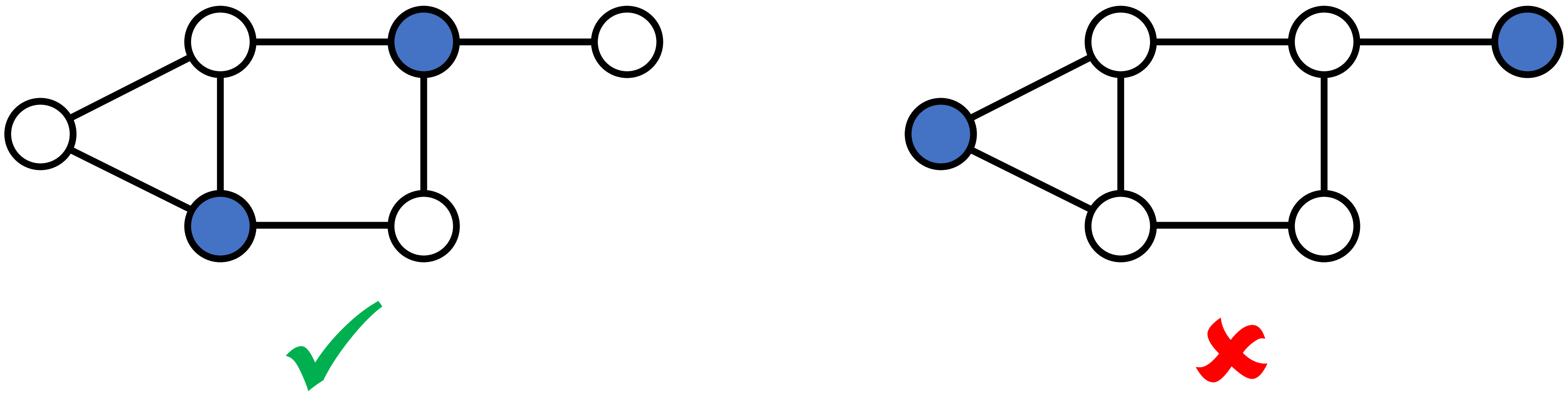
Dominating Set
CSCI 338

Dominating Set

Dominating Set: Given a graph $G = (V, E)$ and integer $k \leq |V|$, is there a subset V' of size $\leq k$, such that every vertex $\in V \setminus V'$ shares an edge with a vertex $\in V'$?



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Dominating Set: For $G = (V, E)$ and $k \leq |V|$, $\exists\, V' \subseteq V$, $|V'| \leq k$, s.t. each $v \in V \backslash V'$ shares an edge with a $u \in V'$?

Claim: Dominating Set $\in NP - C$

Proof:

1. Show Dominating Set $\in NP$.

Given $G = (V, E)$, k and a subset V' of V, confirm that $|V| \geq k$ and that for each vertex $v \in V$, $v \in V'$ or there is some edge $(v, u) \in E$ such that $u \in V'$.

$O(n^3)$ running time $\Rightarrow$ Dominating Set $\in NP$.

