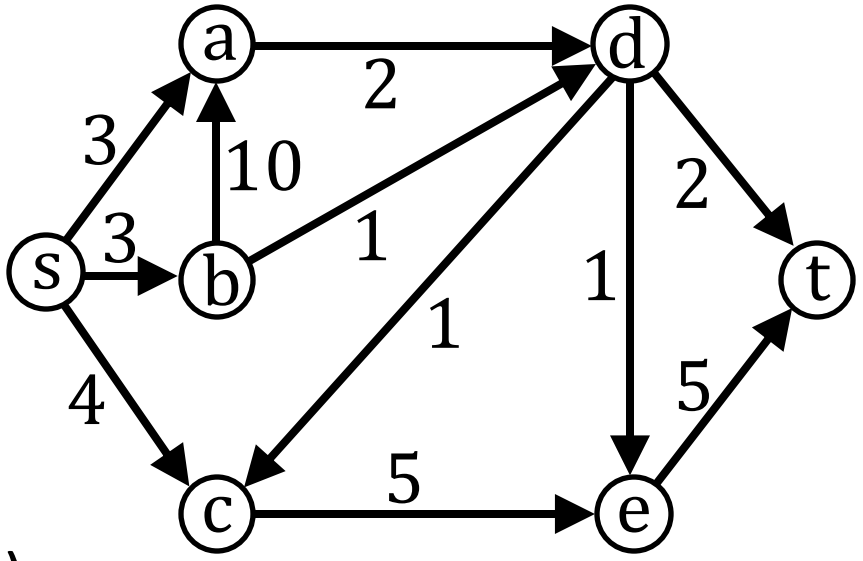
Flow Networks
CSCI 432

Flow Network

Flow Network:

- Directed-edge graph, $G = (V, E)$.
- Non-negative edge capacity, c_e.
- Single source, s, without input edges.
- Single sink, t, without output edges.



An $s - t$ flow is a function $f\colon E \to \mathbb{R}^+$ such that:

- $0 \leq f(e) \leq c_e, \forall e \in E$. (capacity constraint)
- $\sum_{e \epsilon \text{input}(v)} f(e) = \sum_{e \epsilon \text{output}(v)} f(e)\,, \forall v \in V \setminus \{s, t\}$. (conservation of flow constraint: "Everything that goes into a node has to come out, except for s and t")
- Value of flow = $val(f)$ = $\sum_{e \epsilon \text{output}(s)} f(e)$ = $\sum_{e \epsilon \text{input}(t)} f(e)$

Ford-Fulkerson Algorithm

Residual graph for flow f, G_f:

- $\forall e$, if $f(e) < c_e$, let $c_e = c_e - f(e)$.
- $\forall e = (u, v)$, if $f(e) > 0$, create $e' = (v, u)$ with $c_{e'} = f(e)$

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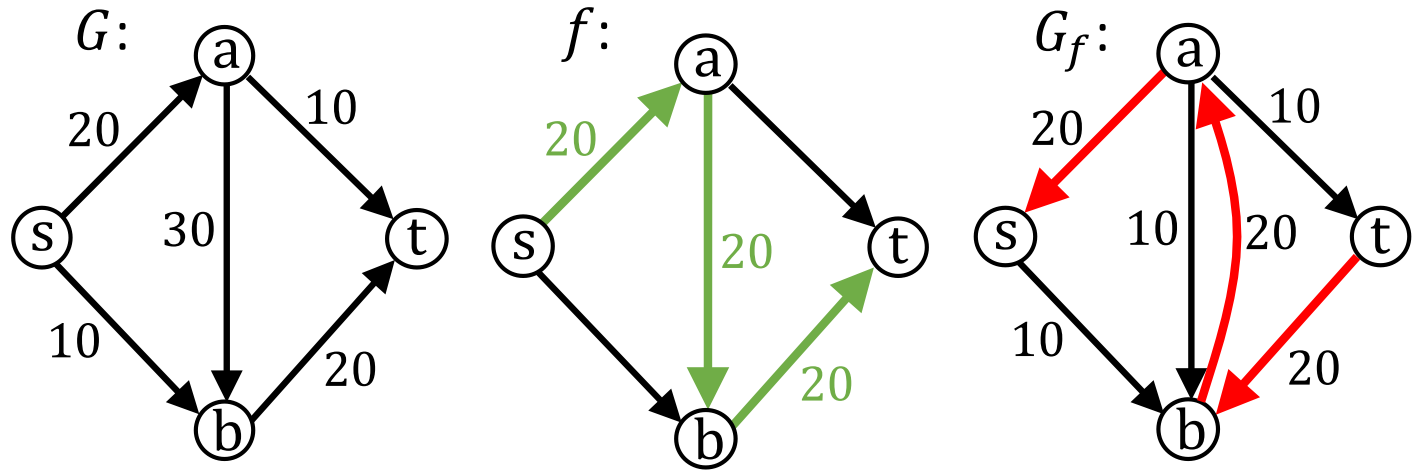

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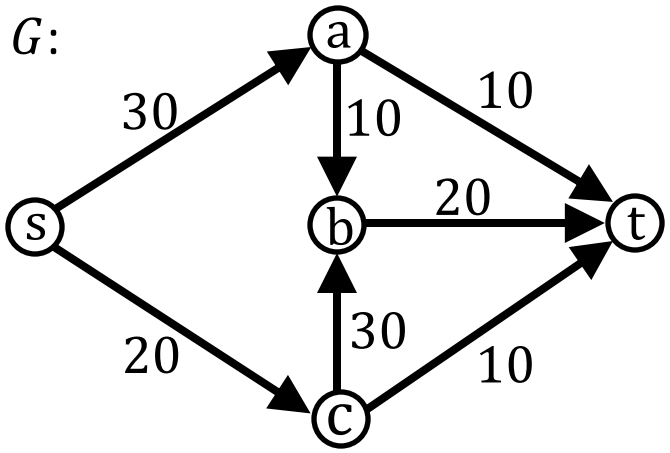

