Linear Programming
CSCI 432

Linear Program (LP)

$x_1 = \text{\# of Rippers sold}$
$x_2 = \text{\# of Ripper Carbons}$

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

Decision Variables:

- Real numbers = solvable in polynomial time (called LP).
- Integers = not (yet?) solvable in polynomial time (called integer linear program – ILP).

Objective:

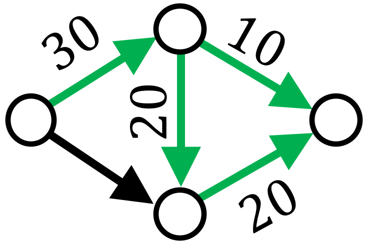
- Can be minimization or maximization.
- Must be linear combinations of variables x_i (e.g. $a_1x_1 + \cdots + a_nx_n$ for constants a_i, not $a_ix_1x_2$).

Constraints:

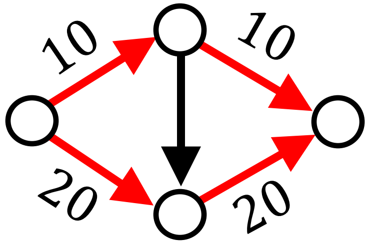
- Can be $\leq$, $\geq$, $=$.
- Must be linear combinations of variables.

Minimum-Cost Flow Problem: Suppose we have a target flow demand d, and a flow network where each edge also has a cost in addition to its capacity. Pushing k flow along edge e incurs the cost $kc(e)$. Find an $s-t$ flow of minimum cost with value d.

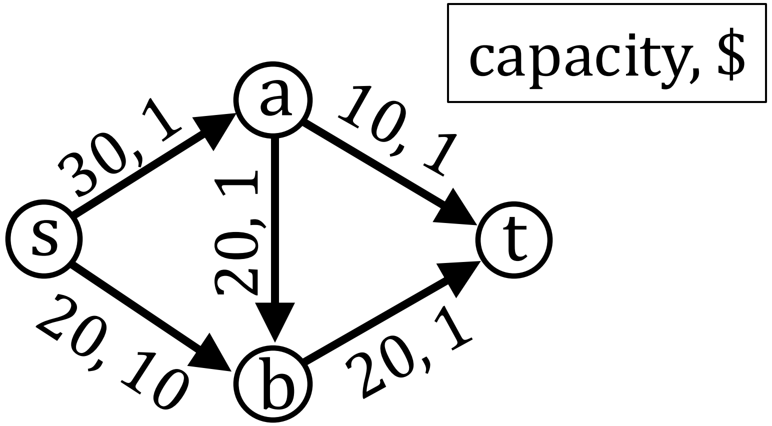
Let $d = 30$



$30 + 20 + 10 + 20 = 80$



$10 + 10 + 200 + 20 = 240$



x_e = Amount of flow on edge e.

Objective: $\min \sum_{e \in E} \text{cost}_e x_e$

Subject to: $x_e \leq \text{capacity}_e, \forall e \in E$

$\sum_{e \in \text{input}(v)} x_e - \sum_{e \in \text{output}(v)} x_e = 0, \forall v \in V \setminus \{s, t\}$

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$x_e \geq 0, \forall e \in E$

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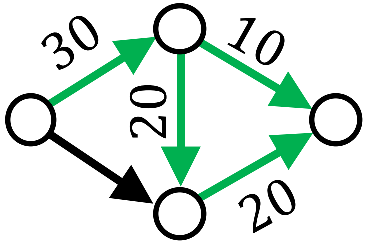

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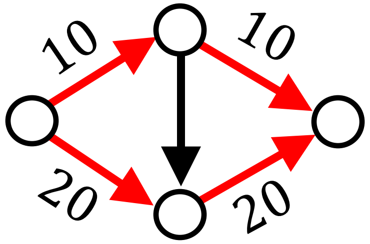