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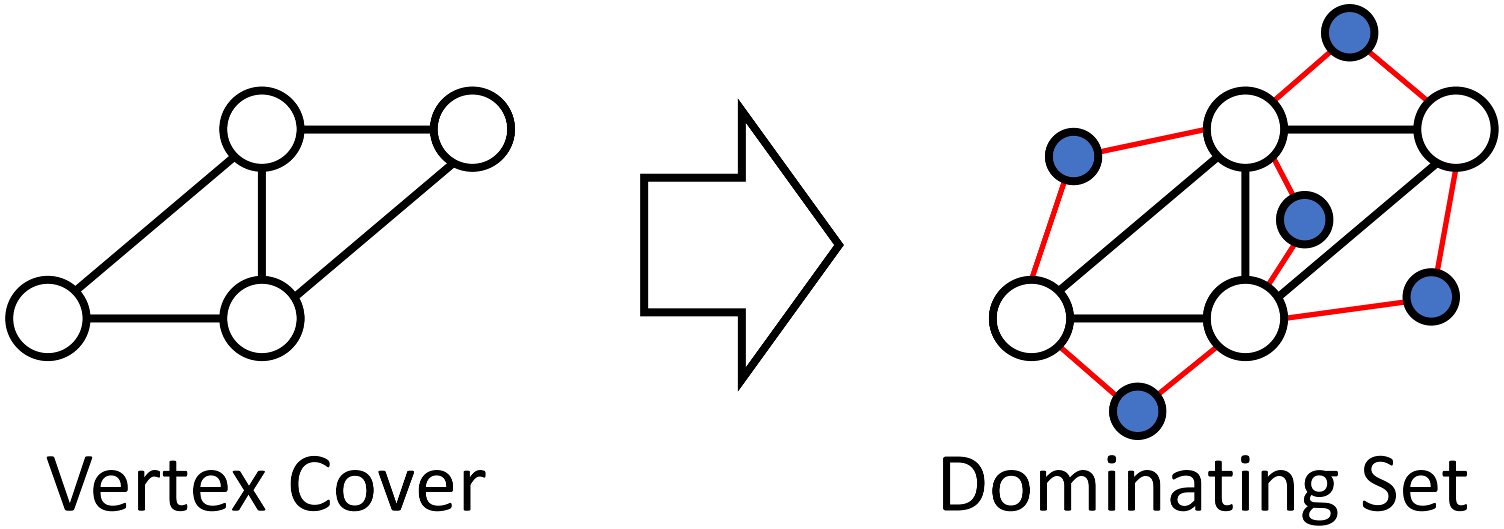
Claim: Vertex Cover $\leq_P$ Dominating Set

Proof: Turn $G = (V, E)$ into $G' = (V', E')$ as follows: For each $e = (u, v) \in E$, add u and v to V' and add e to E'. Also, add a new vertex w to V' and add edges (u, w) and (v, w) to E'. $O(n + n^2 + n^2 + 2n^2) = O(n^2)$ time.



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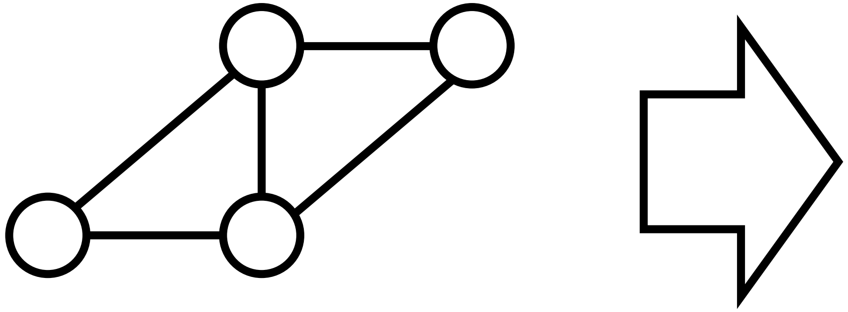
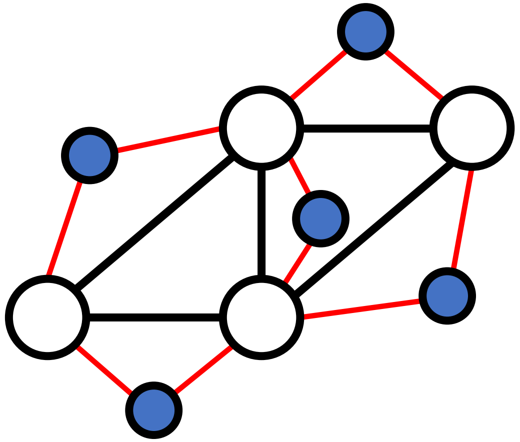
G has a k-VC $\Leftrightarrow$ G' has a k-DS.



Vertex Cover



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Vertex Cover: For $G = (V, E)$ and $k \leq |V|$, $\exists\, V' \subseteq V$, $|V'| \leq k$, s.t. each edge contains an endpoint from V'?

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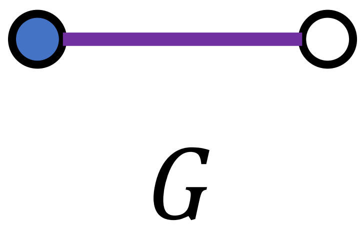

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Dominating Set: For $G = (V, E)$ and $k \leq |V|$, $\exists\, V' \subseteq V$, $|V'| \leq k$, s.t. each $v \in V \backslash V'$ shares an edge with a $u \in V'$?

