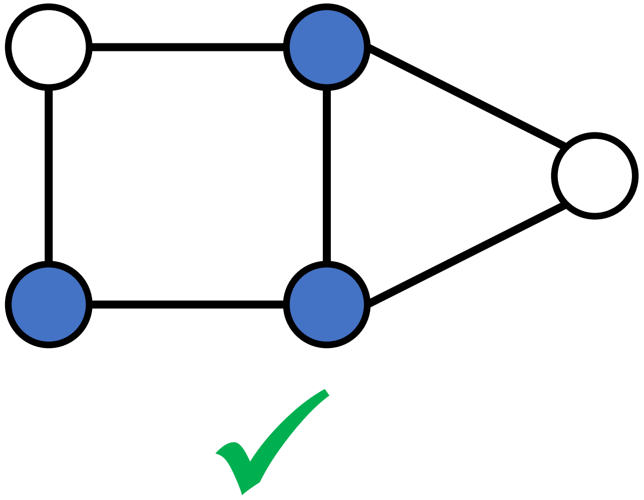
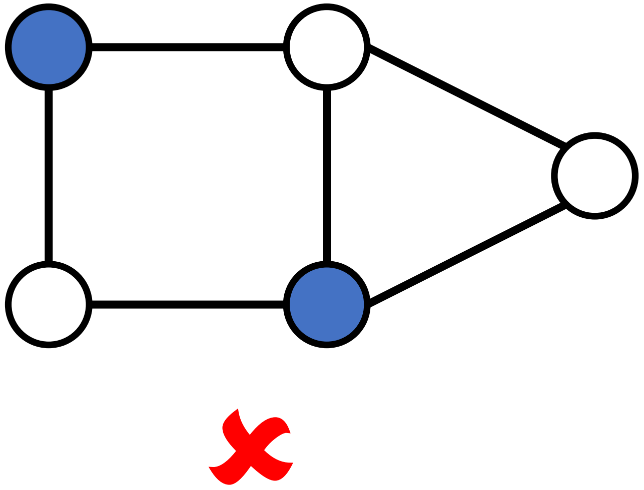
Approximation Algorithms
CSCI 338

Vertex Cover

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Optimization Problem.



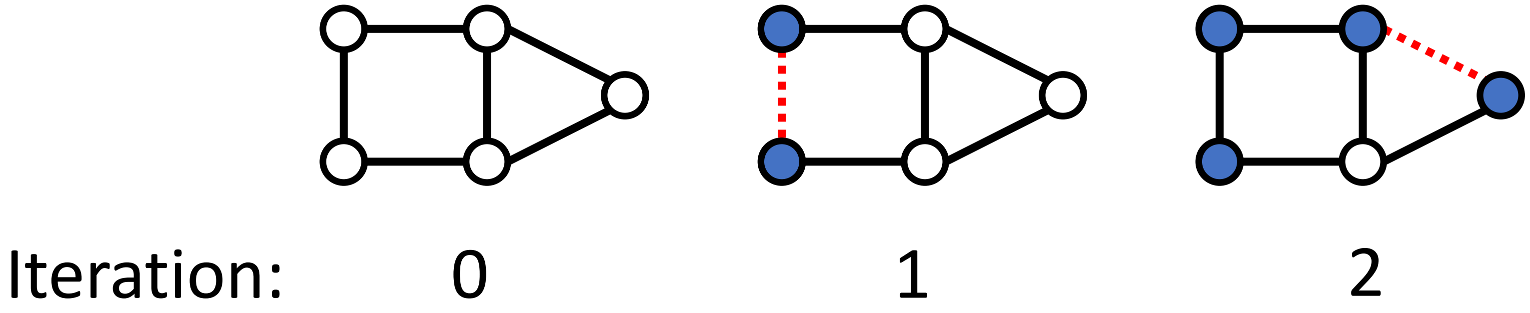


Vertex Cover – Algorithm

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Algorithm:

```
while uncovered edge exists
  select both vertices from uncovered edge
```



Vertex Cover – Performance

```
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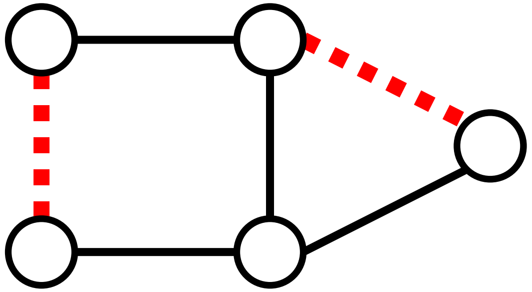
Consider a set of edges, $E' \subset E$, that do not share vertices. Is there a relationship between the minimum vertex cover and $|E'|$?



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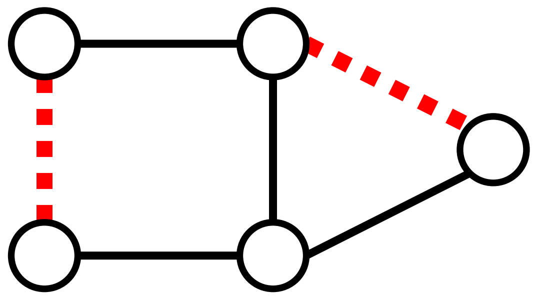
$$|E'| \leq \text{OPT}$$

Size of actual smallest vertex cover.



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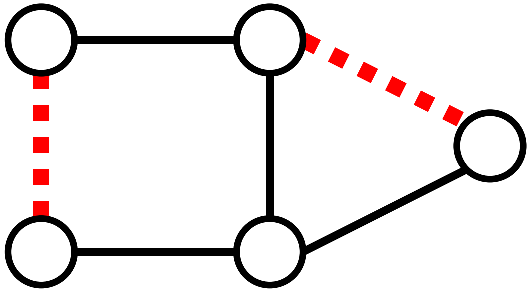
Size of actual smallest vertex cover.

If we selected fewer than one vertex per edge, we would not have a vertex cover, because that edge would not be covered!



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$$\text{ALG} = 2\,|E'|$$

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Consid

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We cannot find optimal vertex covers in poly time unless $P = NP$, but this algorithm is at worst 2-times optimal.

Does t

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Vertex Cover is approximable within the bound $2 - \frac{\log \log|V|}{2 \log|V|}$ and inapproximable within the bound 1.3606.

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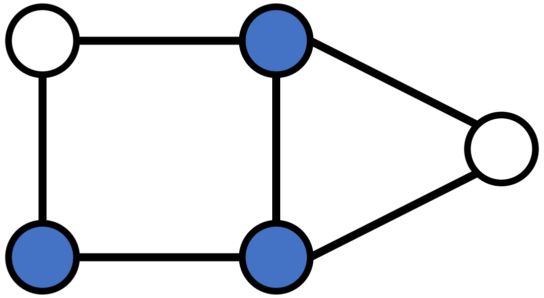
Independent Set

Does the approximation algorithm for Vertex Cover give an approximation algorithm for Independent Set?



