Flow Networks
CSCI 432

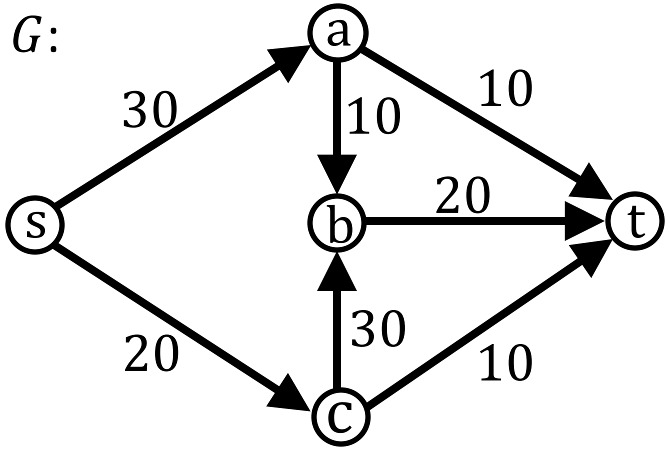
Ford-Fulkerson Algorithm

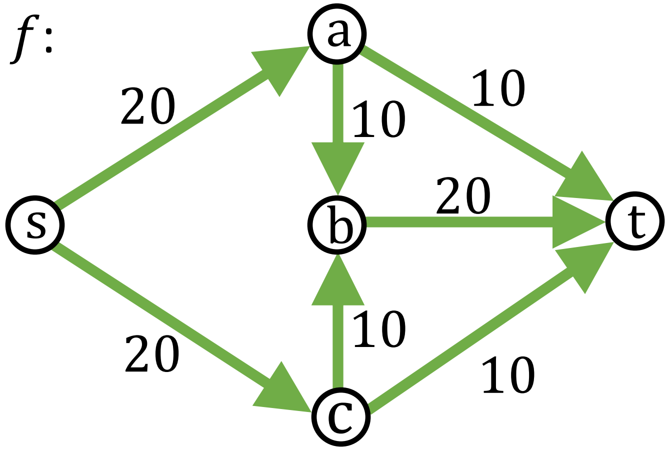
Residual graph for flow f, G_f:

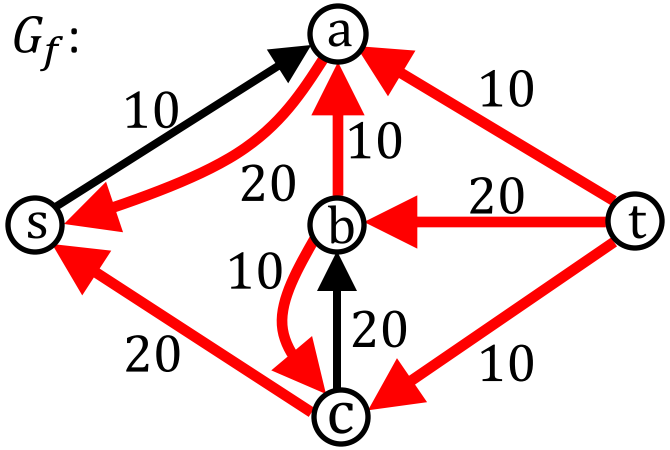
- $\forall e$, if $f(e) < c_e$, let $c_e = c_e - f(e)$.
- $\forall e = (u, v)$, if $f(e) > 0$, create $e' = (v, u)$ with $c_{e'} = f(e)$

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = simple s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```







Work Scheduling

Problem: We need to make holiday schedules for our employees. Each employee has a set of holidays that they are able to work. Each employee should work at most 3 holidays. We want to maximize the number of holidays covered.

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Apollo 13 Filter Problem:

"We need to fit this into the hole for this, using nothing but that"

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work scheduling problem **Max Flow solver** **valid flow network components**

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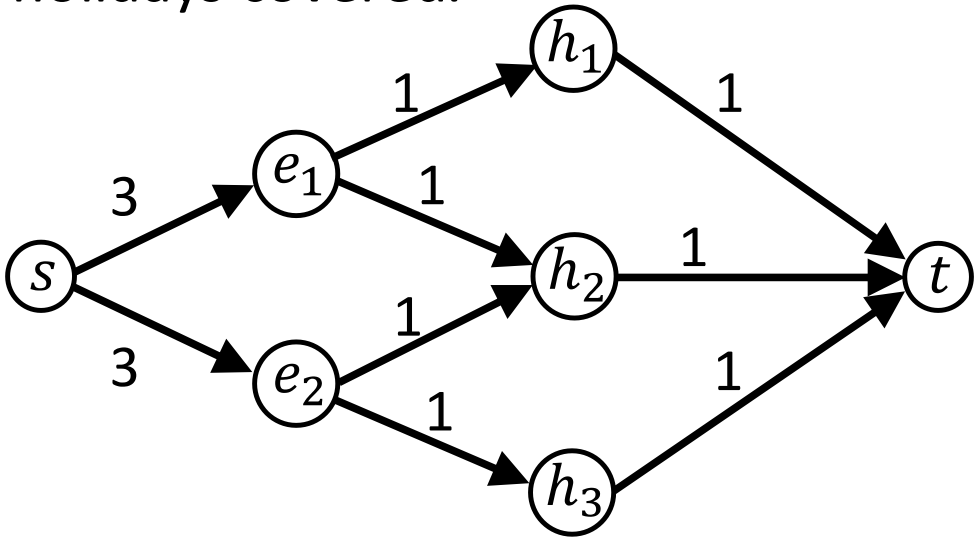

Algorithm:

1. Build flow network:

a) Make a node for each employee, a node for each holiday, a source, and a sink.

b) Connect the source to each employee node with a capacity of 3.

c) Connect each holiday node to the sink with a capacity of 1.

d) If an employee is able to work a holiday, connect them with a capacity of 1.

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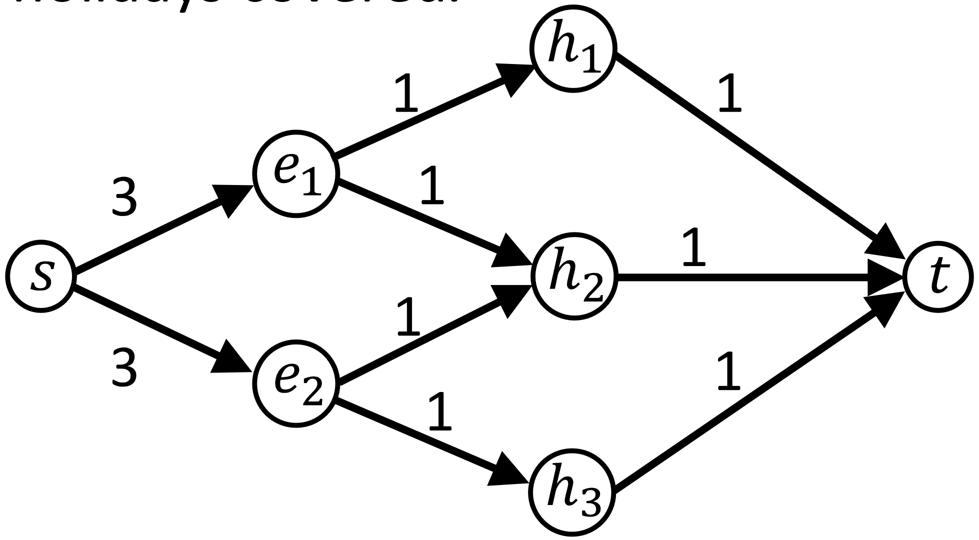