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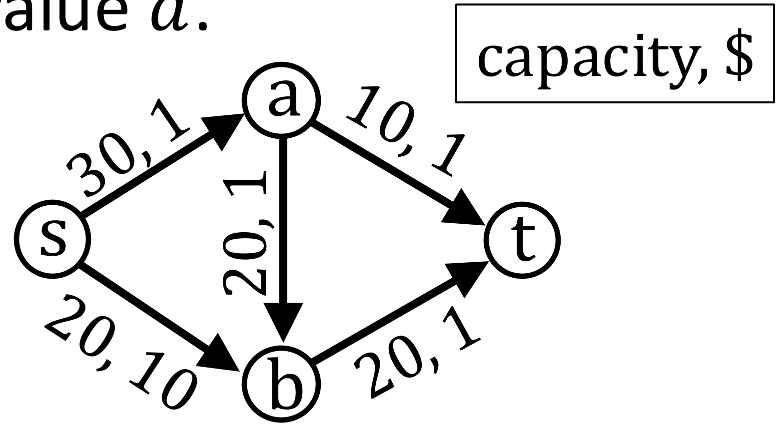

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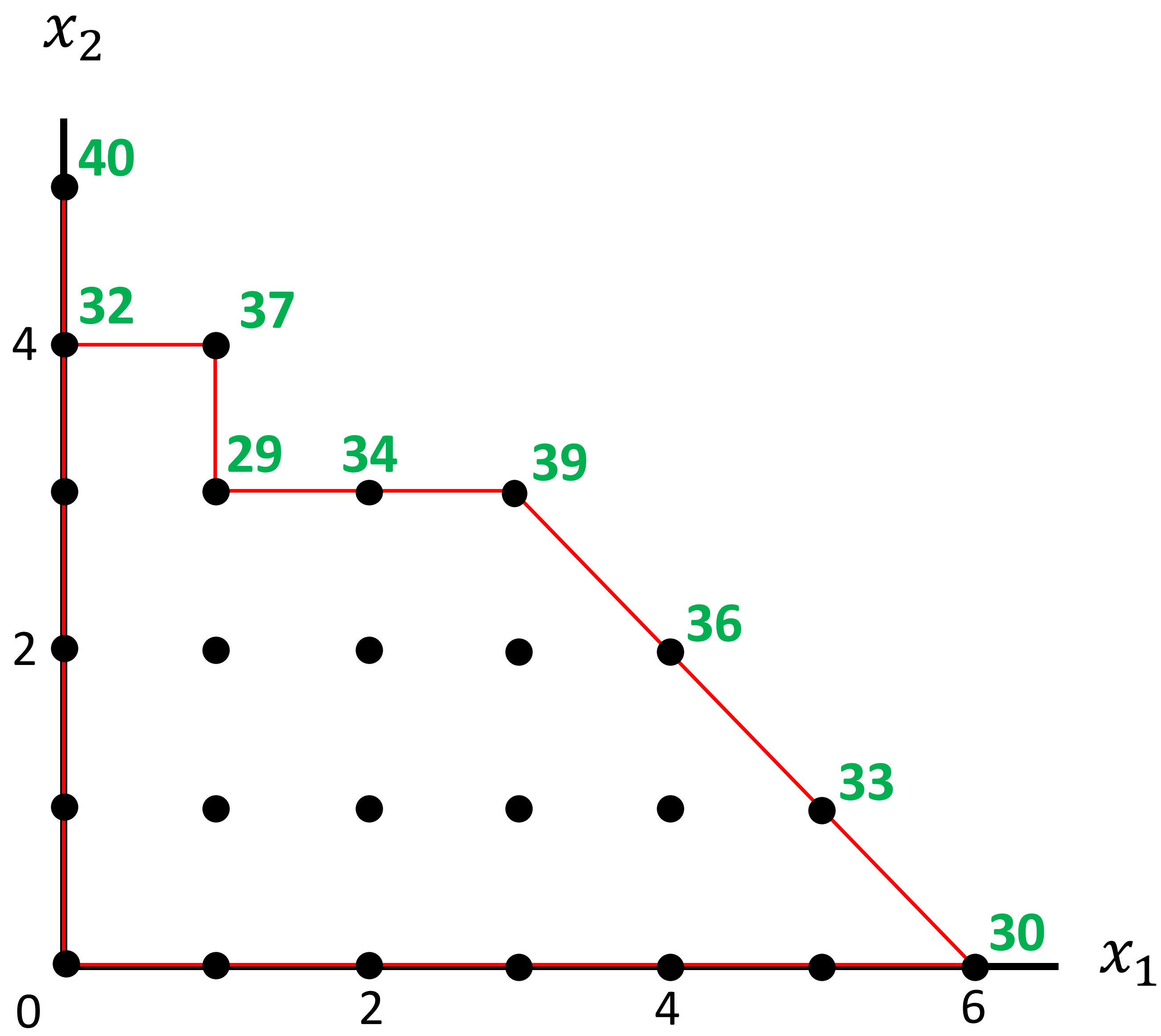

