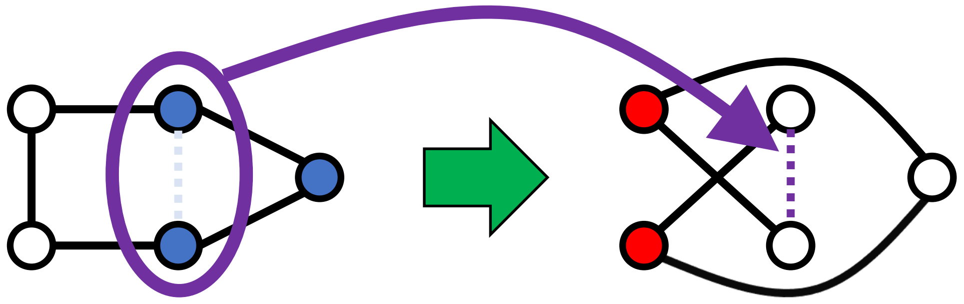

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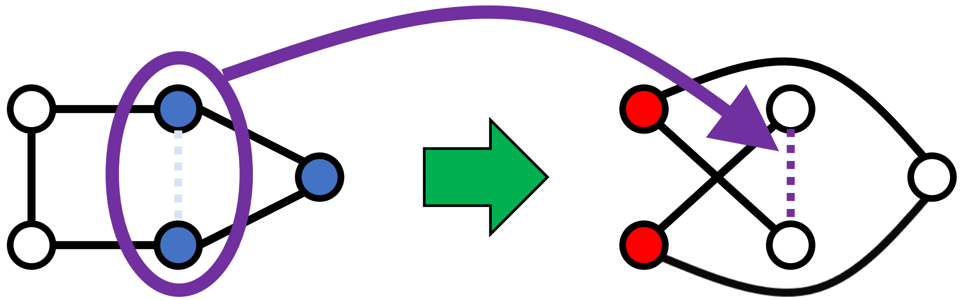

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Project 3

Report:

- Introduction
- Algorithm descriptions
  - How do they work?
  - How do you know they are valid?
  - How do you know they are optimal (if they are)?
  - Running times?
- Evaluation description
  - What metrics are you testing?
  - How many iterations of each scenario did you do?
  - How did you generate your graphs?
- Results
  - Plots?
  - What are your conclusions?