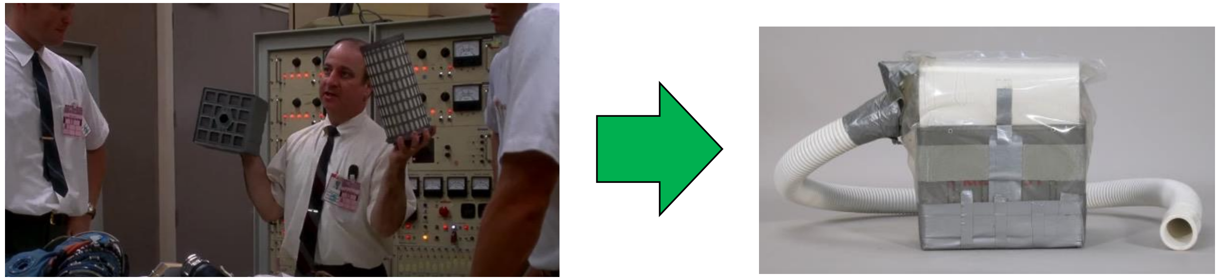

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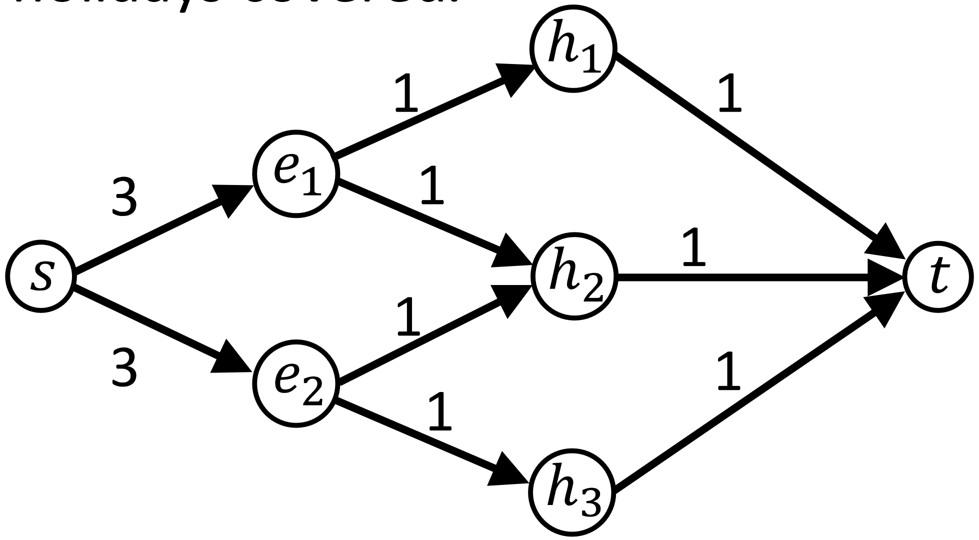

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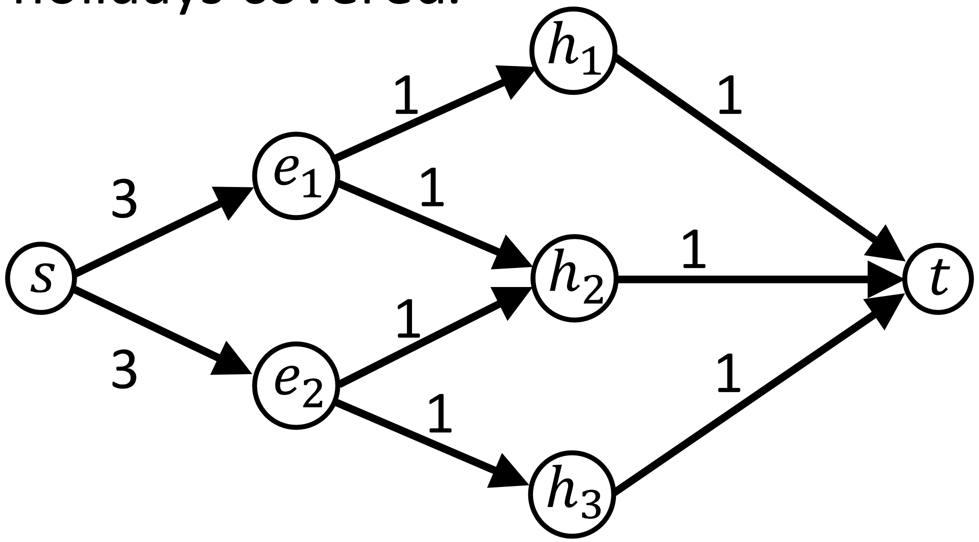

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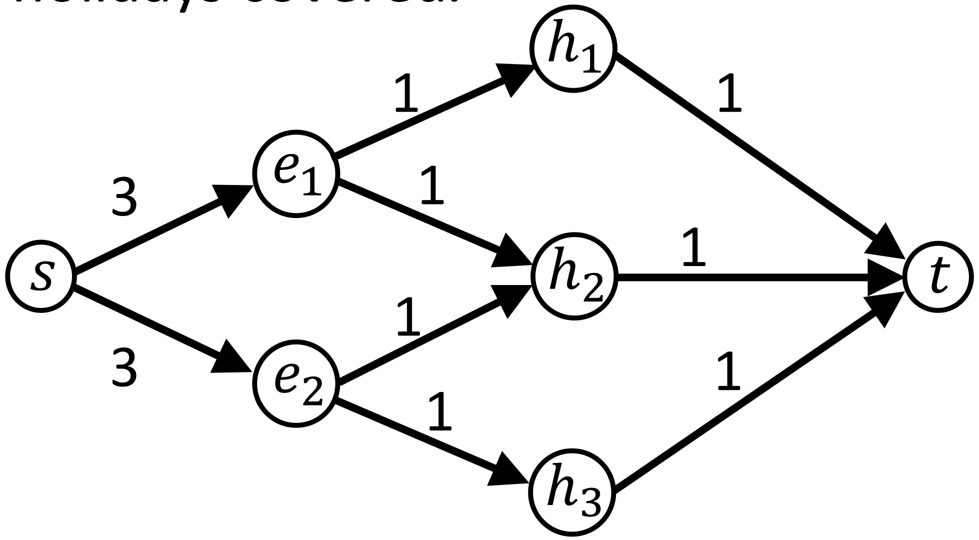

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<u>Algorithm:</u>

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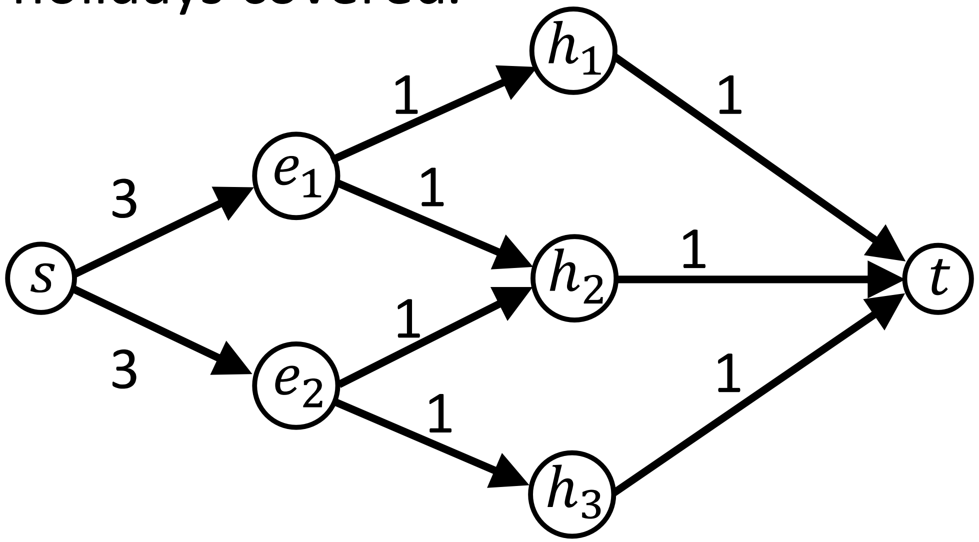
2. Find max flow.
3. <Translate flow to answer>