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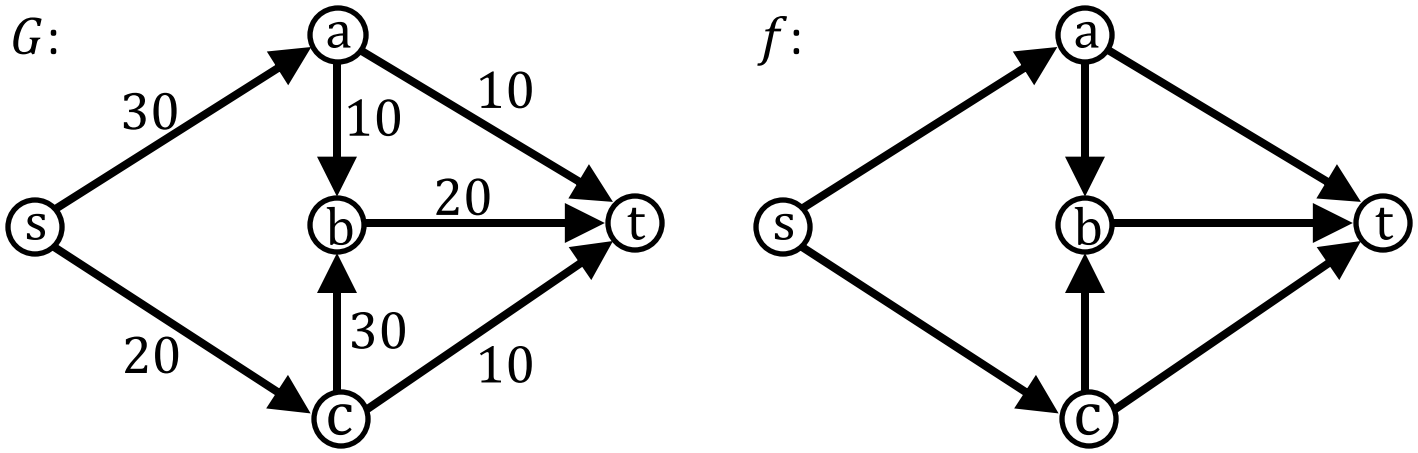

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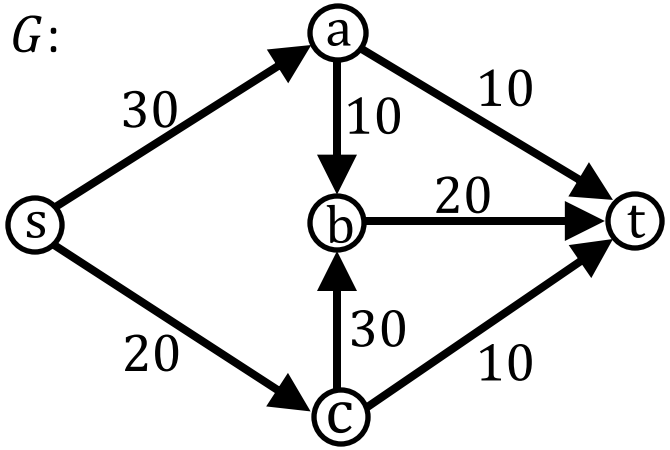

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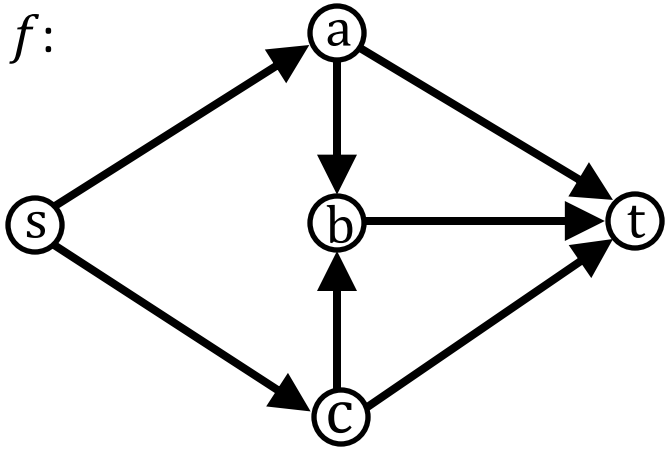

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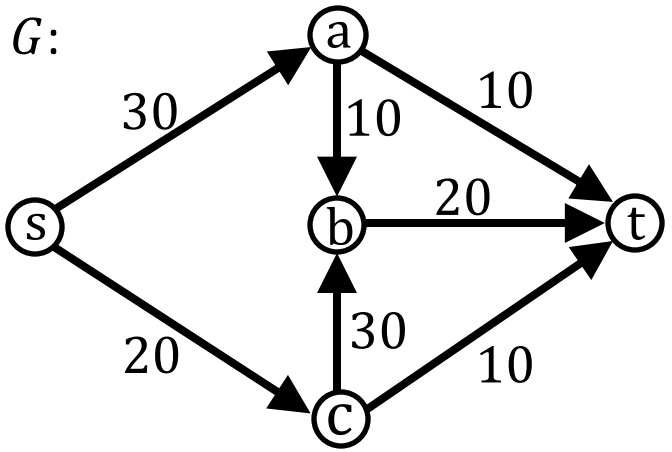

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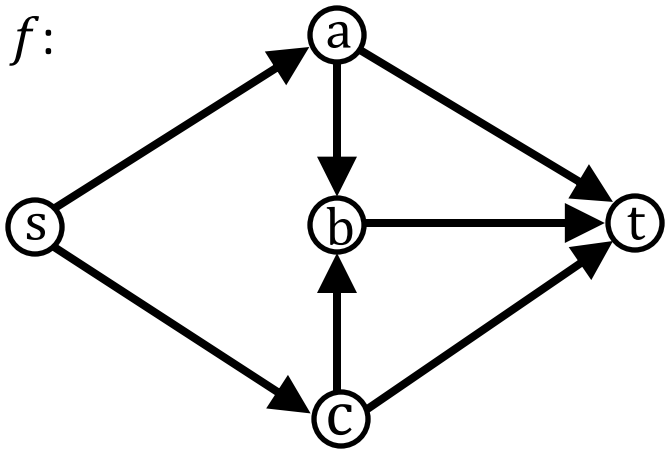

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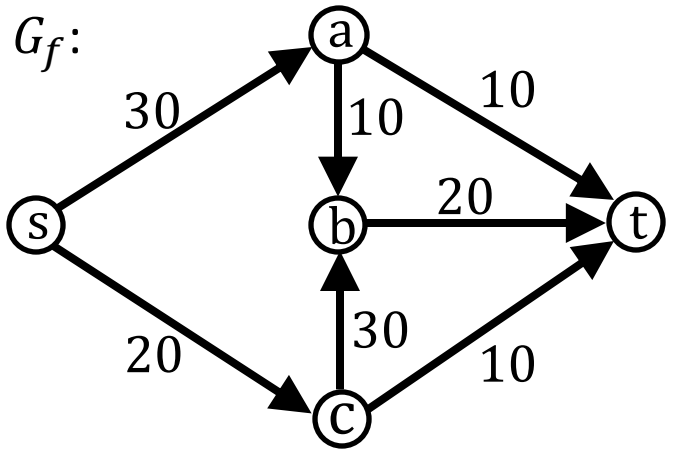