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G has a k-VC $\Leftrightarrow$ G' has a k-DS.

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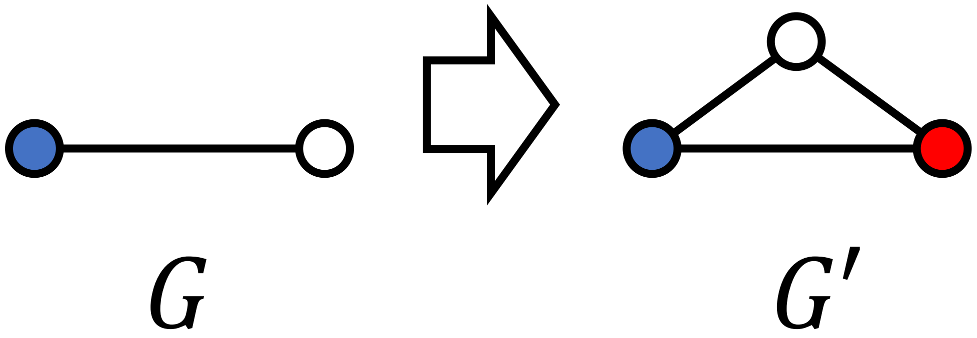

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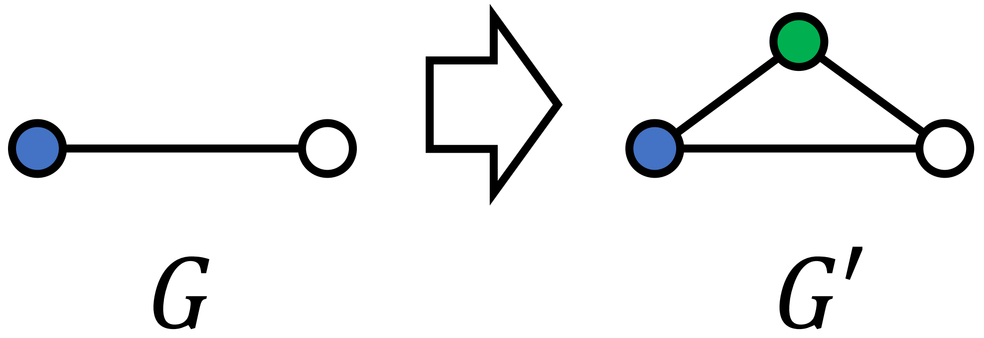

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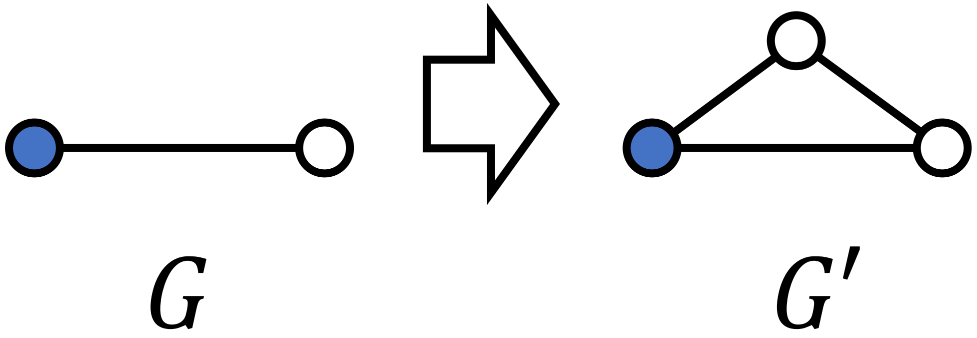