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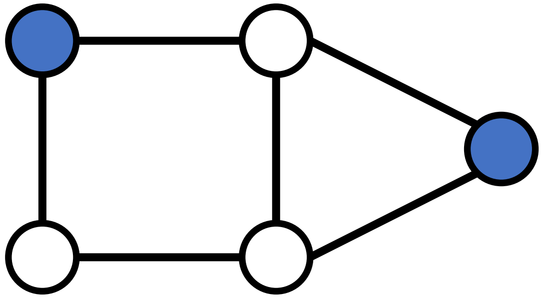

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Minimum Vertex Cover = Maximum Independent Set

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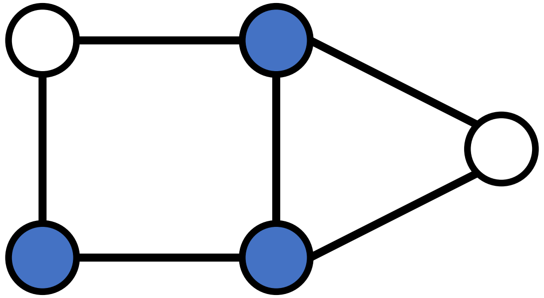

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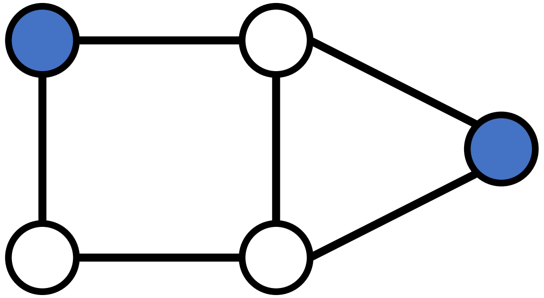


Independent Set

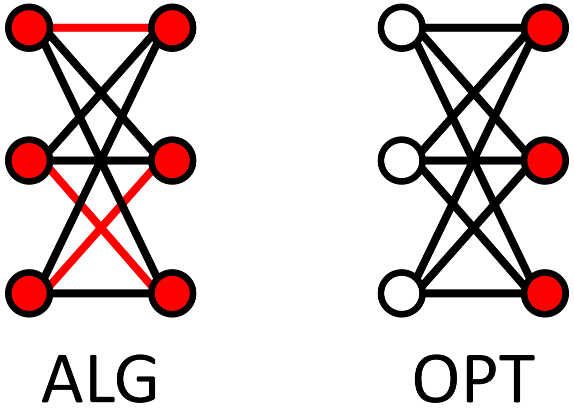
Minimum Vertex Cover = Maximum Independent Set

$$ALG_{VC} \leq 2\ OPT_{VC} \Rightarrow n - ALG_{VC} \geq \frac{1}{?}\ OPT_{IS}$$

Independent Set

Does the approximation algorithm for Vertex Cover give an approximation algorithm for Independent Set?

Complete Bipartite Graph



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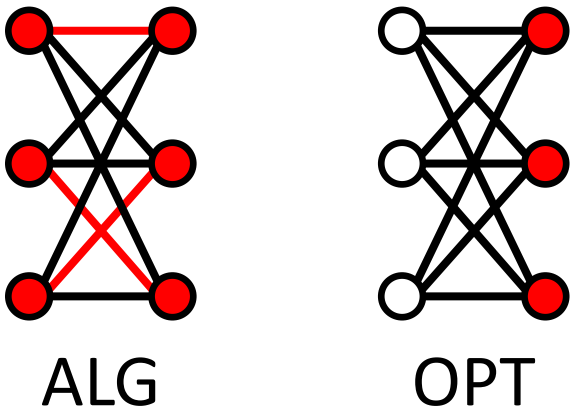