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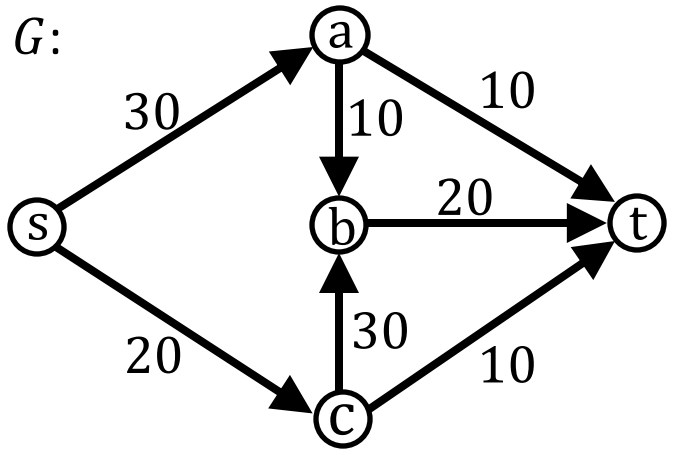

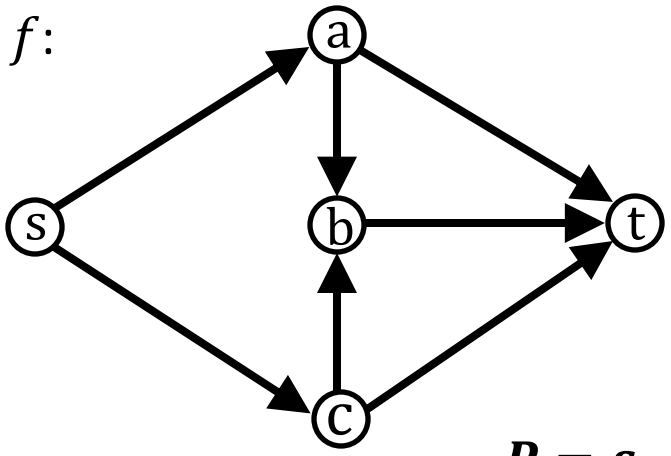

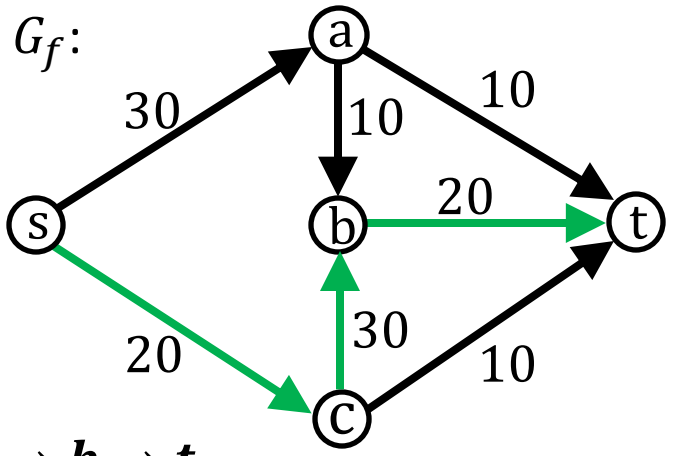

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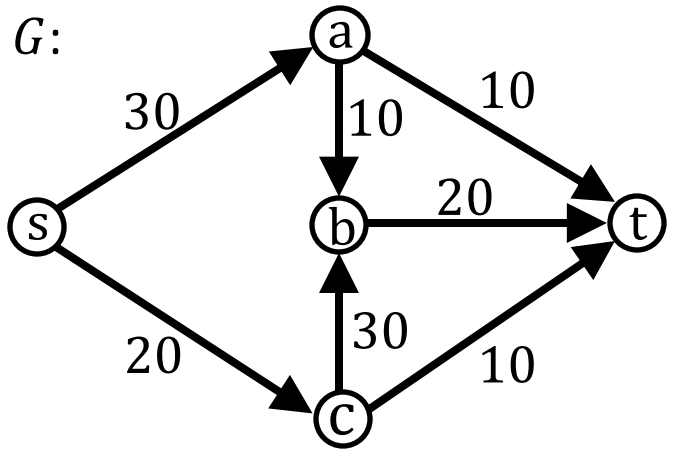

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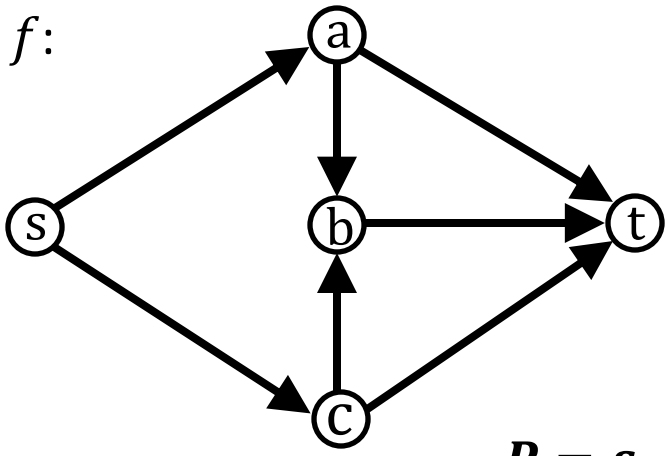

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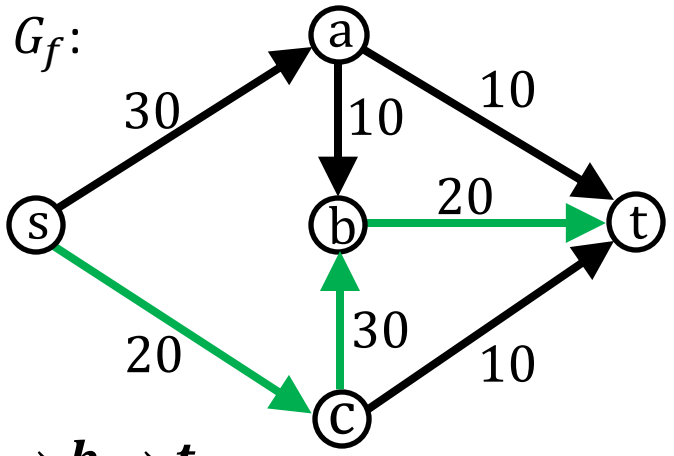