$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

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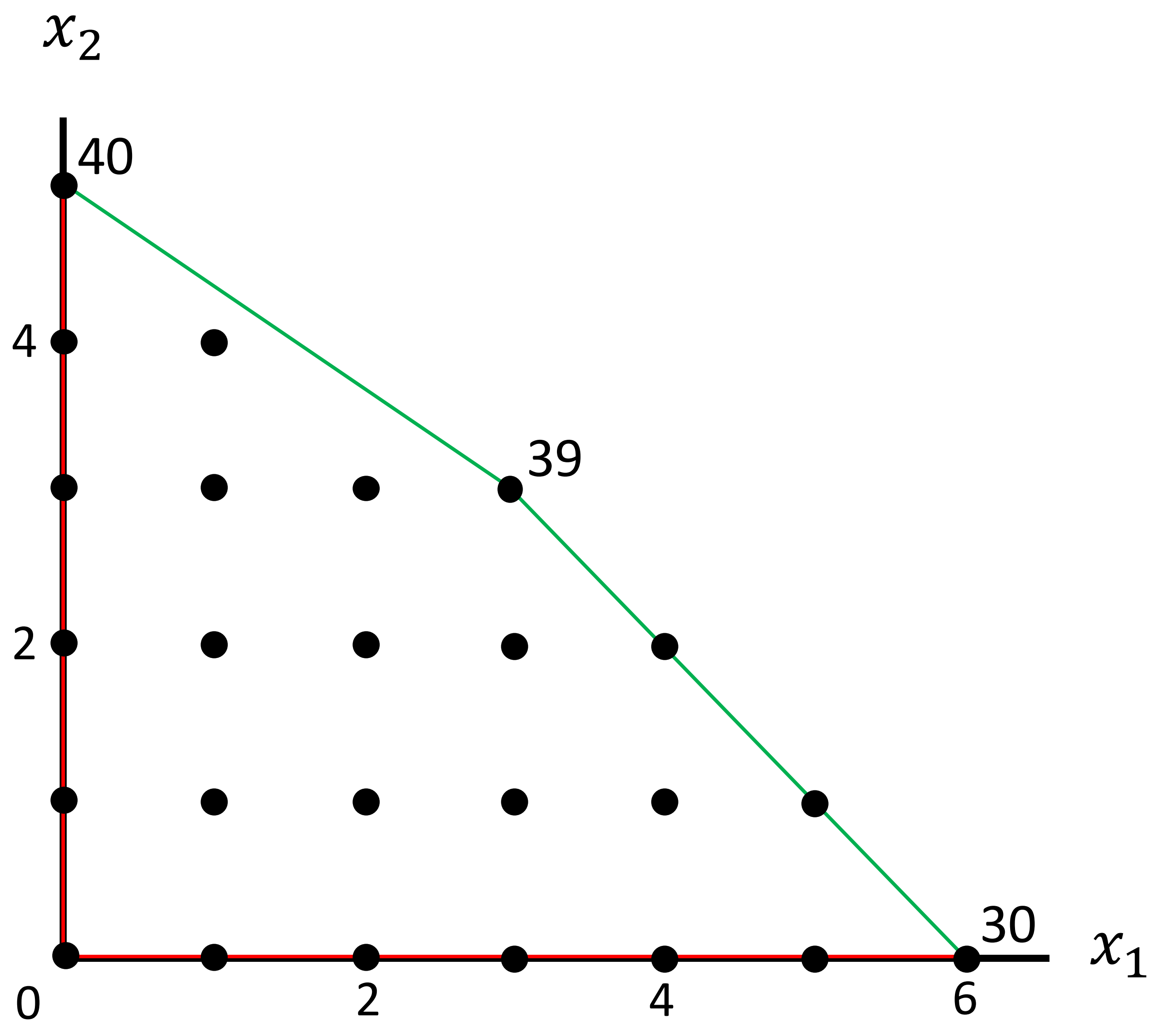
Integer feasible region:

- Not convex.
- local optimum $\neq$ global optimum.