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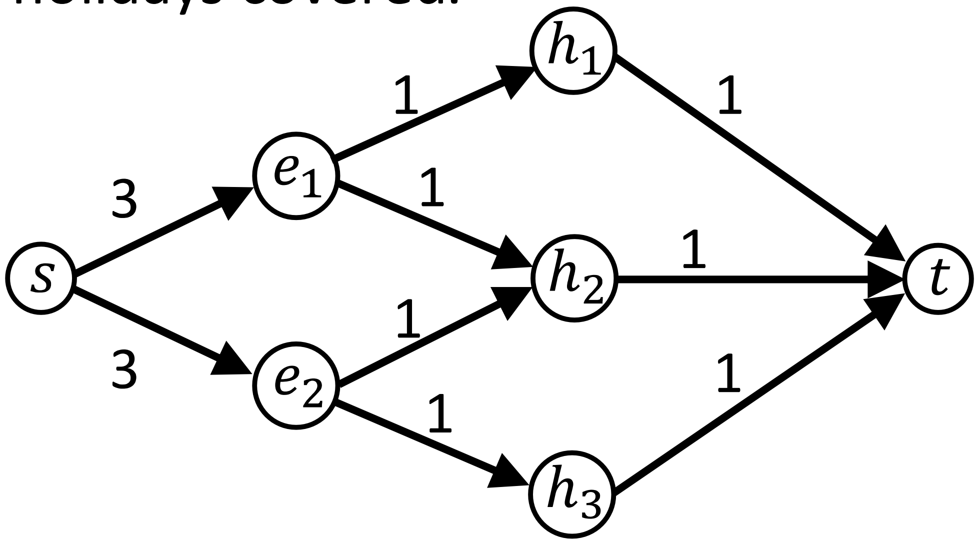

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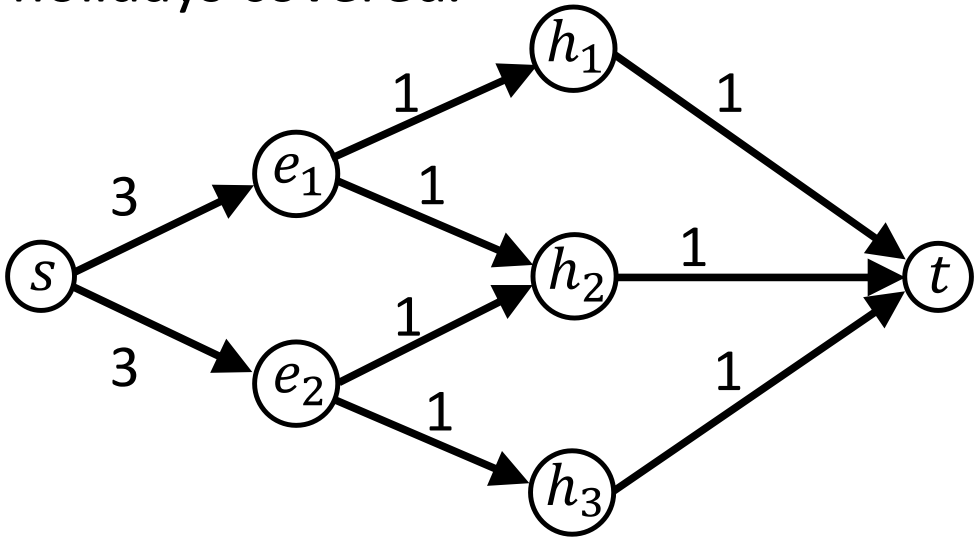

Running Time:

$$O(\text{time to build G}) + O(\text{Max Flow}) + O(\text{time to make answer})$$

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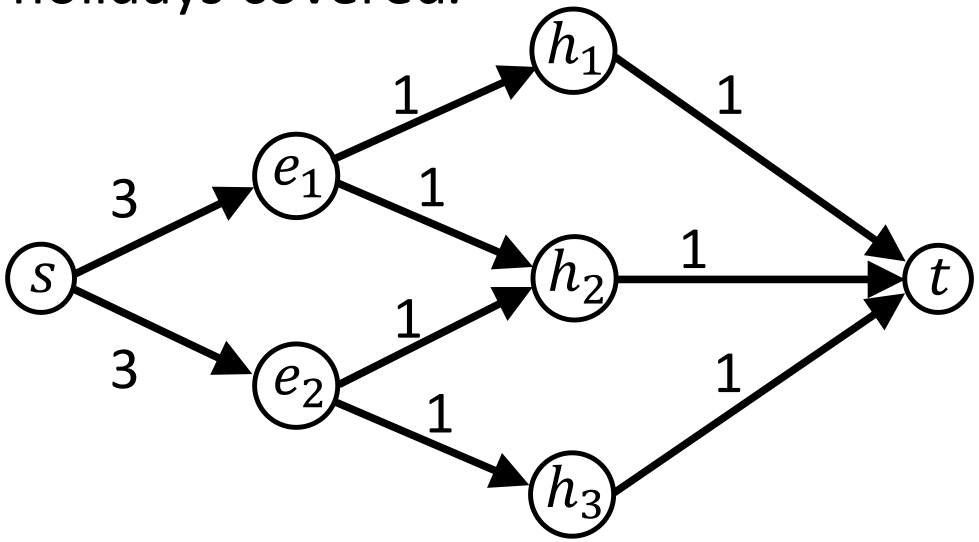

Running Time:

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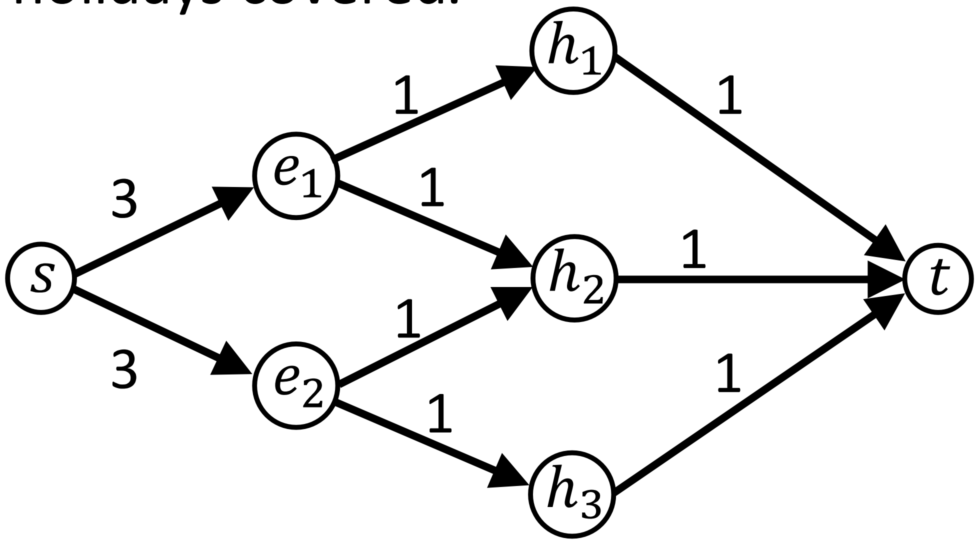

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For $G = (V, E), |G| =$?

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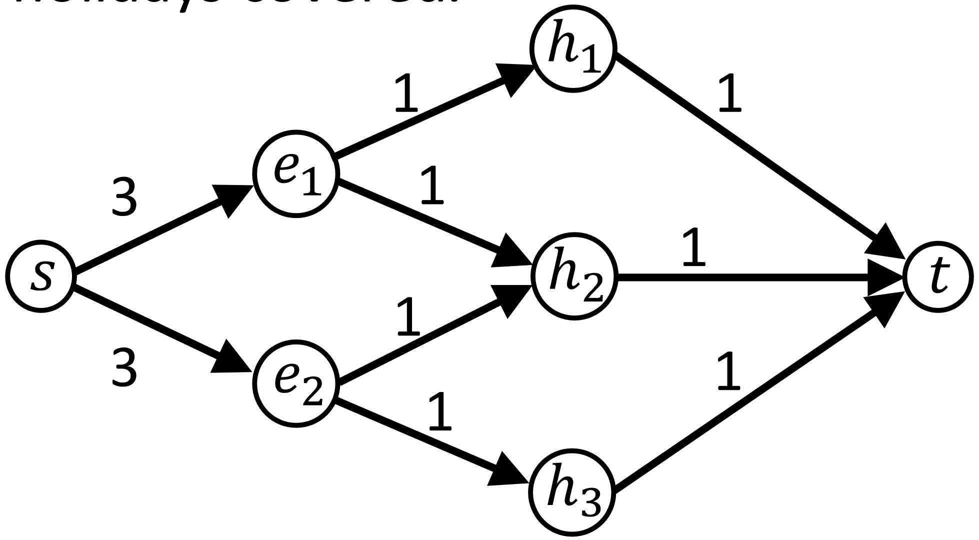

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Running Time:

n = # employees, m = # holidays

For $G = (V, E), |V| = n + m + 2, |E| = n + nm + m$

So, building $G \in O(nm)$.