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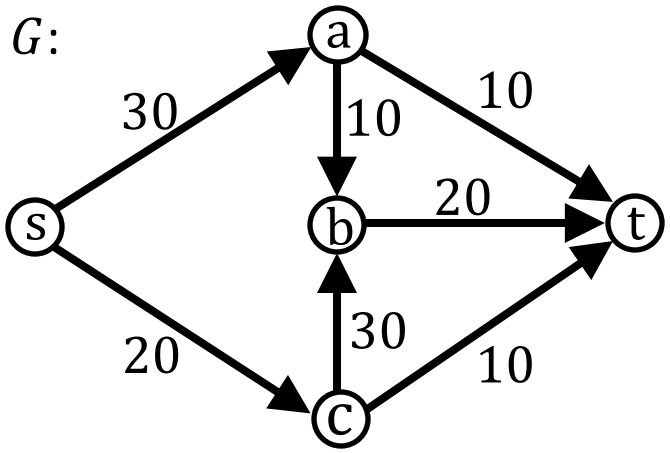

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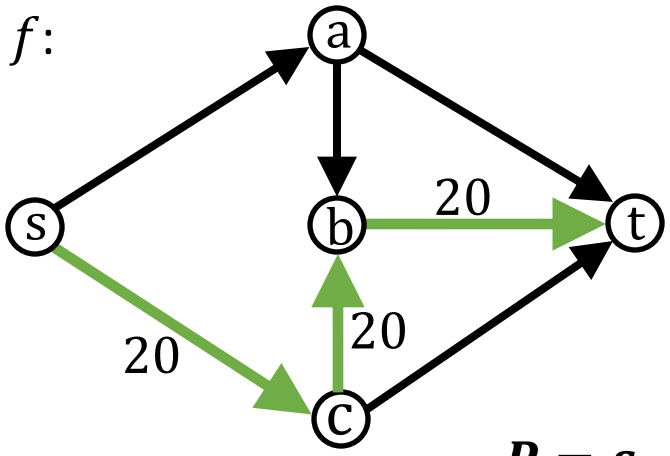

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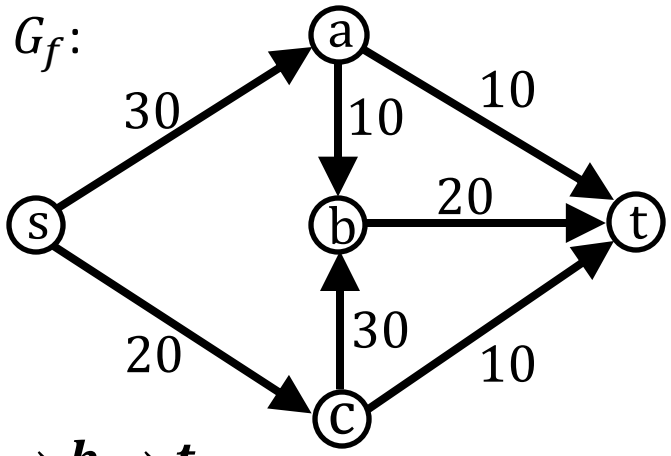

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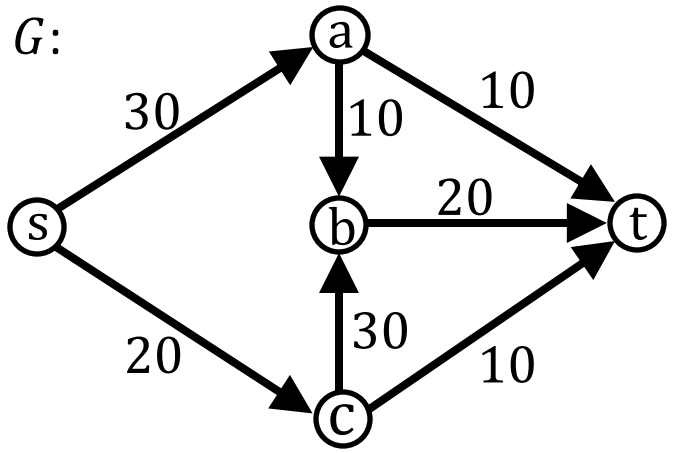