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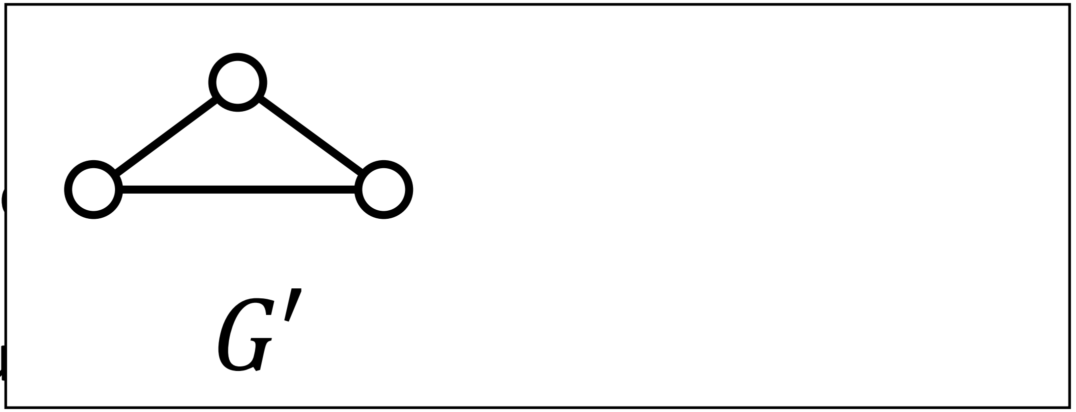

$\Leftarrow$ Suppose G' has an k-DS C.

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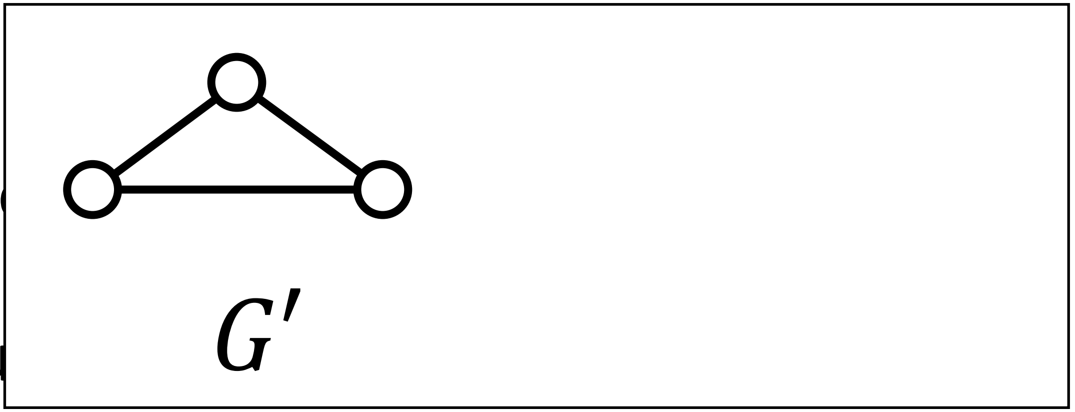

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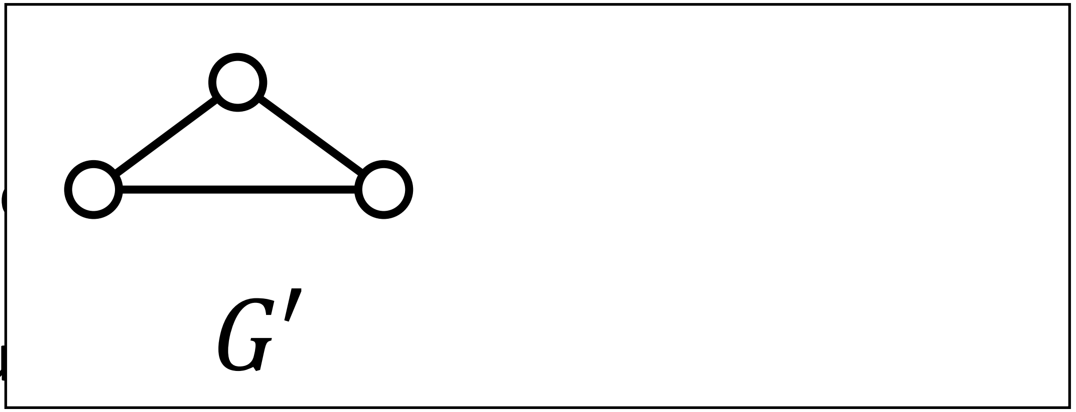