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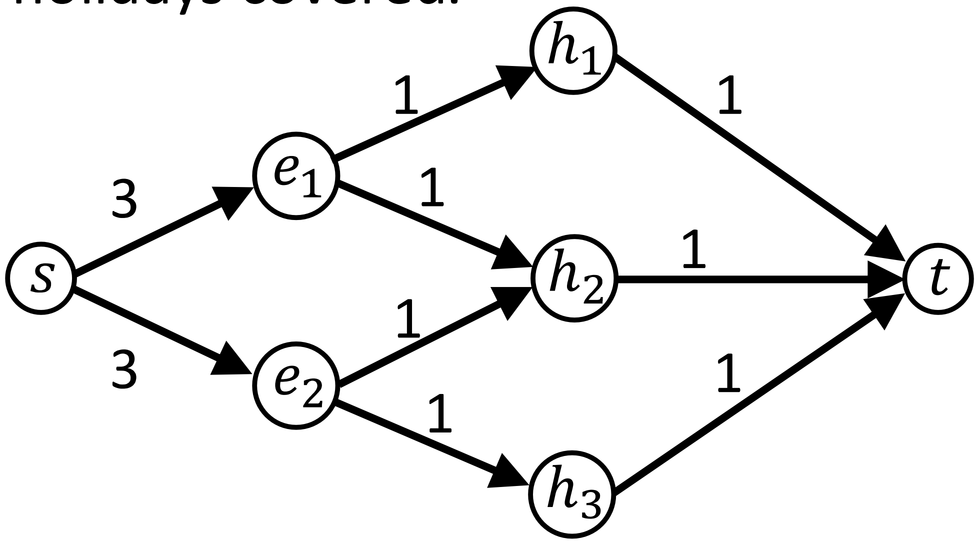

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Total running time ∈ ?

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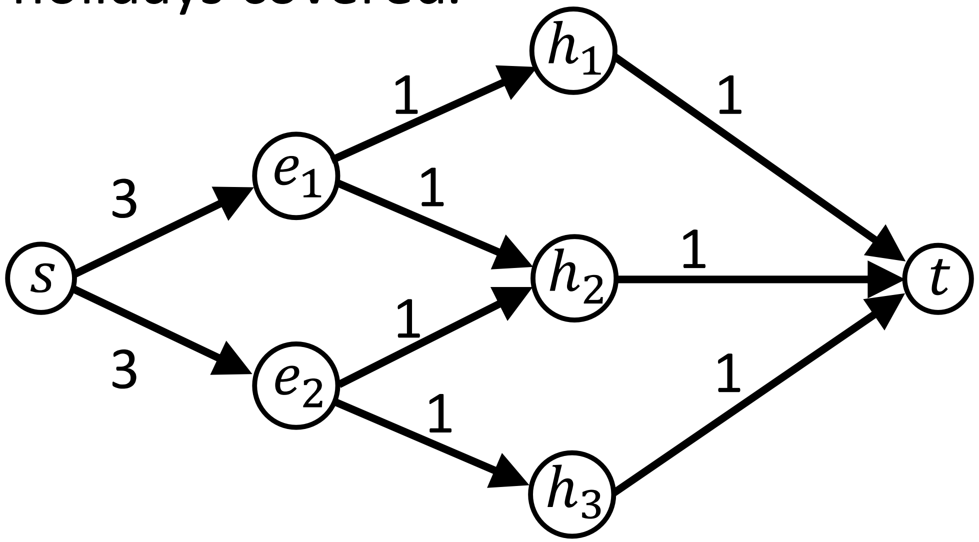

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So, building $G \in O(nm)$.

Total running time $\in O(nm) + O(\text{Max Flow}) + O(nm) \in O(nm) + O(\text{max flow})$

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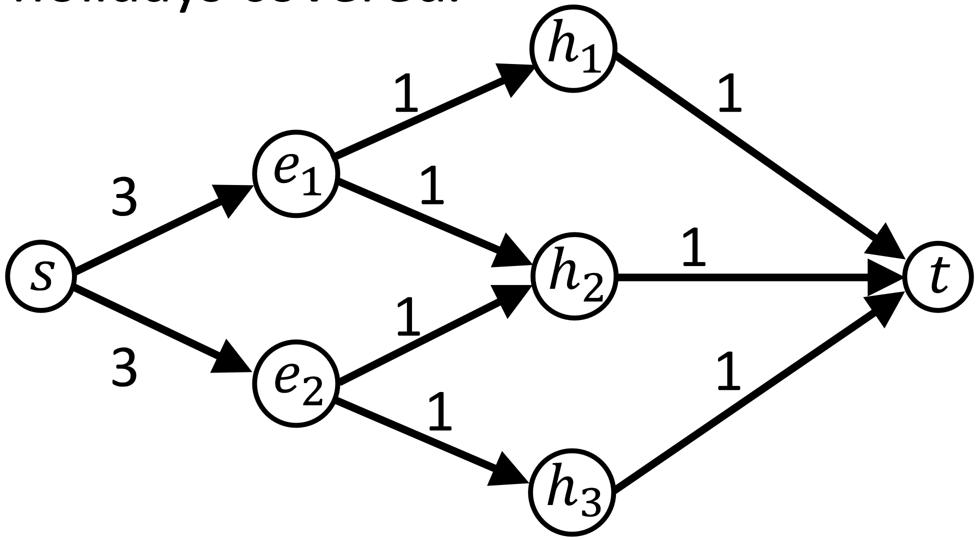

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<u>Validity:</u>

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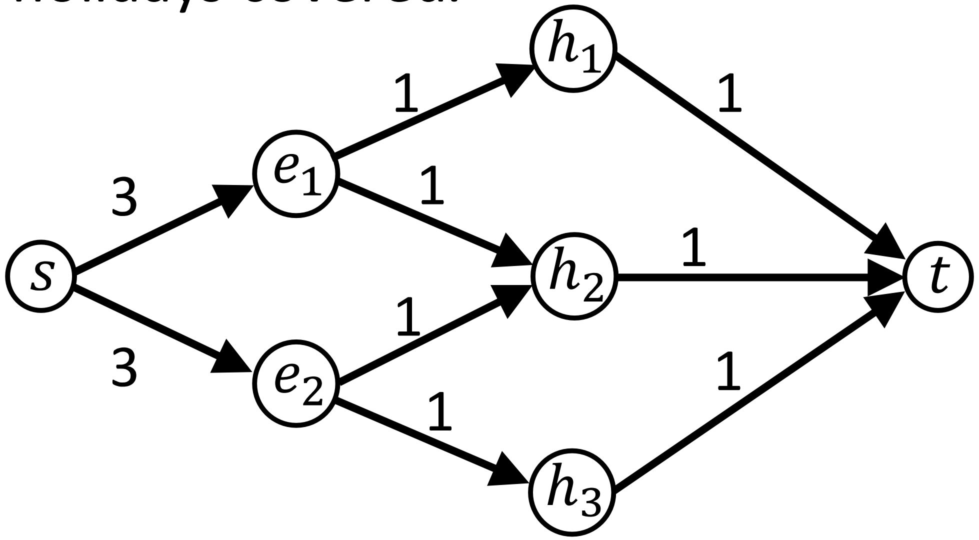

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Employees cannot be assigned a holiday they are not able to work (no edges exist between incompatible pairs).