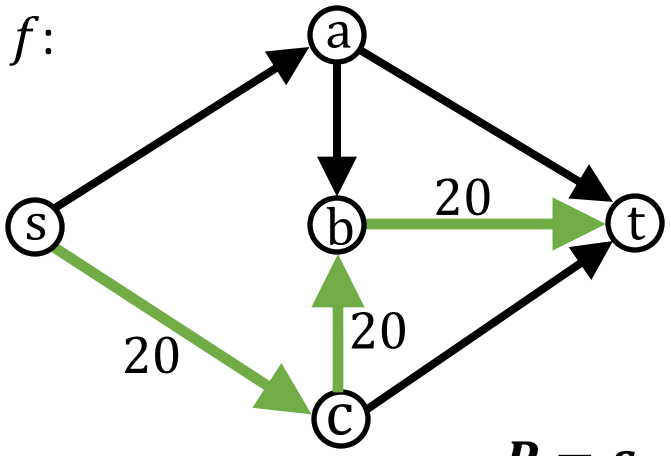

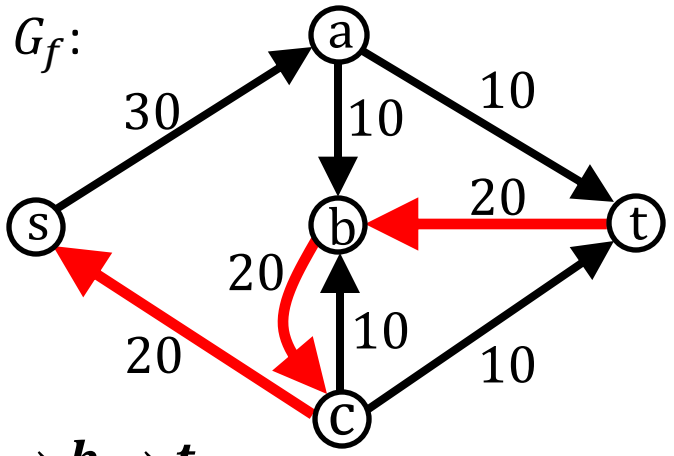

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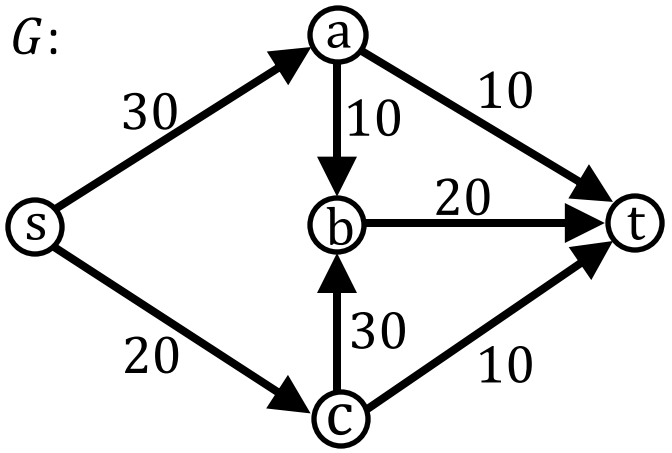

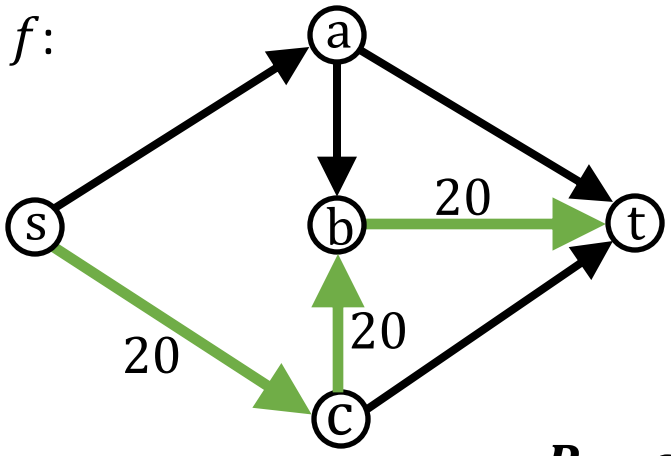

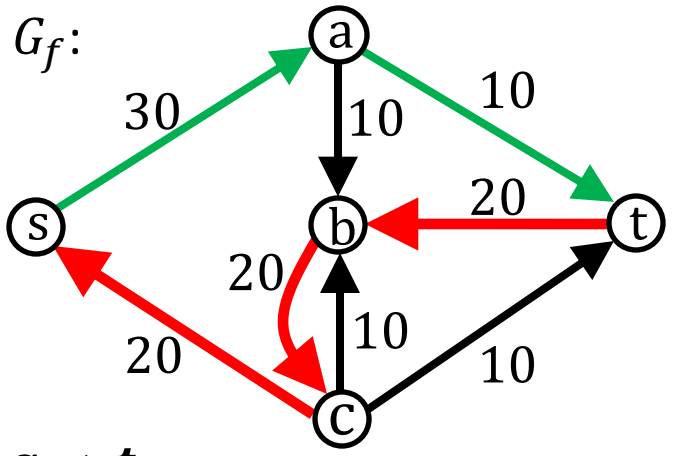

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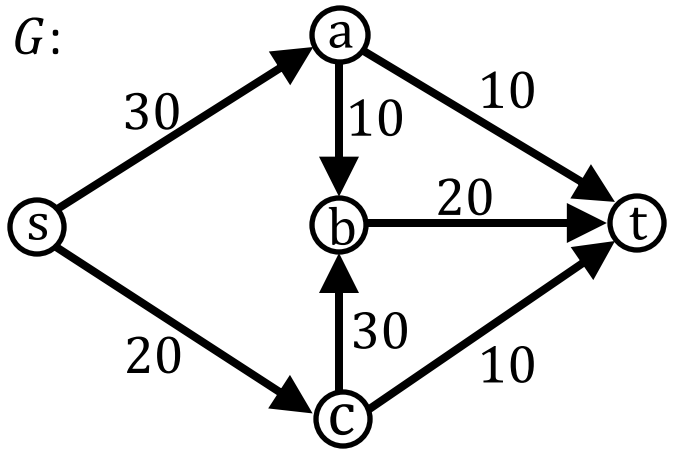