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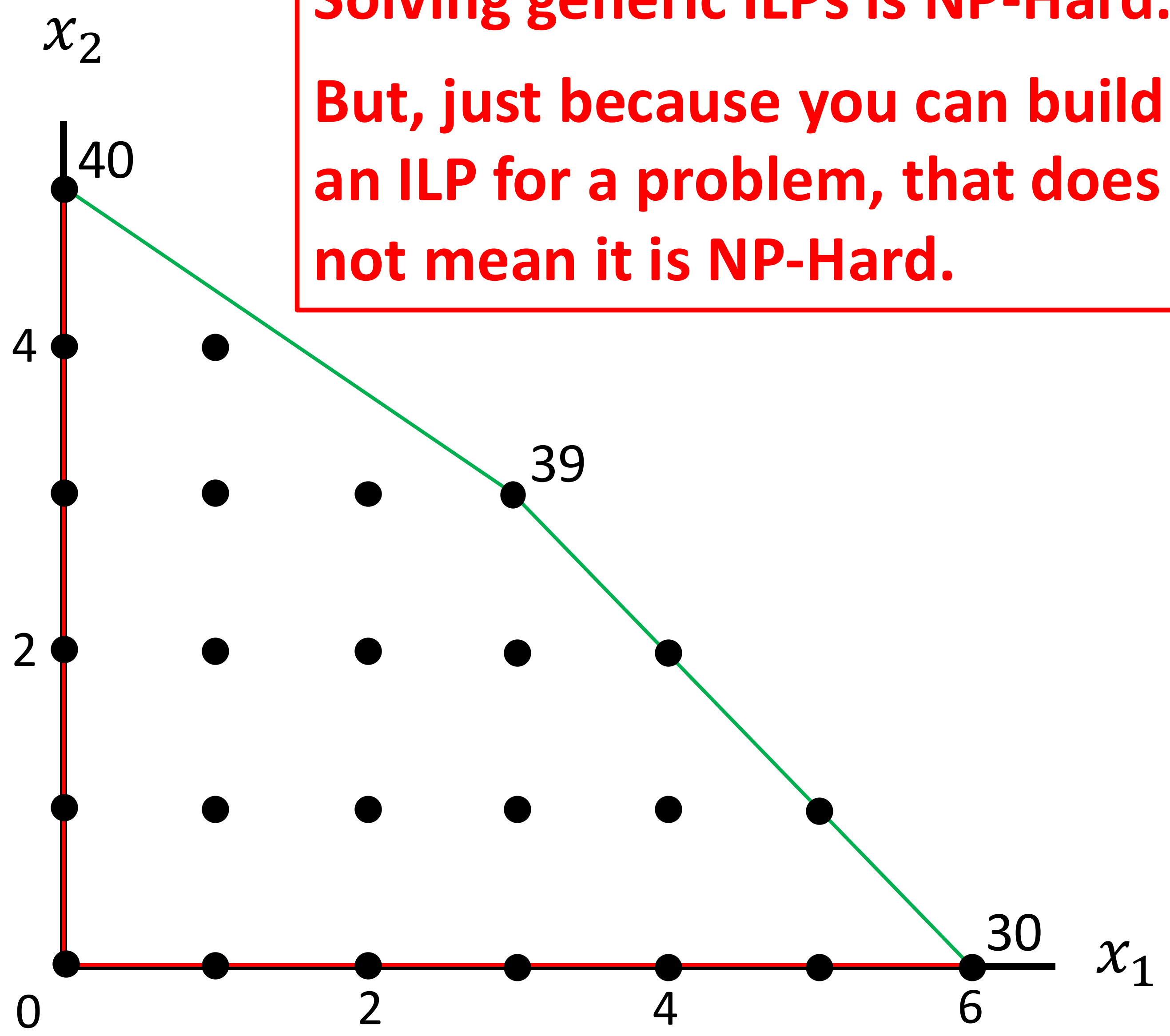
Minimal convex hull:

- Convex.
- local optimum = global optimum.
- $O\left(n^{\lfloor d/2 \rfloor}\right)$ faces, n = # points and d = # dimensions



Solving generic ILPs is NP-Hard.

But, just because you can build an ILP for a problem, that does not mean it is NP-Hard.