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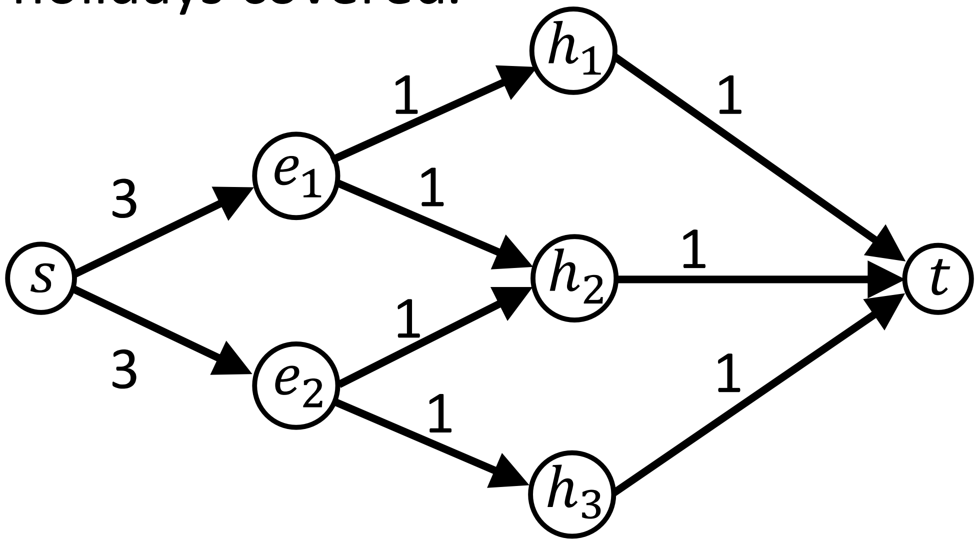

Validity:
Employees cannot be assigned a holiday they are not able to work (no edges exist between incompatible pairs). Employees cannot be assigned to work more than 3 holidays (integer edge capacities yield integer flow values).

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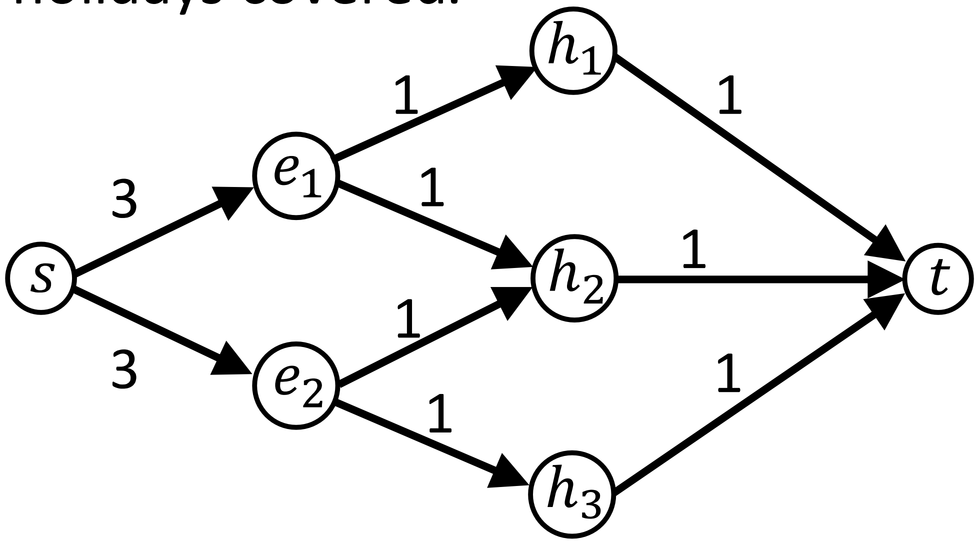

<u>Optimality:</u>

??

We need to show the algorithm's answer is optimal for the problem, not that the max flow found is optimal for the flow network we made.

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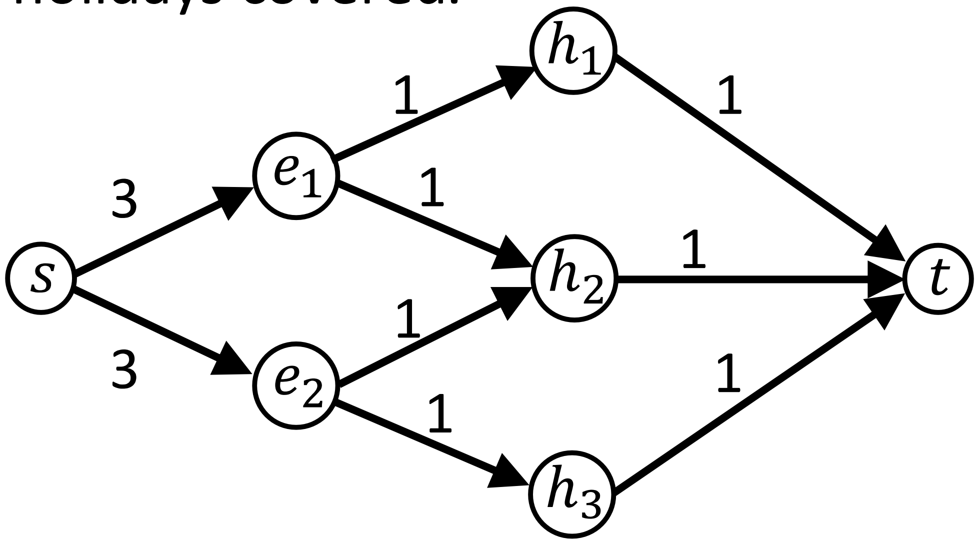

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Suppose max flow $F < OPT \leq m$.

\# holidays

Optimal answer to the problem.

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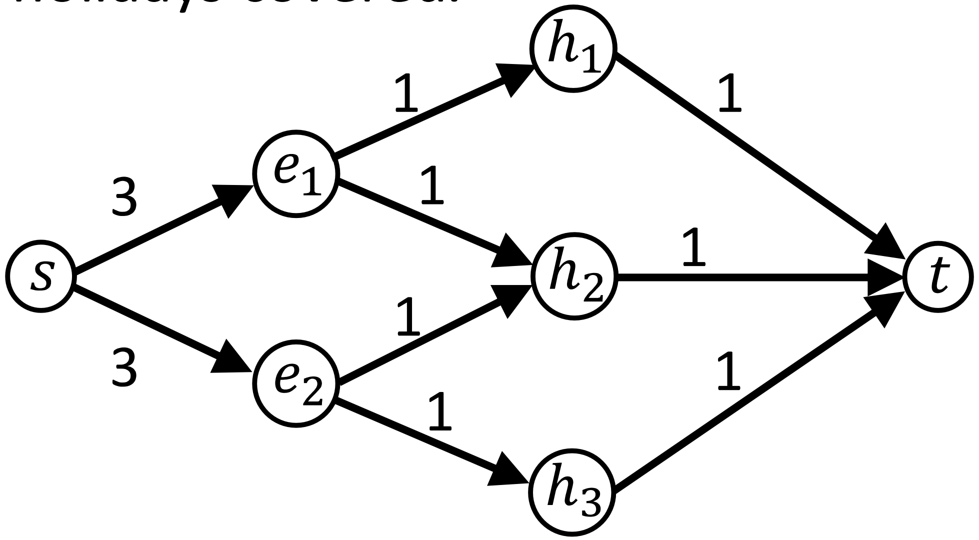

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Plan: Turn OPT into a valid flow with value larger than F. CONTRADICTION!!!

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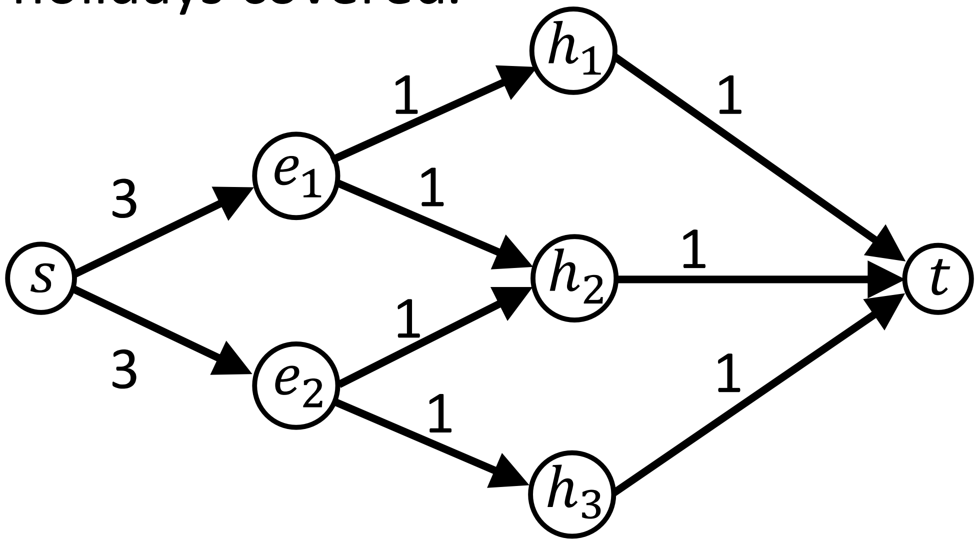

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Reducing the number of employees working a single holiday to 1 keeps that holiday covered.