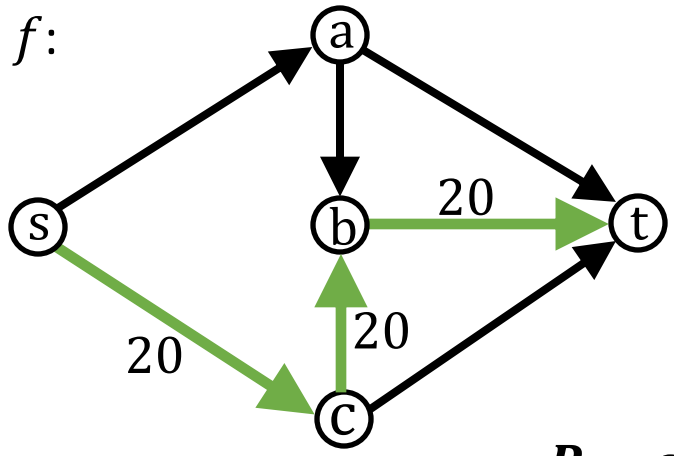

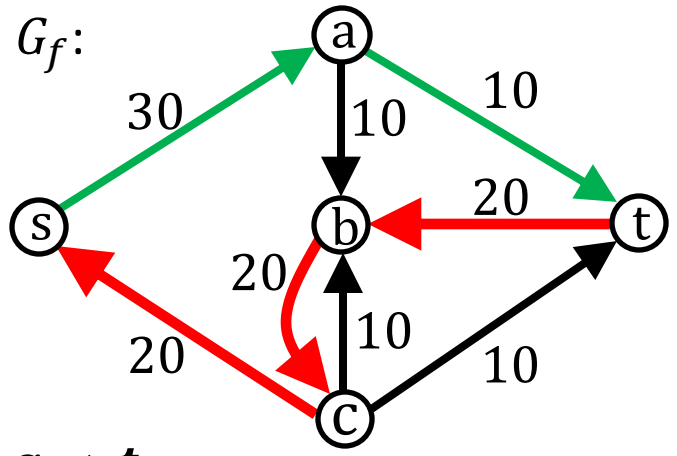

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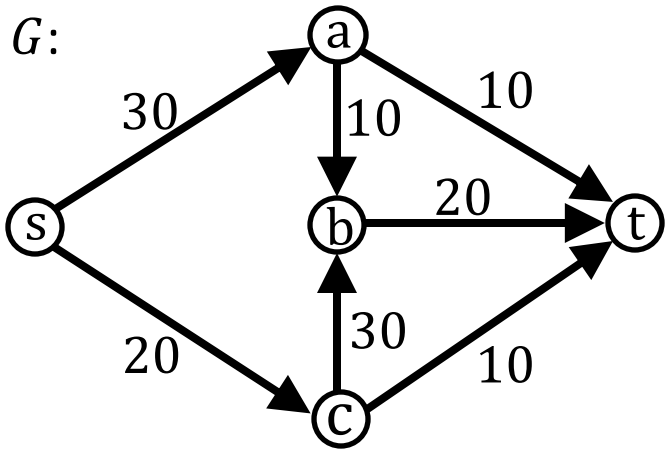

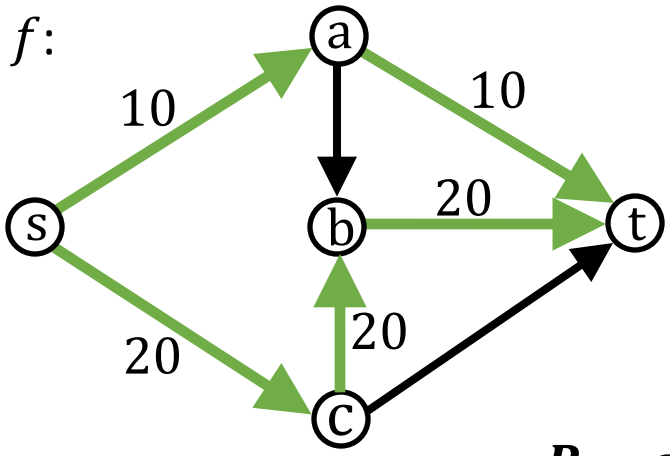

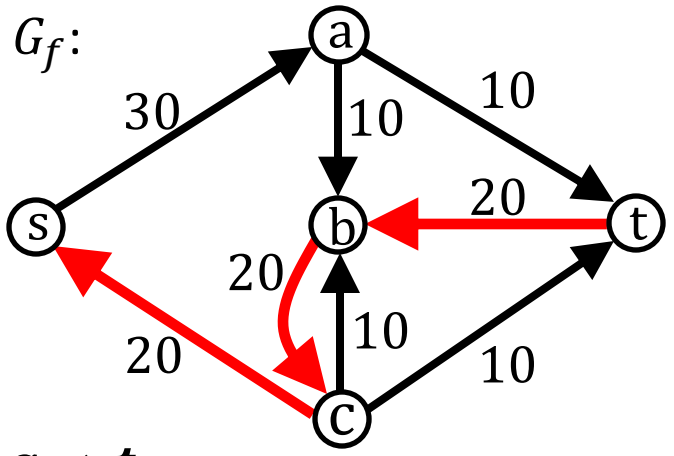

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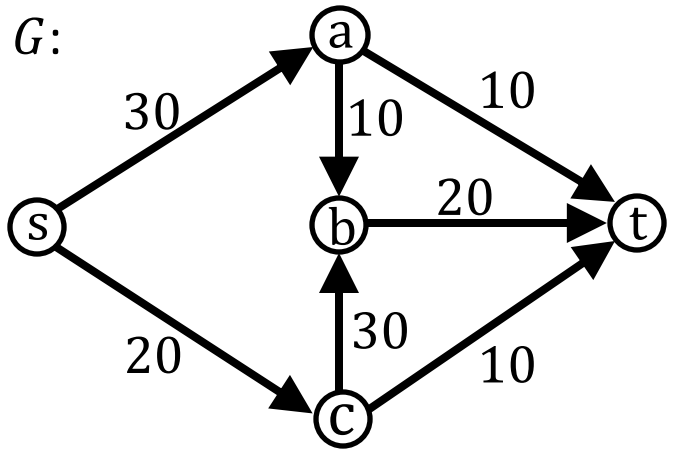