$$ALG_{VC} \leq 2\ OPT_{VC} \Rightarrow n - ALG_{VC} \geq \frac{1}{?}\ OPT_{IS}$$

$$n \leq 2\frac{n}{2} \qquad\qquad 0 \geq \frac{1}{?}\frac{n}{2}$$

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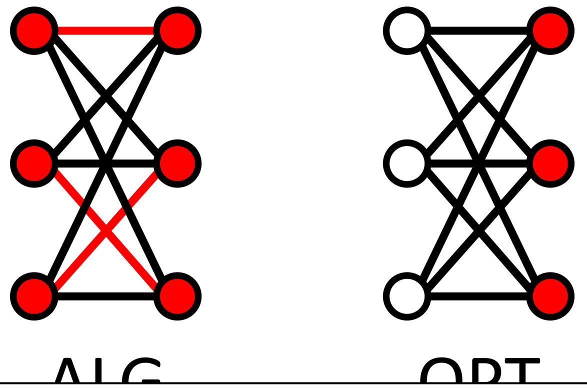

Independent Set is inapproximable within the bound $|V|^{1-\varepsilon}$, for any $\varepsilon > 0$.

$n \leq 2\frac{n}{2}$

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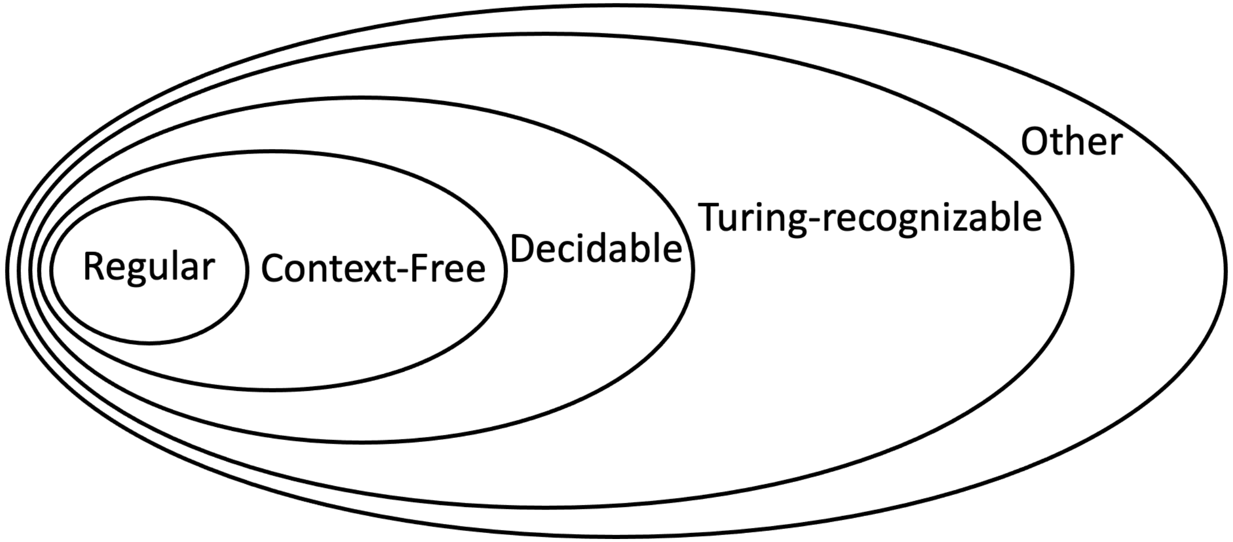
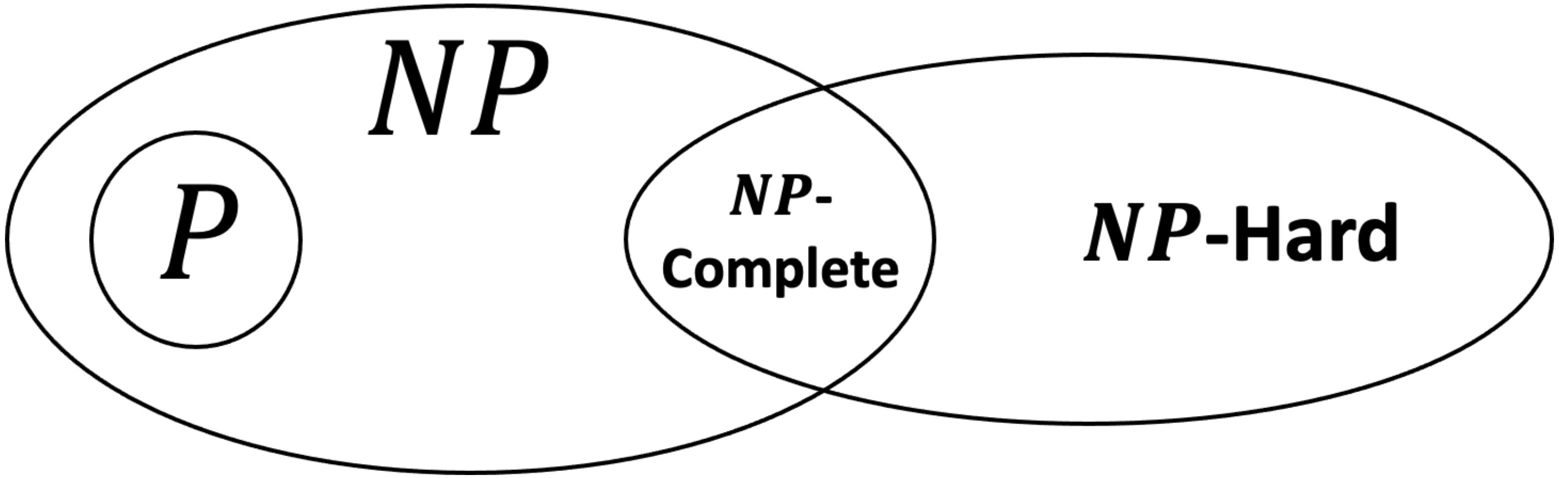
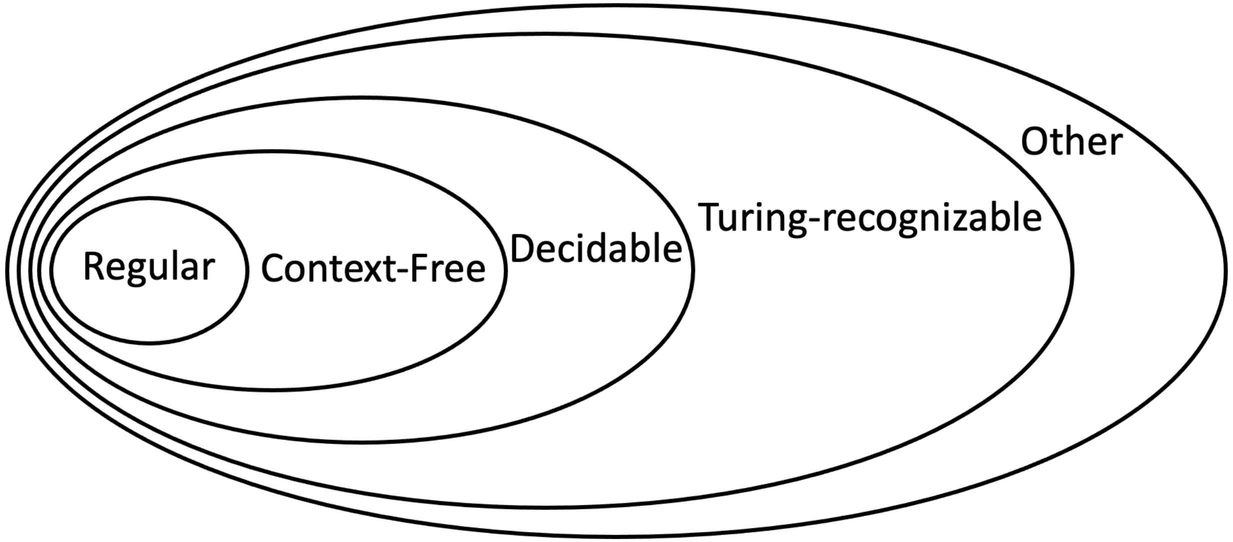


Computability Hierarchy



Complexity Hierarchy



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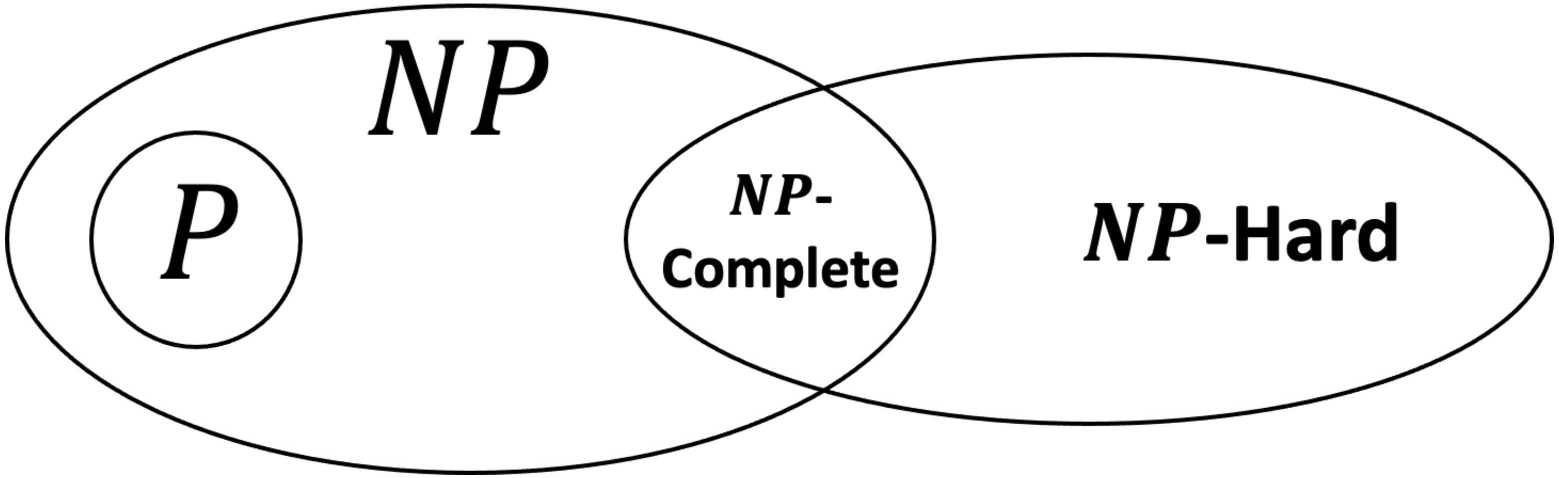

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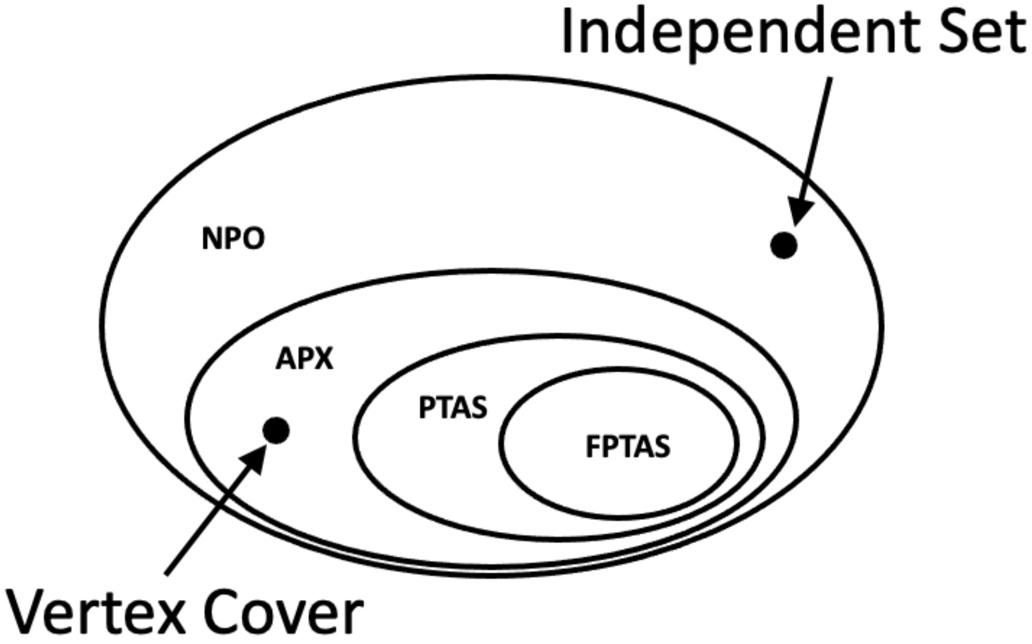
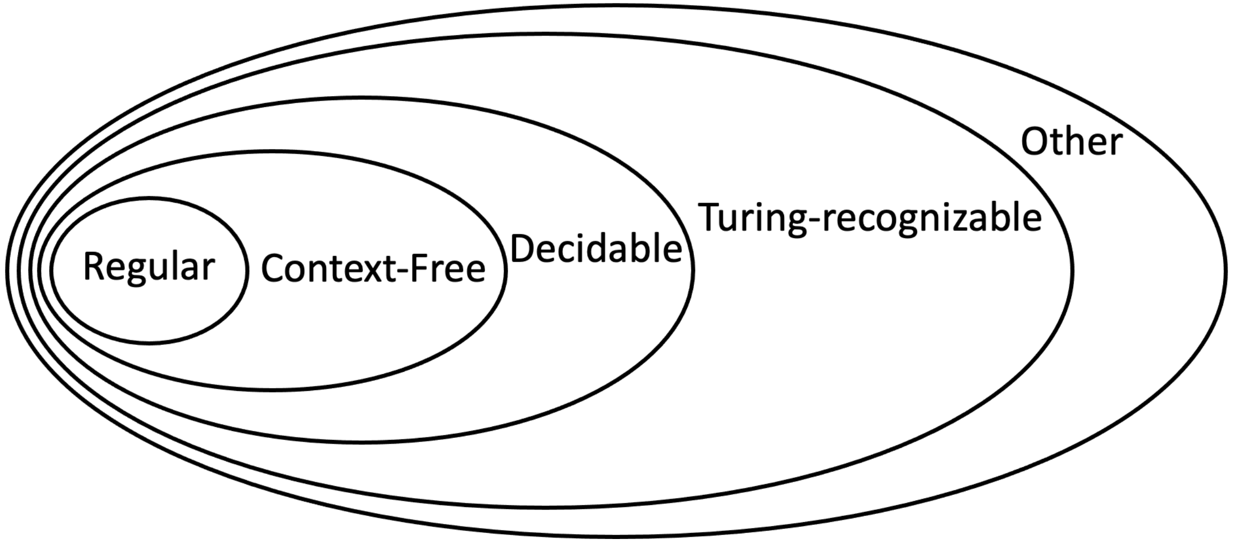