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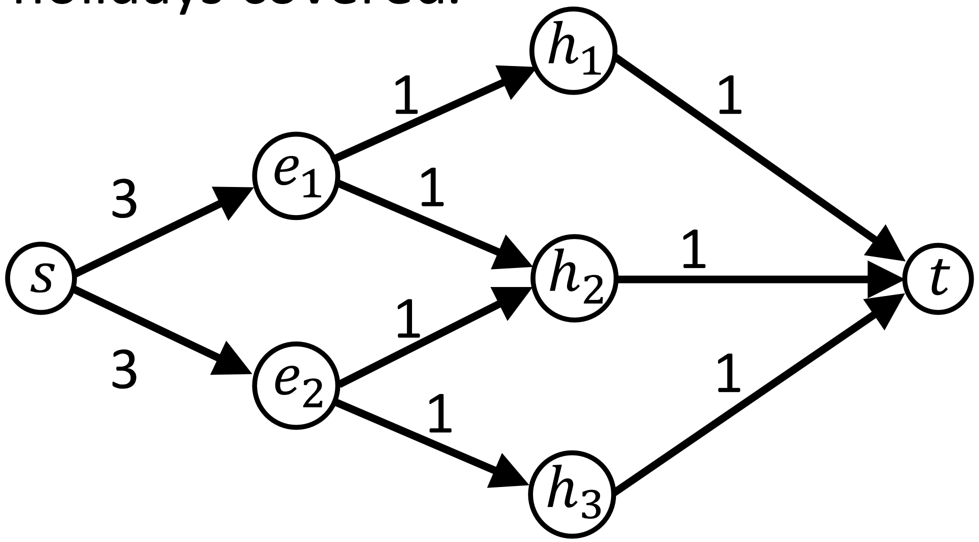

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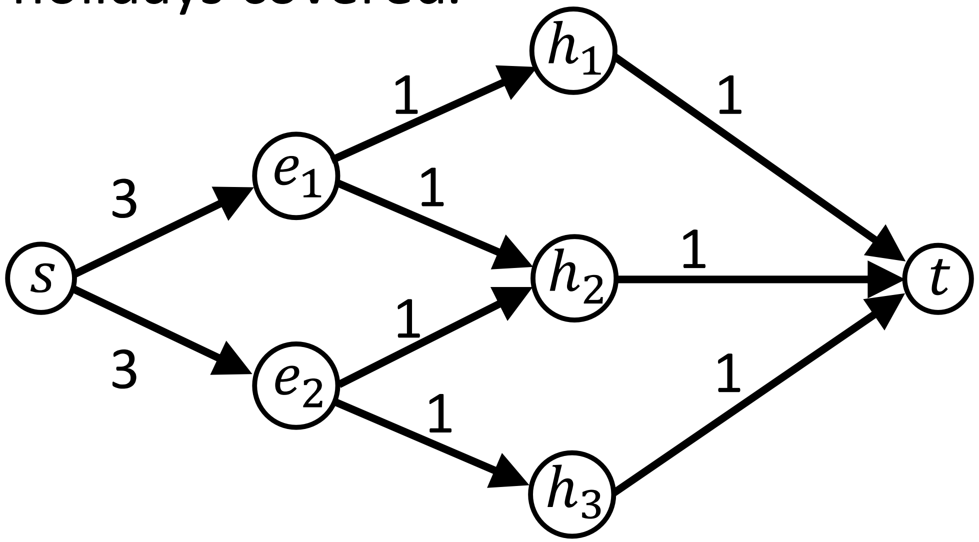

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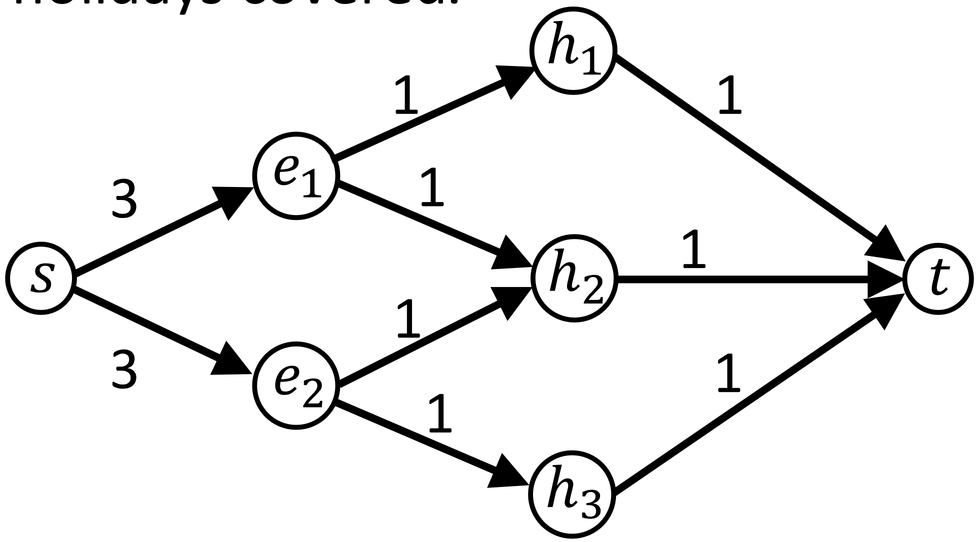

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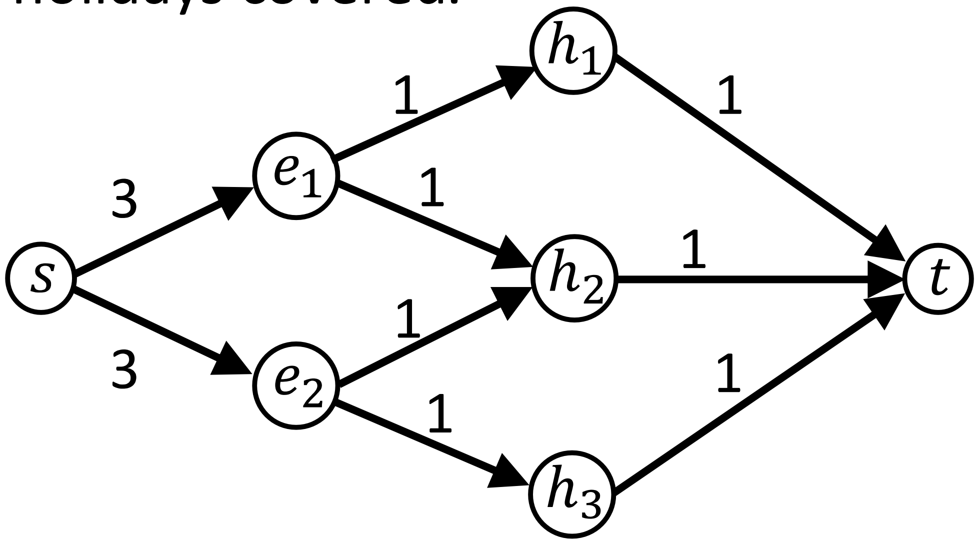

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Now we need to argue this flow is valid.