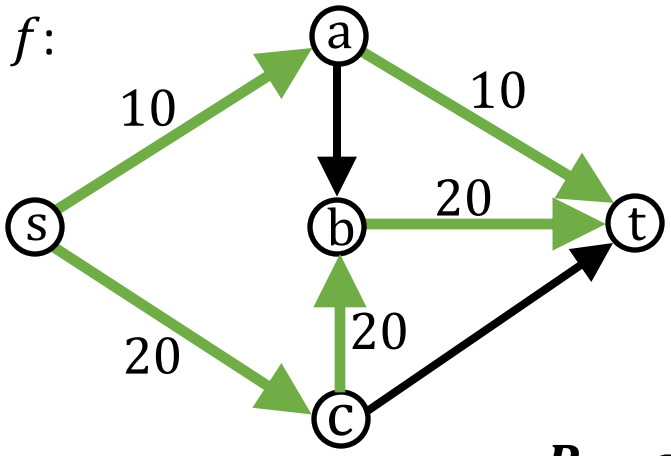

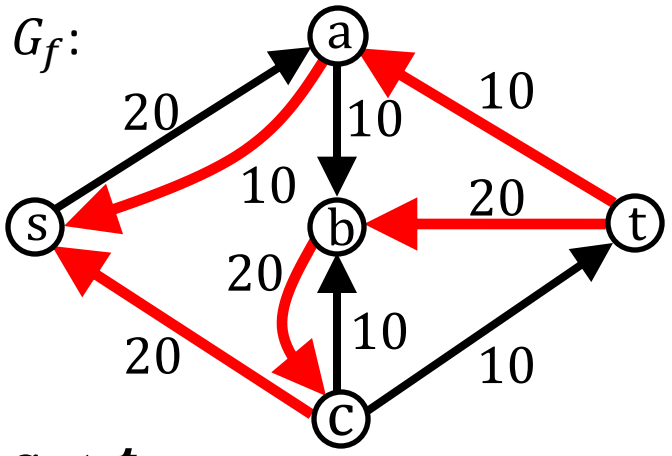

$\boldsymbol{P = s \to a \to t}$

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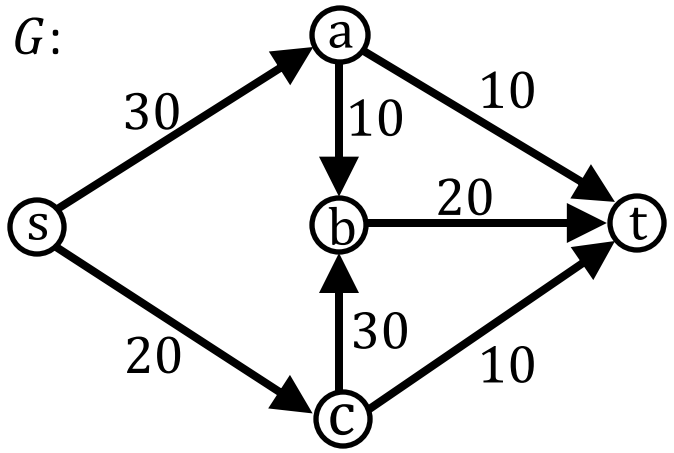

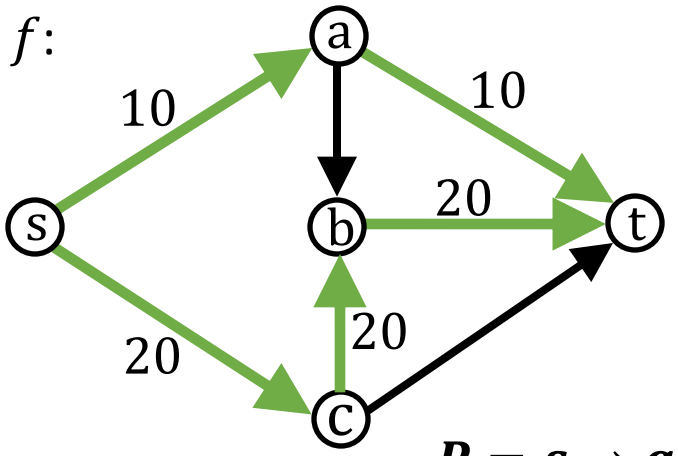

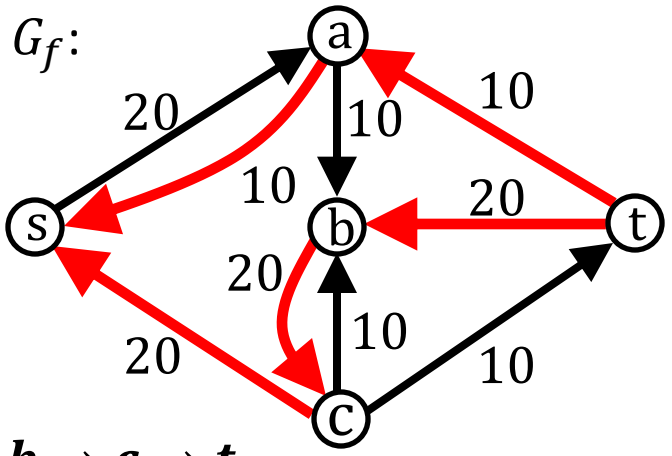

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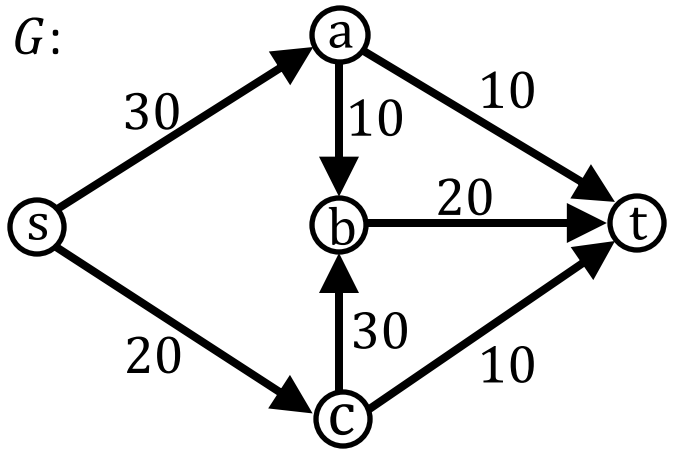