Approximability Hierarchy



Computability Hierarchy

TSP Approximation Algorithm

TSP: Given a weighted graph, find a least cost cycle that visits each vertex exactly once.

Algorithm?

TSP Approximation Algorithm

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Suppose H is an α-approximation algorithm for TSP.

I.e. $H(G) =$ Hamiltonian Cycle C_H, where cost(C_H) $\leq \alpha$ cost(C_{OPT})

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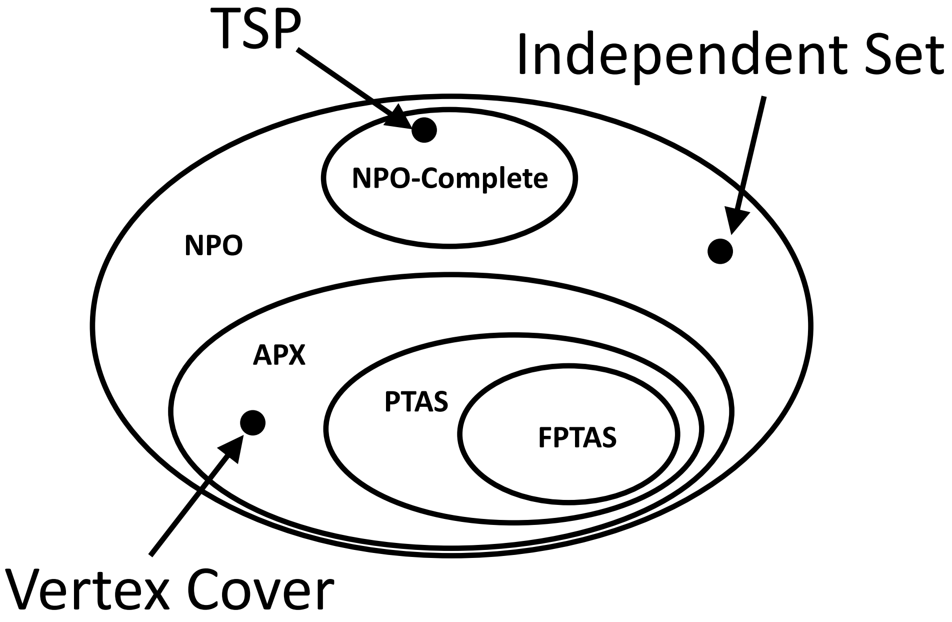

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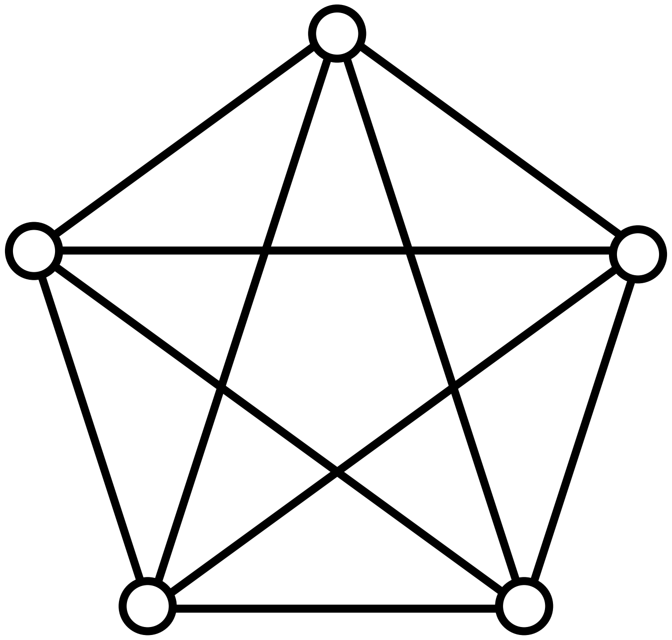
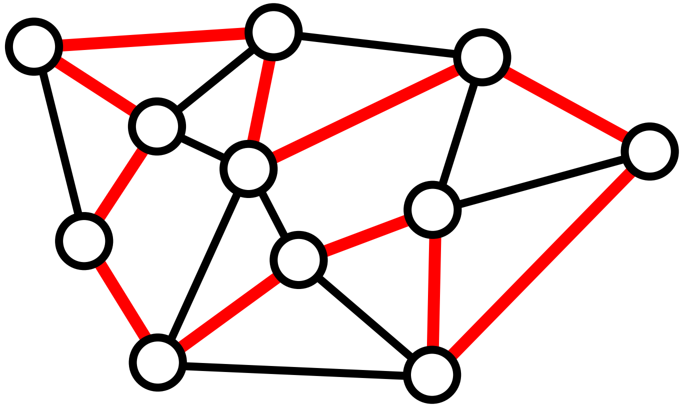
Special Case - Metric TSP

Metric TSP: Given a complete weighted graph that satisfies the triangle inequality, find a least cost cycle that visits each vertex exactly once.