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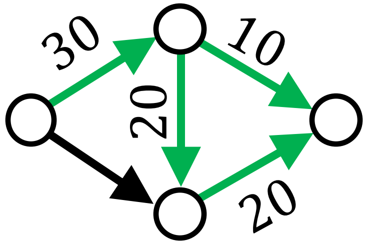

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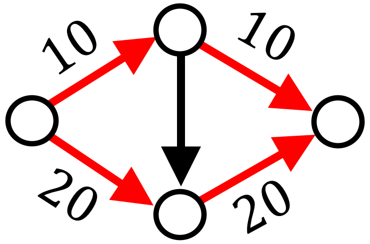

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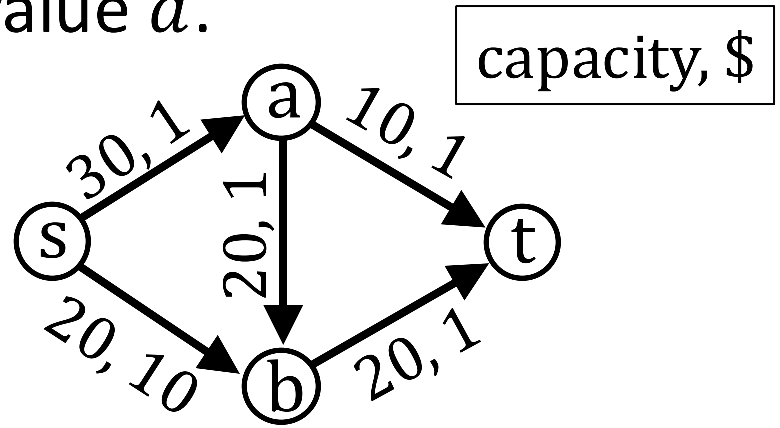

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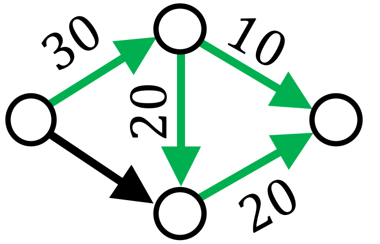

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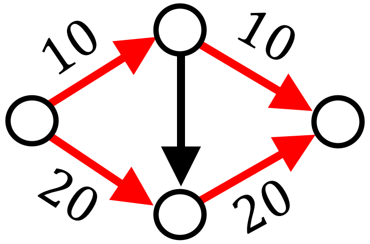

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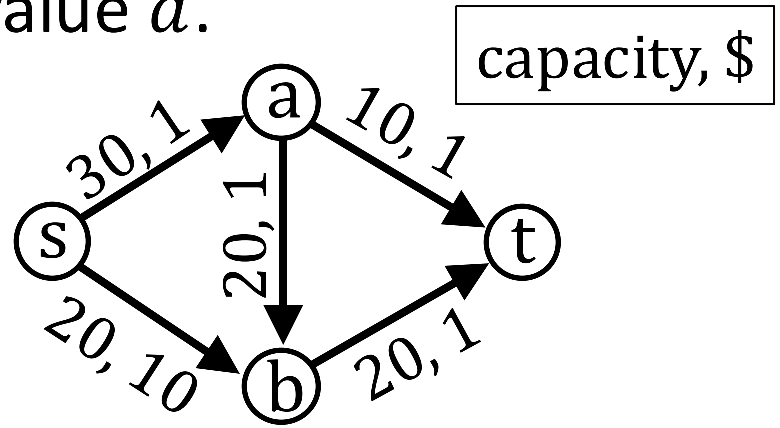

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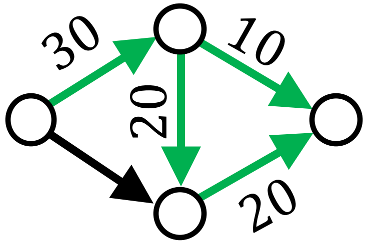

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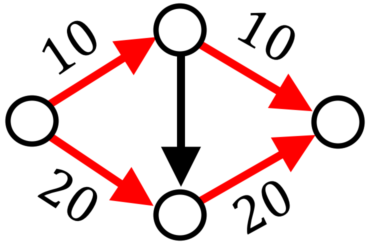