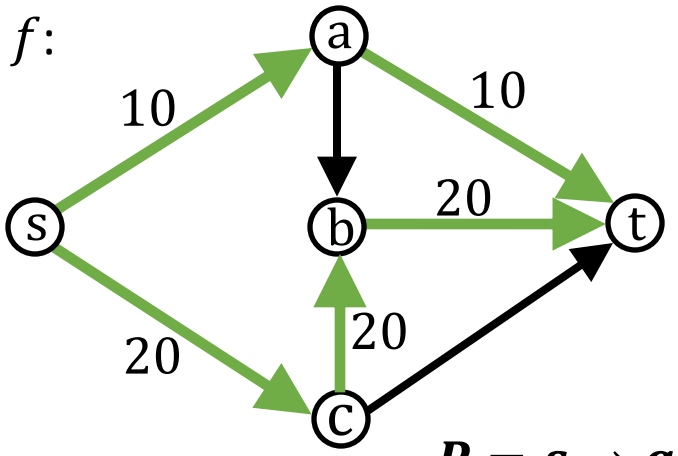

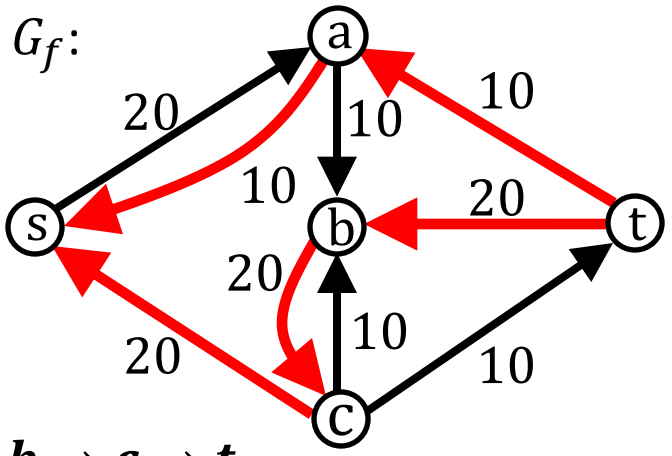

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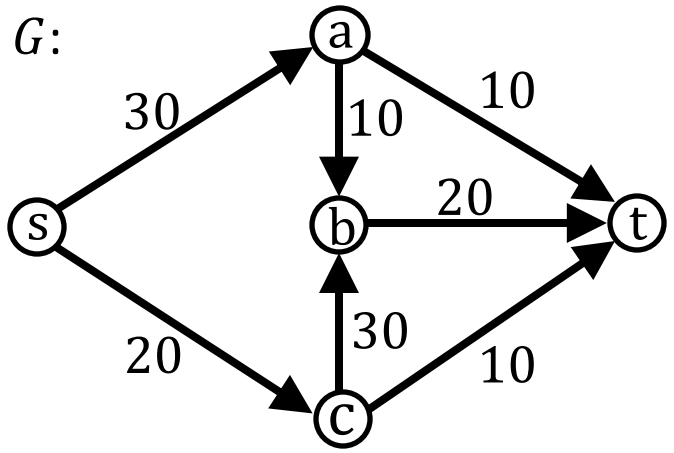

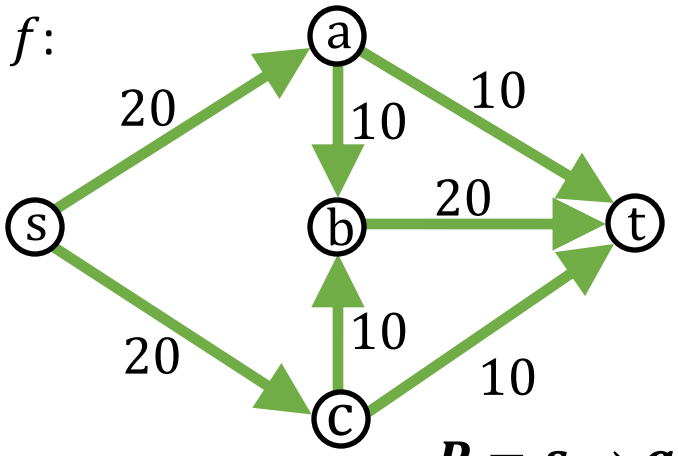

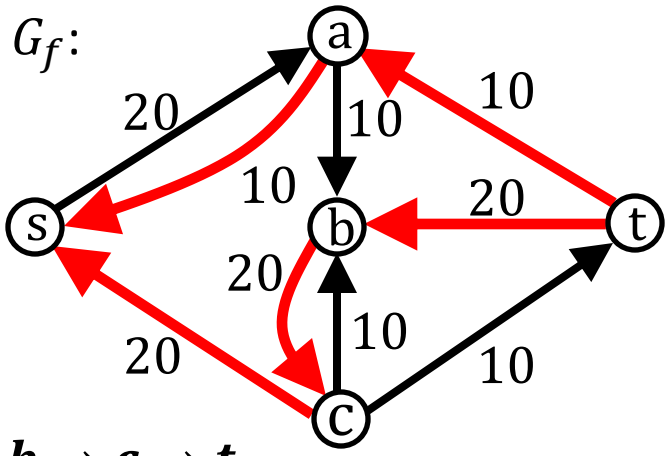

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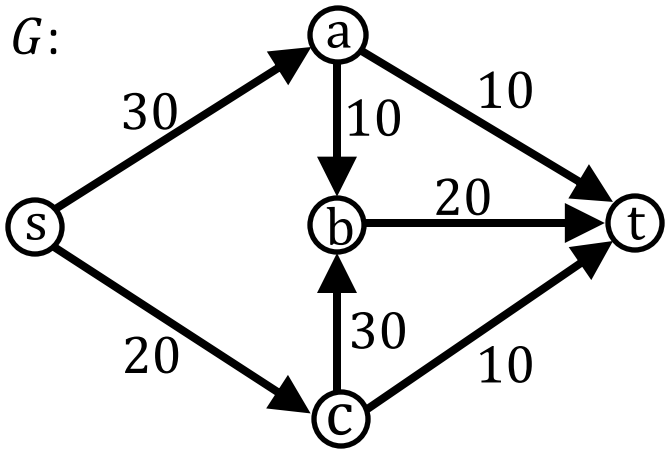

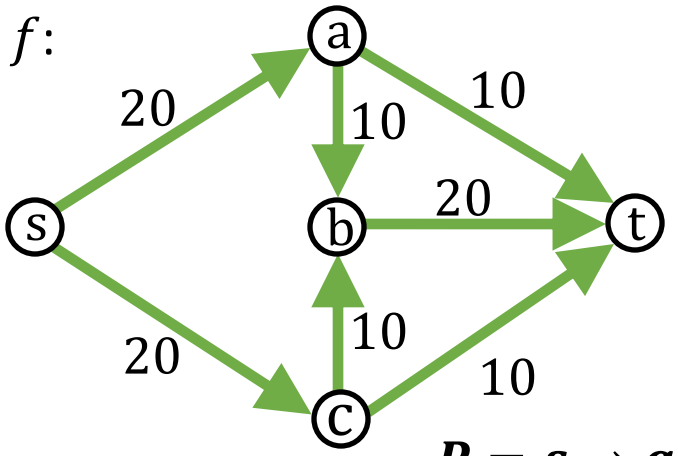

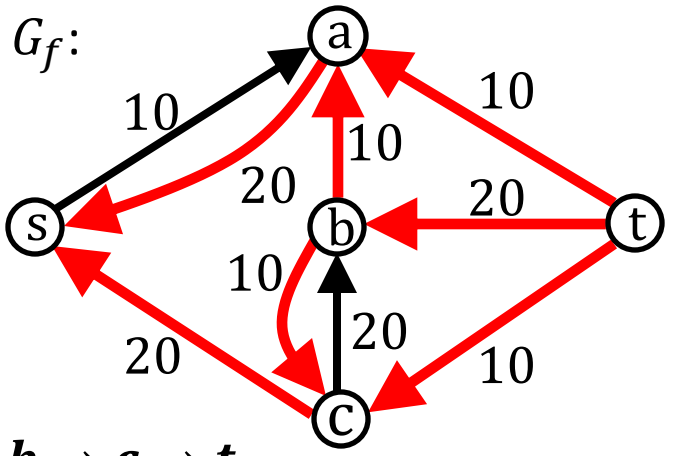

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Need to show:

1. Validity.
2. Running time.
3. Finds max flow.