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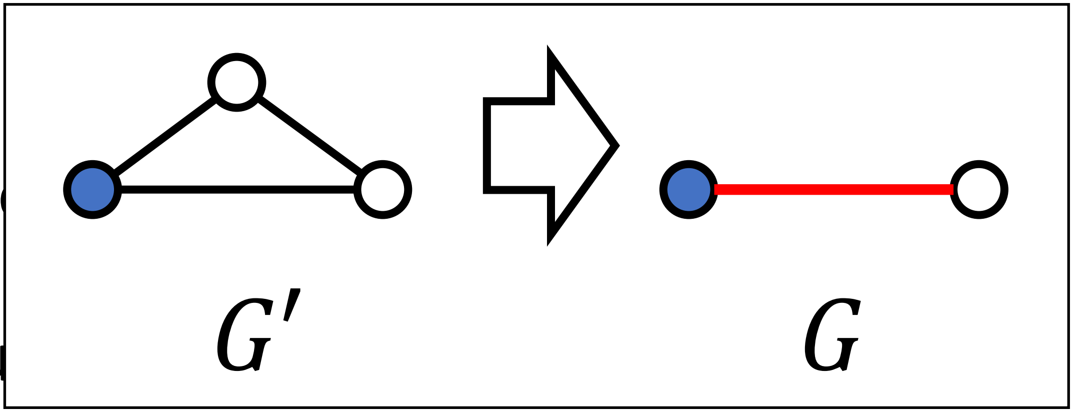

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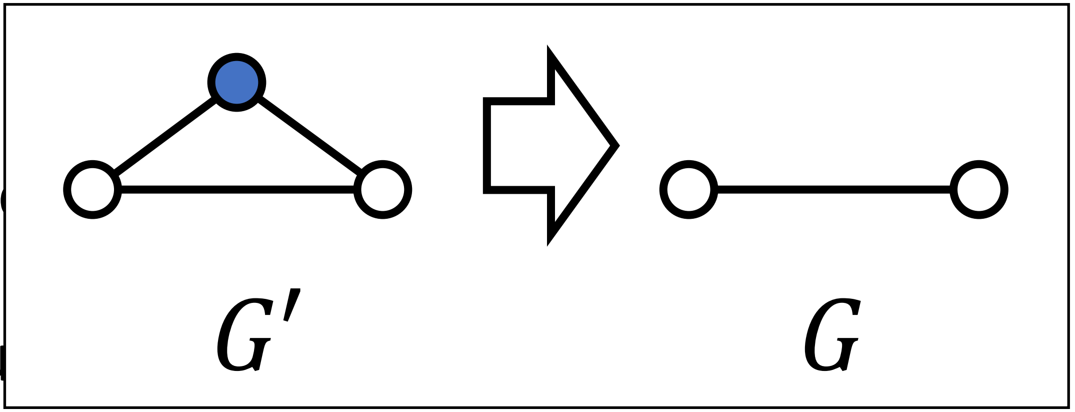

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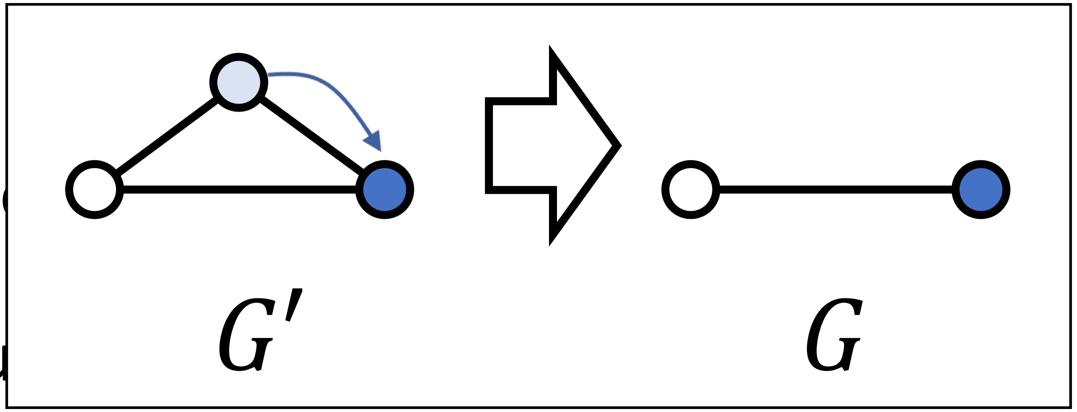

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Claim: Dominating Set $\in NP - C$