Triangle Inequality: $\text{cost}(u, v) \leq \text{cost}(u, w) + \text{cost}(w, v)$



Special Case - Metric TSP



Find some structure that is:

1. Easy to compute.
2. Related to TSP.
3. Lower bound on OPT.

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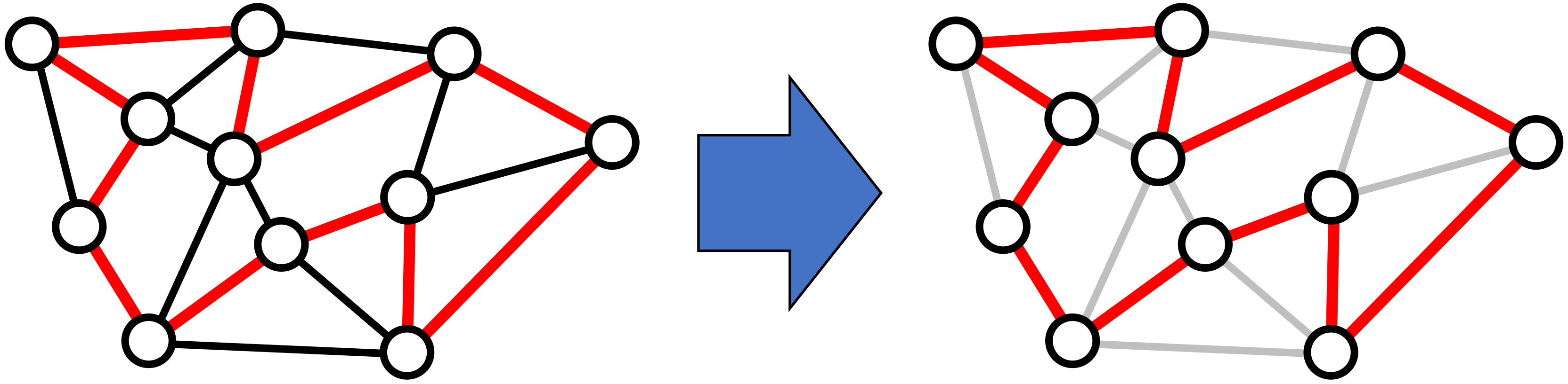

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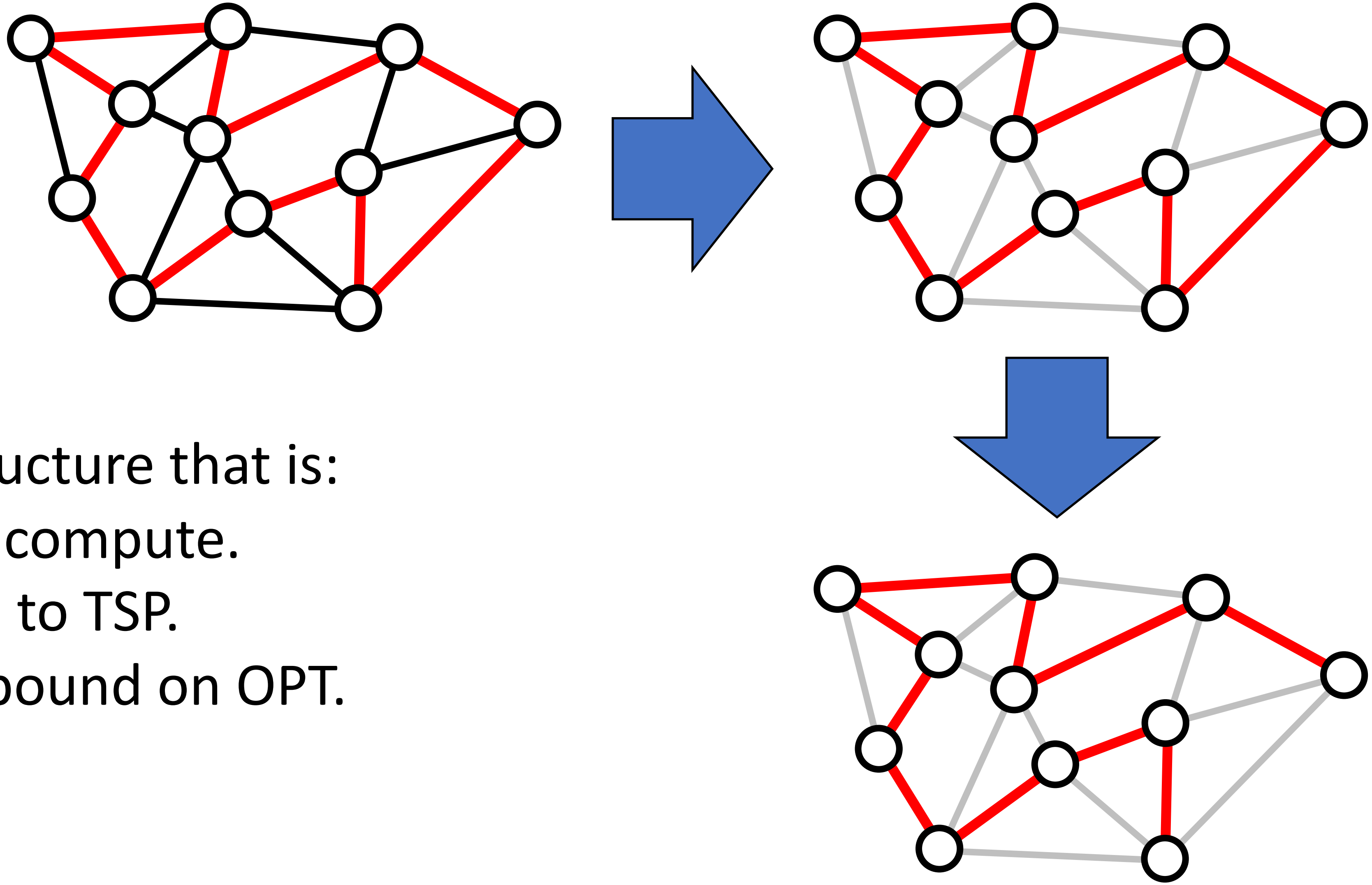