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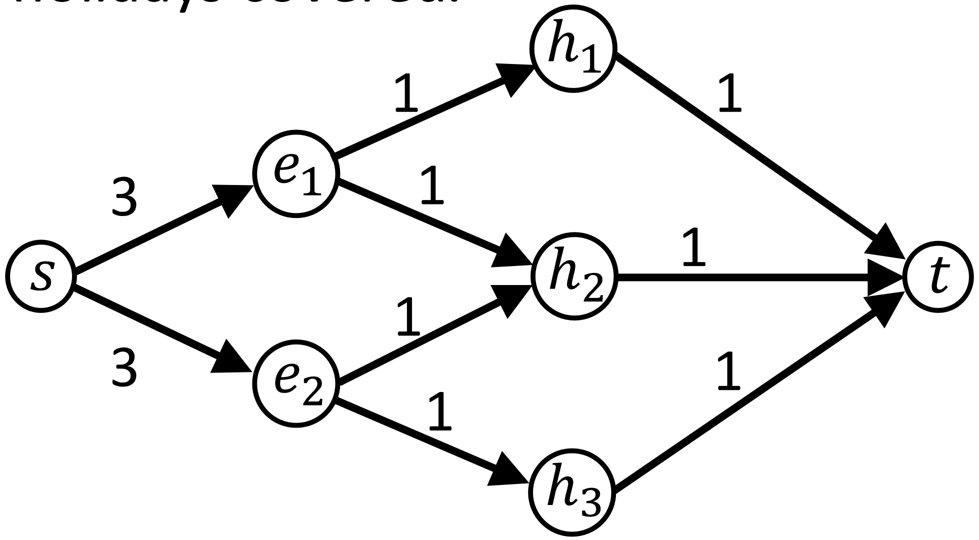

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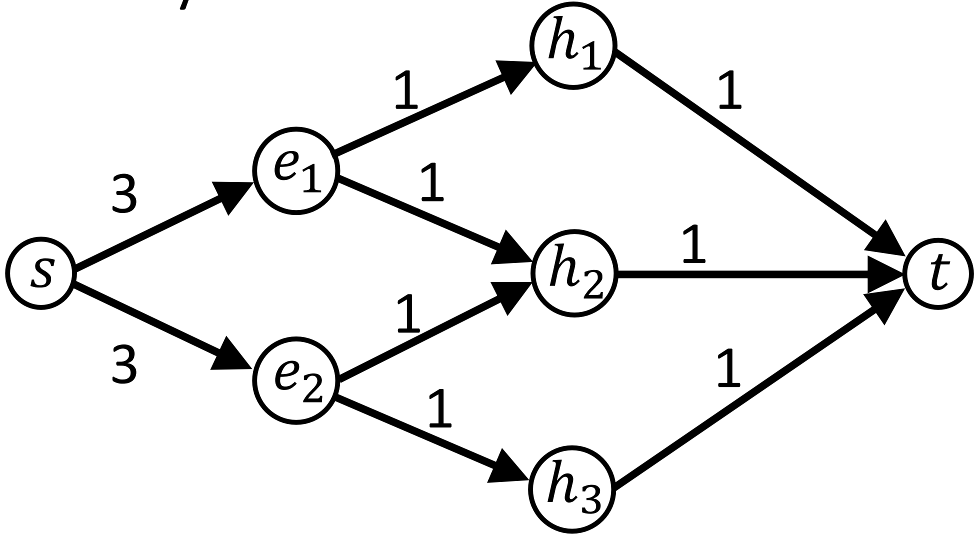

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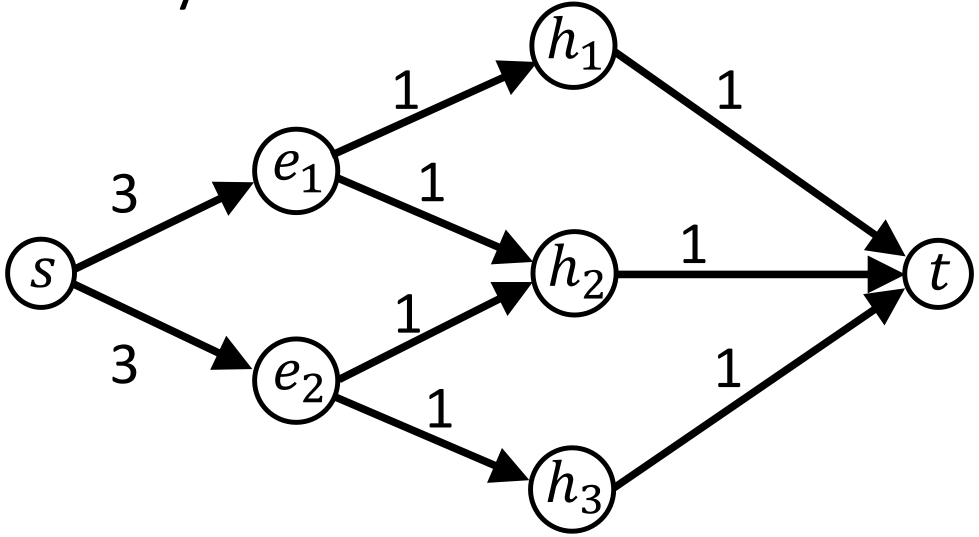

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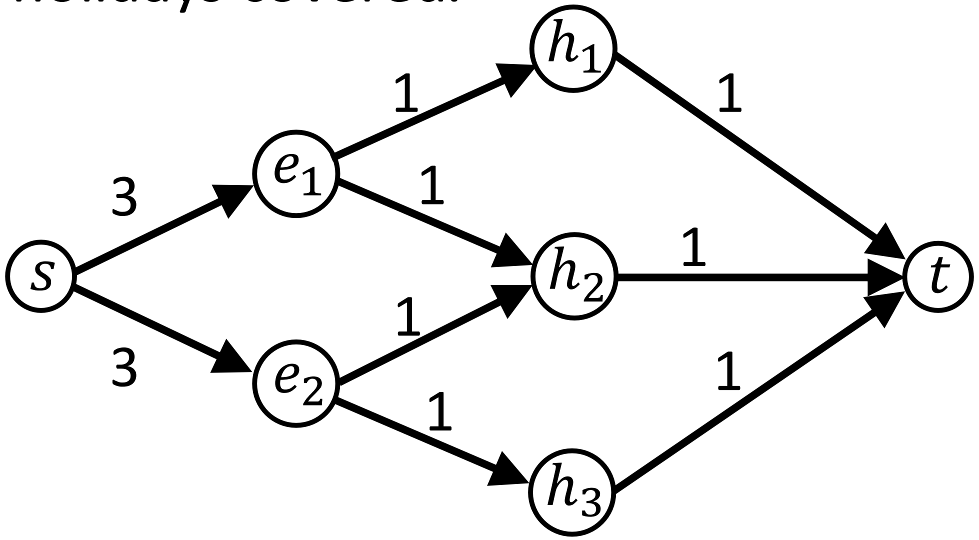

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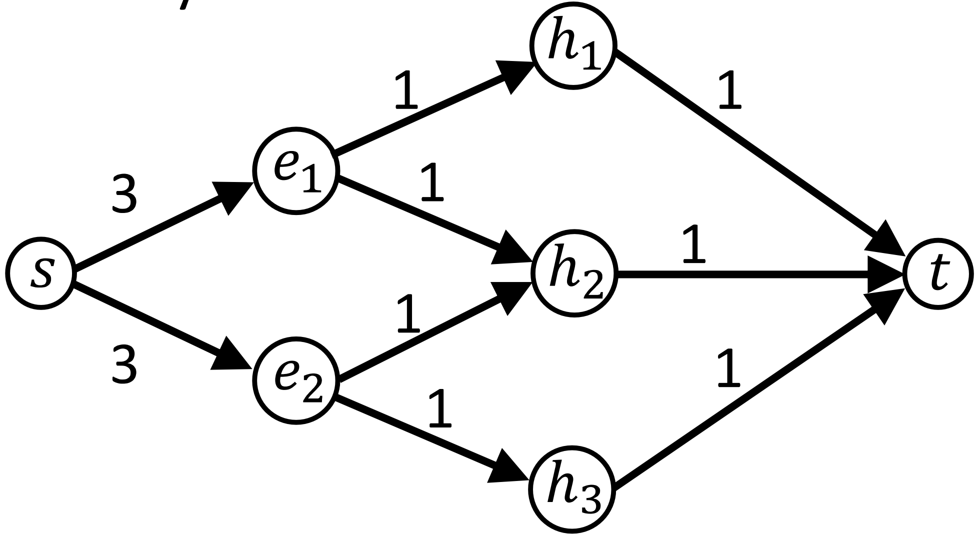