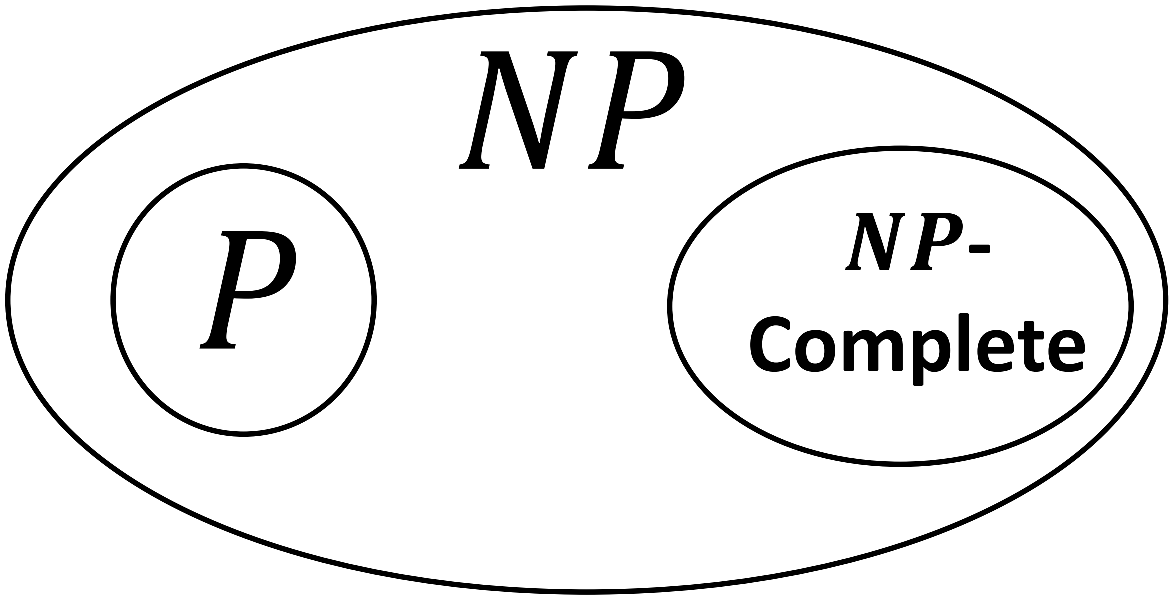
Proof:

1. Show Dominating Set $\in NP$. ✓

2. Show $A \leq_P$ Dominating Set, for some $A \in NP - C$. ✓

$\therefore$ Dominating Set $\in NP - C$

Coping with NP-Completeness



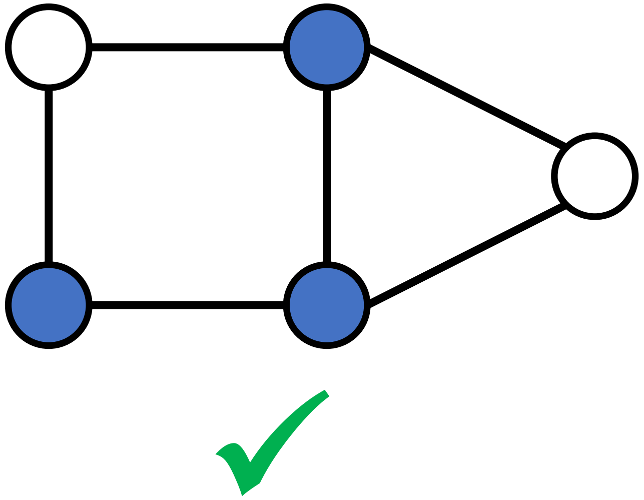
Techniques to handle NP-Complete problems:

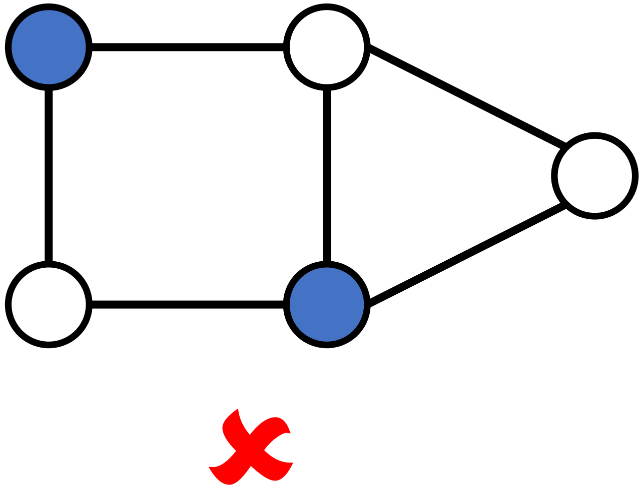
1. Brute Force.
2. Heuristics.
3. Approximation Algorithms.
4. Fixed-parameter Tractable Algorithms.