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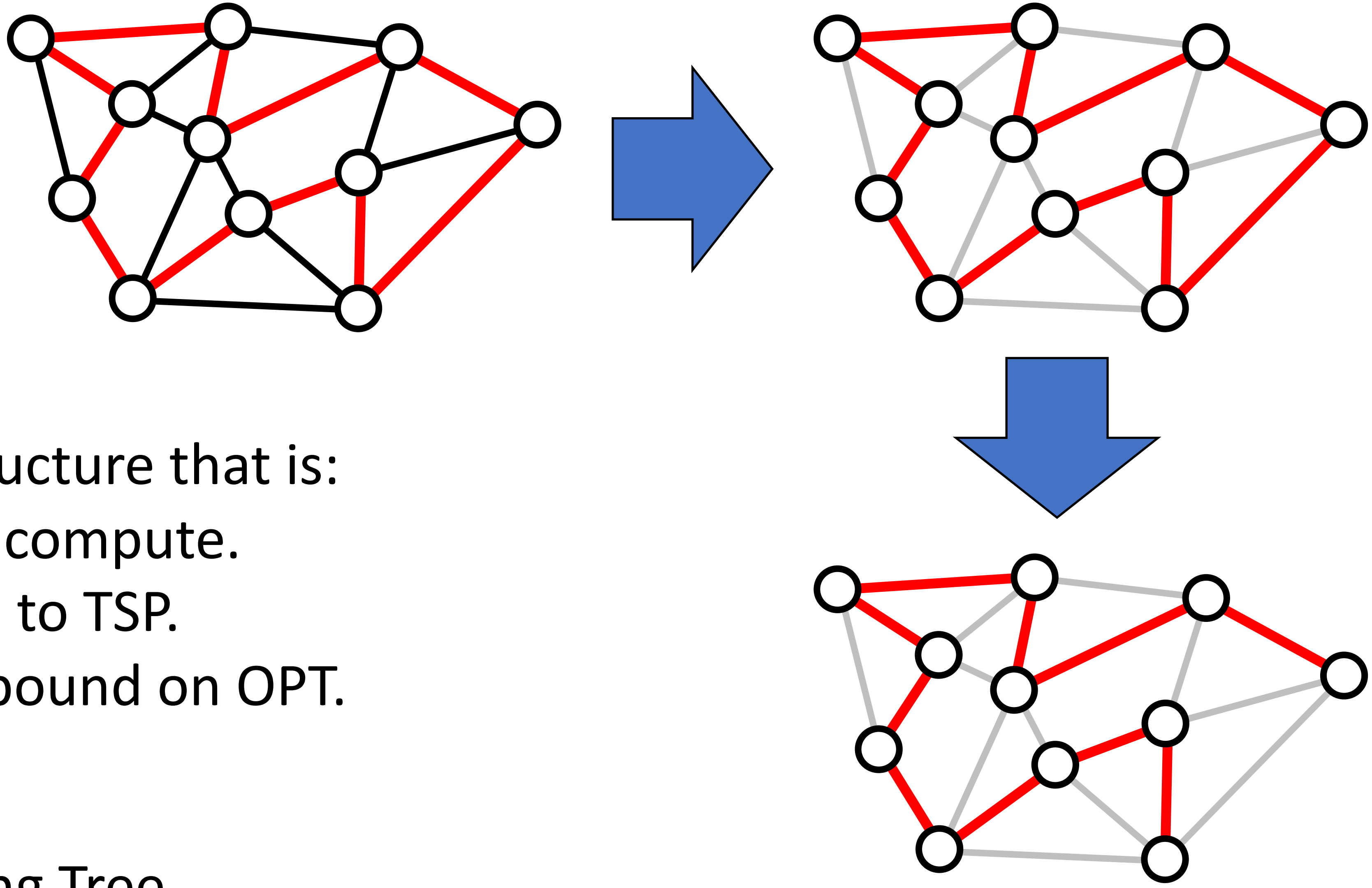

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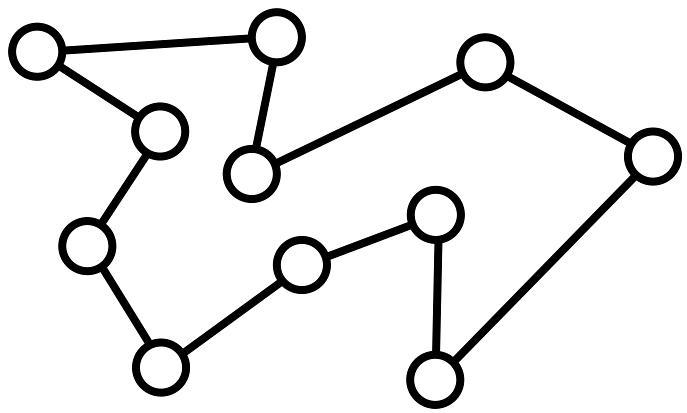
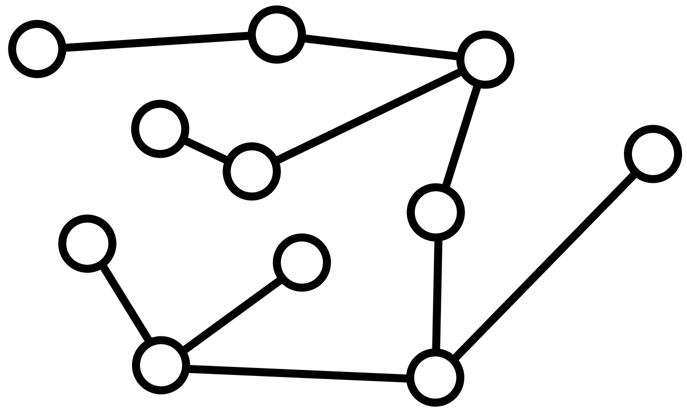
Spanning Tree

Special Case - Metric TSP

Relationship between OPT and cost of MST?



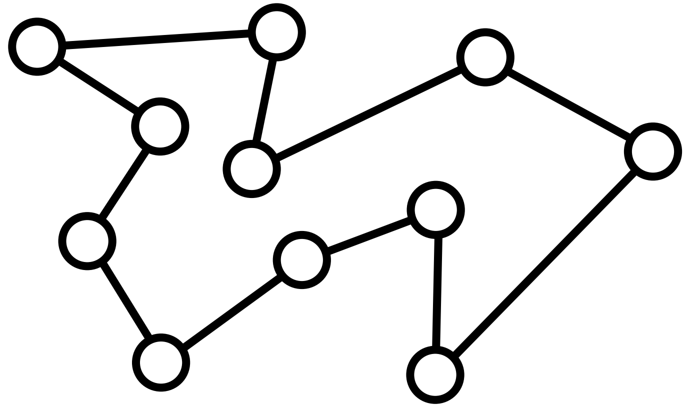


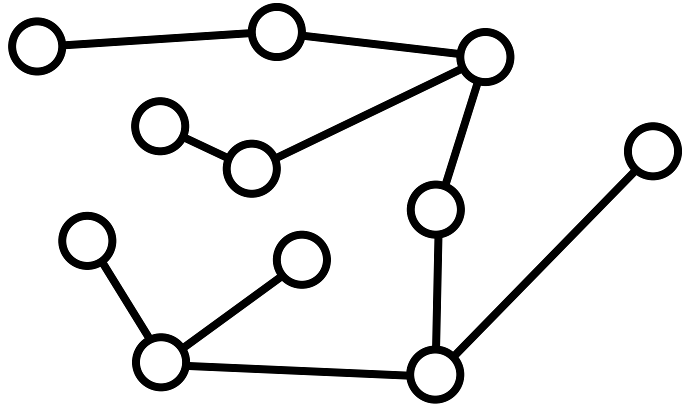
Special Case - Metric TSP

Relationship between OPT and cost of MST?