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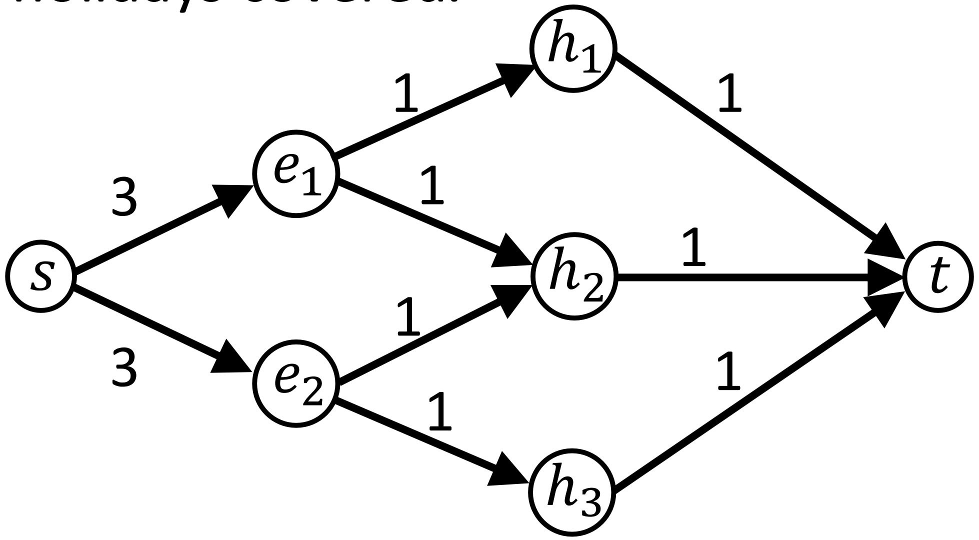

Algorithm:

1. Build flow network.
2. Find max flow.
3. If employee has outgoing edge carrying flow, assign them to work that holiday.

Optimality:

Suppose max flow $F < OPT \leq m$. That means employees can be assigned to cover OPT holidays. First, remove redundant holiday coverage. Make edge for each assignment carry 1. Make edge leaving each covered holiday carry 1. Make edge into each employee carry sum of the flow leaving. Conservation of flow holds at employees because that's how we set incoming flow. Conservation at holiday nodes holds because if a holiday is covered, it's done with a single employee. Also, no capacities are violated. $\Rightarrow$ Valid Flow

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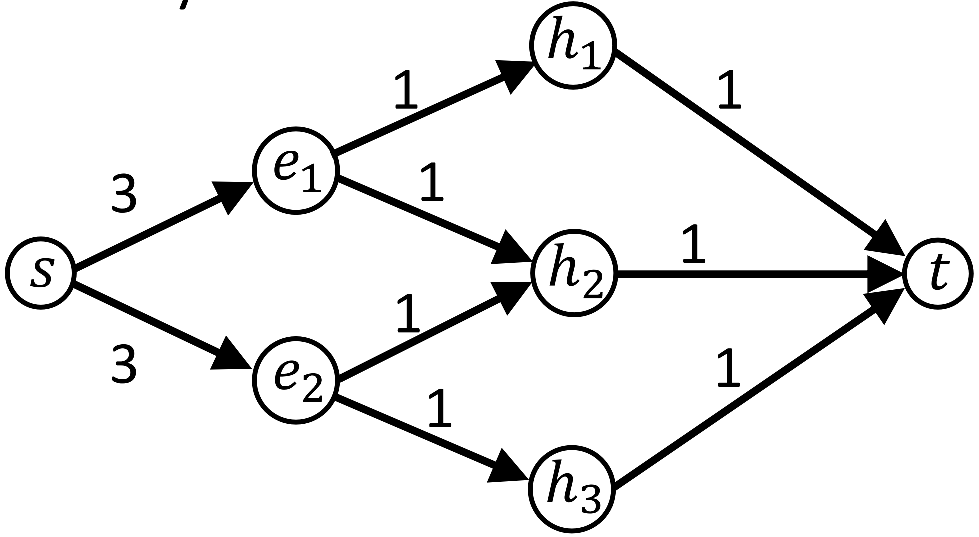

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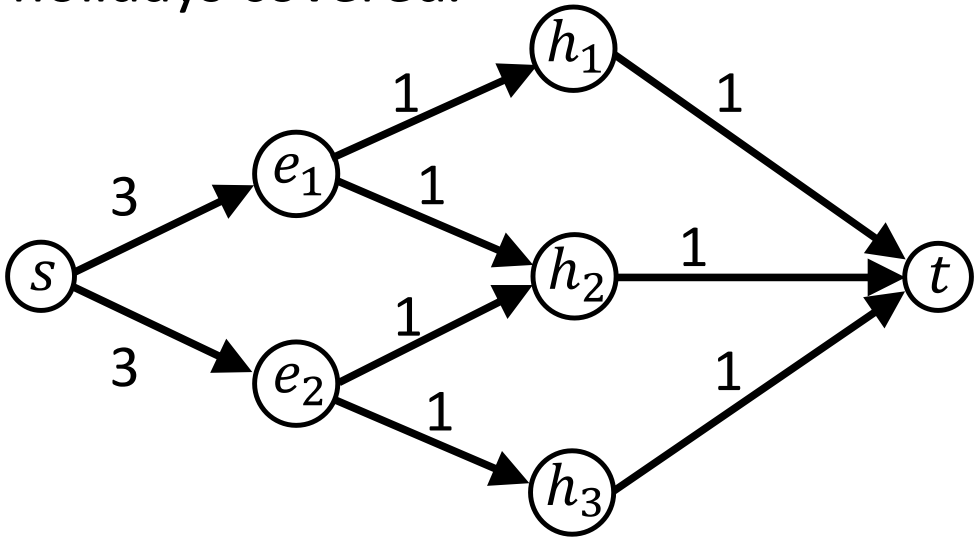

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Work Scheduling

Problem: We need to make holiday schedules for our employees. Each employee has a set of holidays that they are able to work. Each employee should work at most 3 holidays. **Each holiday needs exactly one employee working it.**

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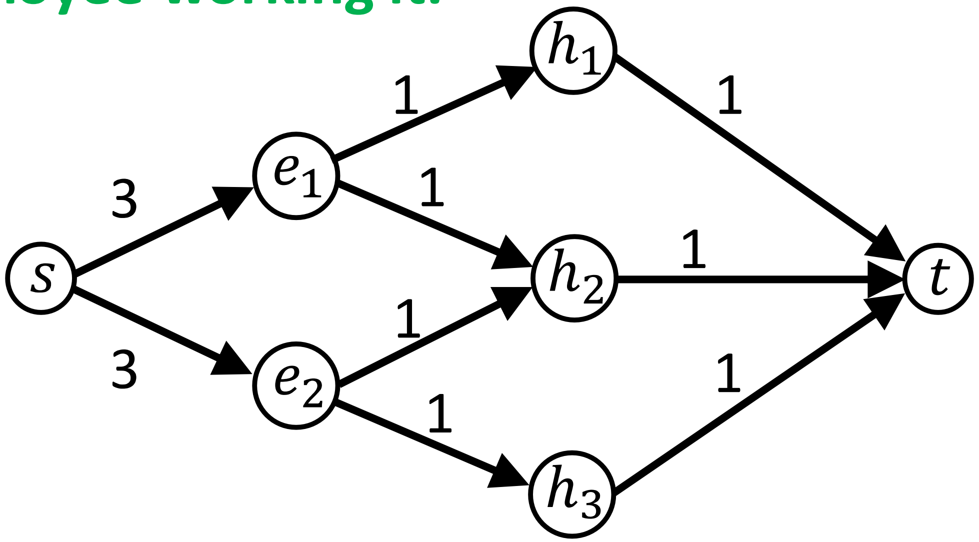

Need to change anything?

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Problem: We need to make holiday schedules for our employees. Each employee has a set of holidays that they are able to work. Each employee should work at most 3 holidays. **Each holiday needs exactly one employee working it.**

Algorithm:

1. Build flow network.
2. Find max flow.
3. If max flow < # holidays, infeasible instance.
4. If employee has outgoing edge carrying flow, assign them to work that holiday.