Vertex Cover

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Optimization Problem.





Vertex Cover – Algorithm

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Algorithm:

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while uncovered edge exists
  select both vertices from uncovered edge
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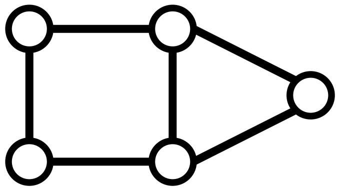

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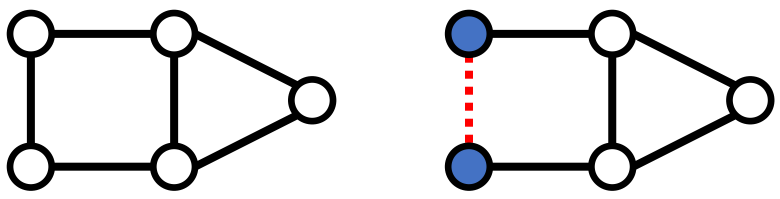

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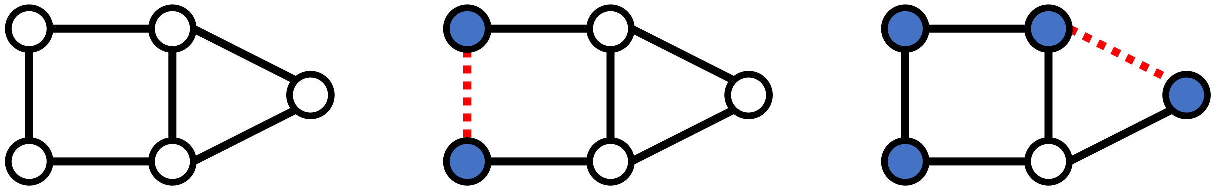

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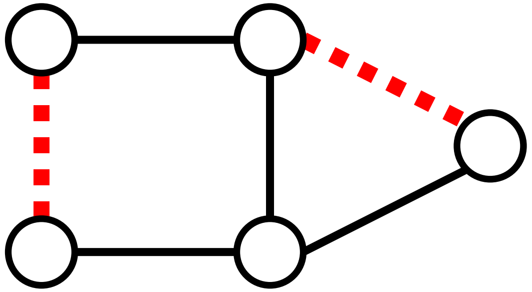
Consider a set of edges, $E' \subset E$, that do not share vertices.



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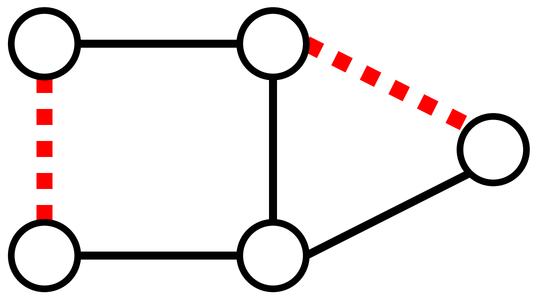
Consider a set of edges, $E' \subset E$, that do not share vertices. Is there a relationship between the minimum vertex cover and $|E'|$?



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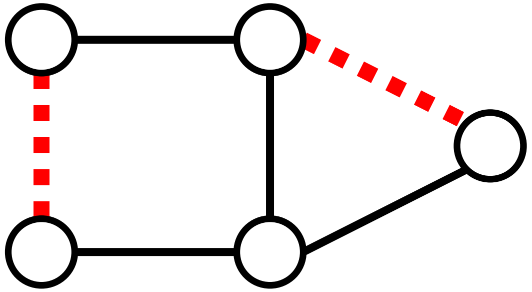
Size of actual smallest vertex cover.



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If we selected fewer than one vertex per edge, we would not have a vertex cover, because that edge would not be covered!