$\text{cost(MST)} \leq \text{OPT}$

How to turn MST into a cycle?





Special Case - Metric TSP

Relationship between OPT and cost of MST?

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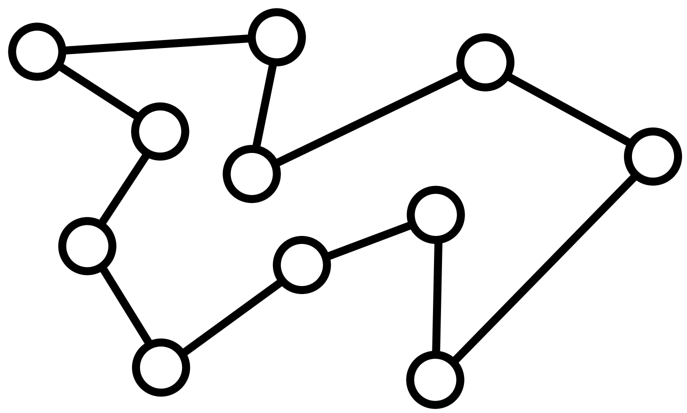
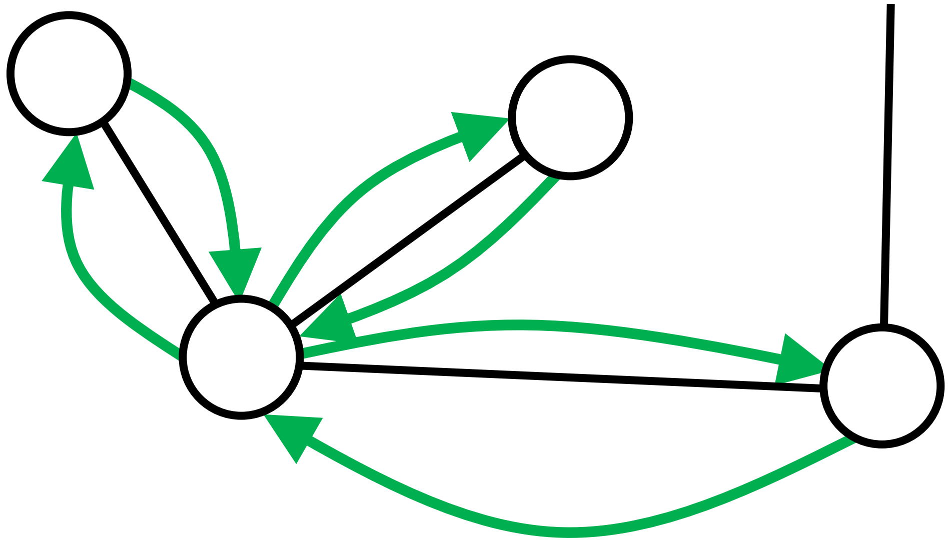
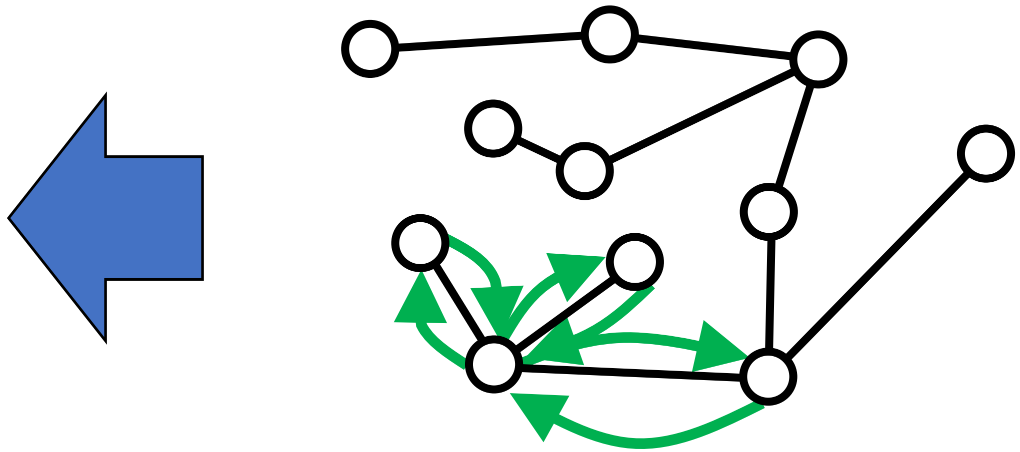
How to turn MST into a cycle?

What is the cost of this cycle?







Special Case - Metric TSP

Relationship between OPT and cost of MST?

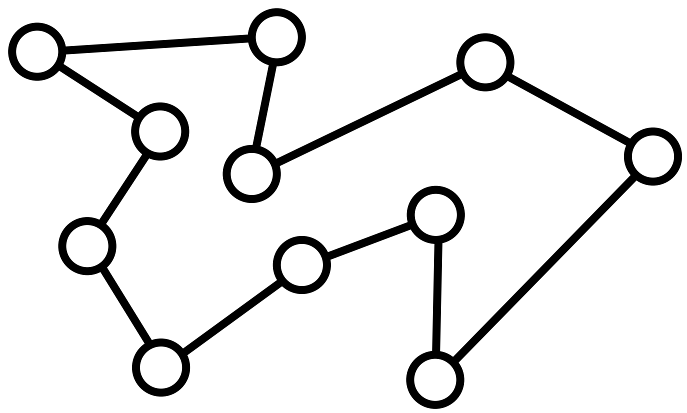
$$\text{cost(MST)} \leq \text{OPT}$$

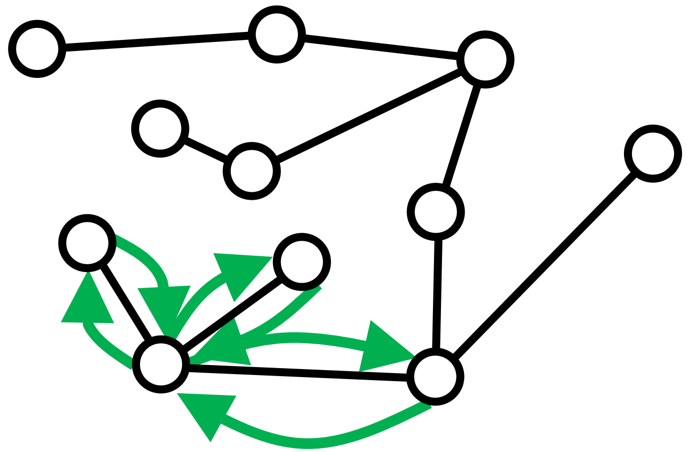
How to turn MST into a cycle?

What is the cost of this cycle?

$$\text{ALG} = 2\ \text{cost(MST)}$$

Relationship between ALG and OPT?





Special Case - Metric TSP

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How to turn MST into a cycle?

What is the cost of this cycle?

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Relationship between ALG and OPT?

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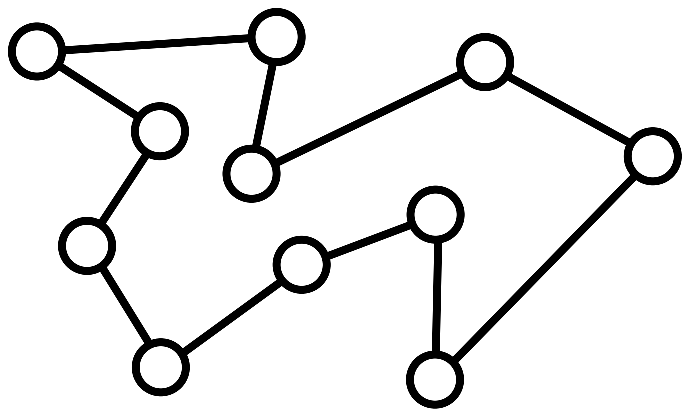
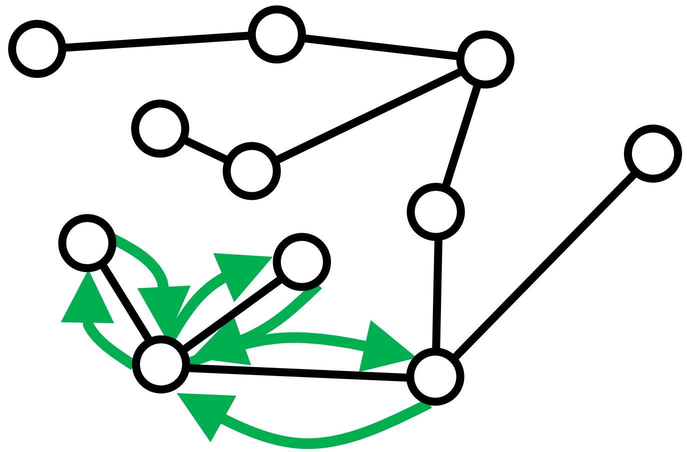
Any problems?





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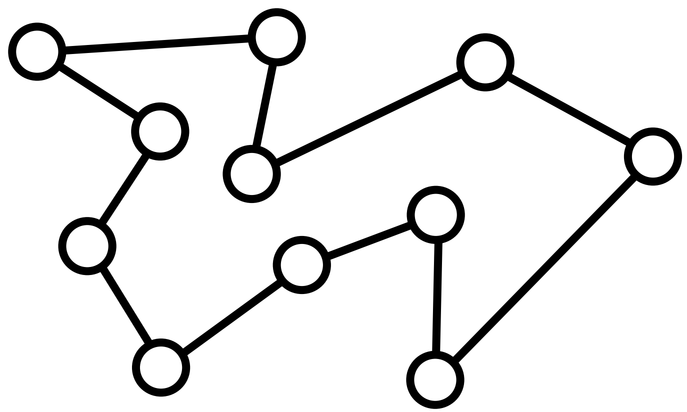


How to turn MST into a cycle?

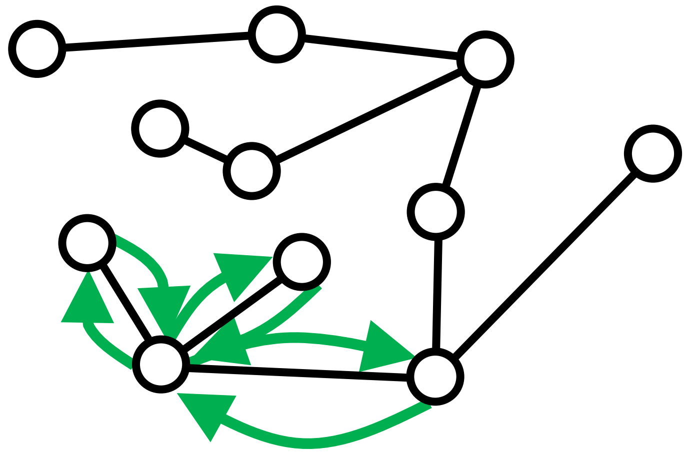
What is the cost of this cycle?

$$\text{ALG} = 2 \text{ cost(MST)}$$

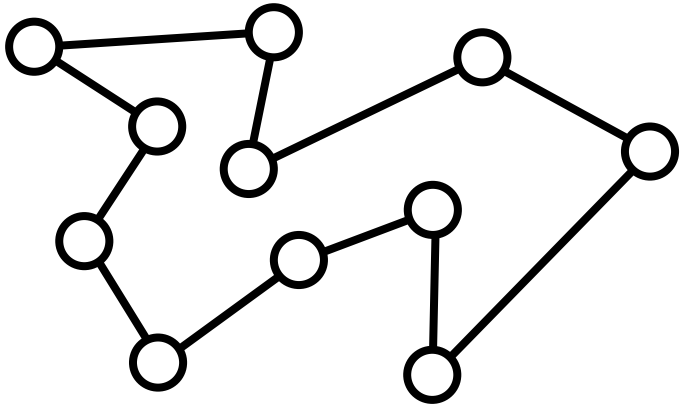
Relationship between ALG and OPT?

$$\text{ALG} = 2 \text{ cost(MST)} \leq 2 \text{ OPT}$$

How can we eliminate double visits (without messing up the cost)?



Special Case - Metric TSP



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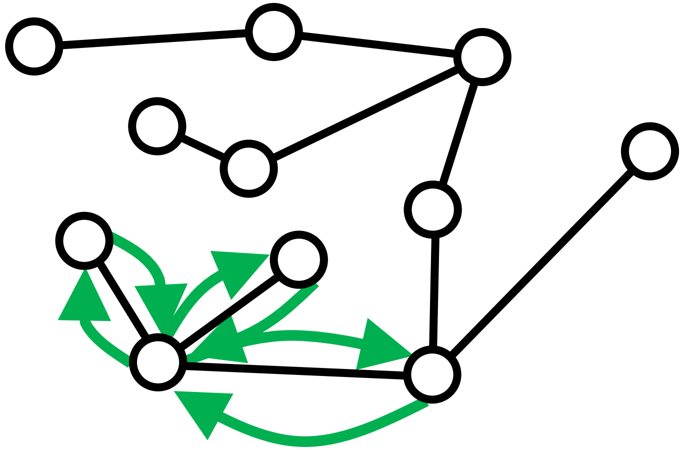

How can we eliminate double visits (without messing up the cost)?

Skip to next unvisited vertex. Can only decrease cost (triangle inequality).

$\text{dist}(u, v) \leq \text{dist}(u, w) + \text{dist}(w, v)$

Special Case - Metric TSP

Metric TSP: Given a complete weighted graph that satisfies the triangle inequality, find a least cost cycle that visits each vertex exactly once.

$$\text{ALG} = 2\ \text{cost(MST)} \leq 2\ \text{OPT}$$