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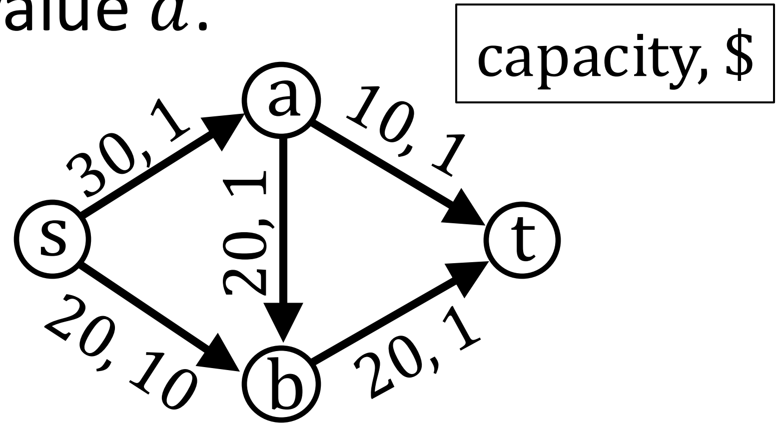

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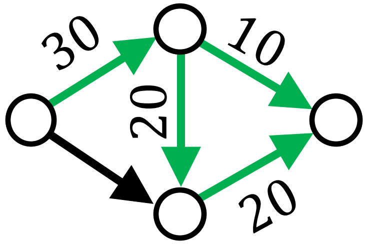

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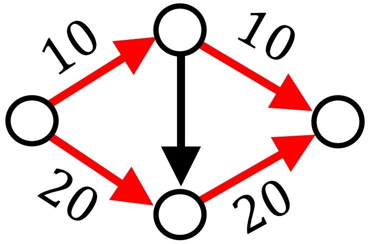

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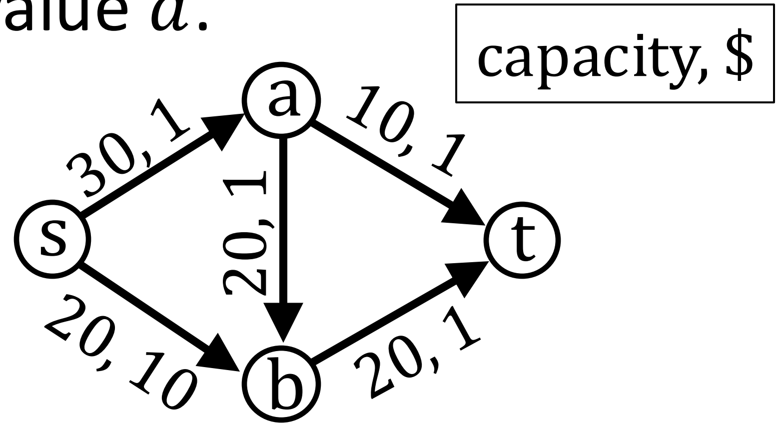

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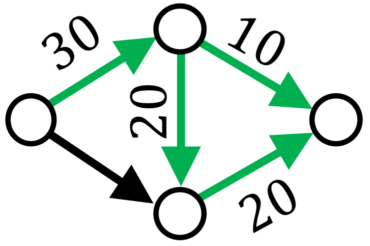

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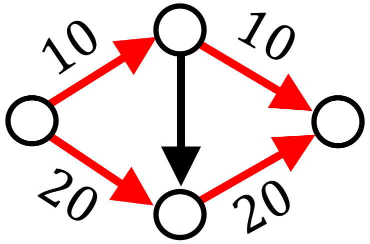

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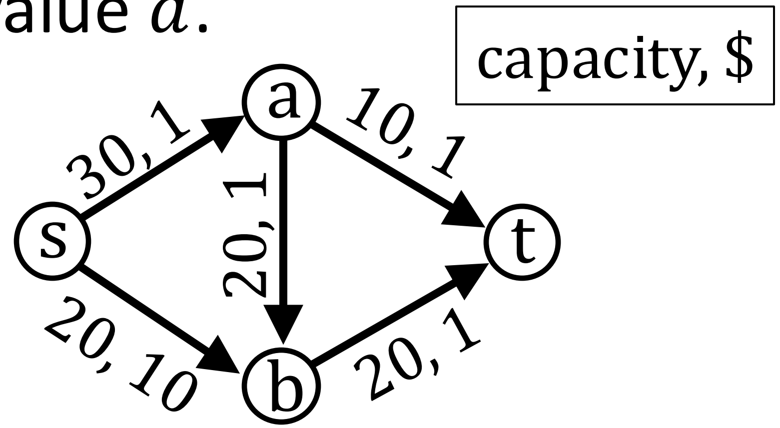

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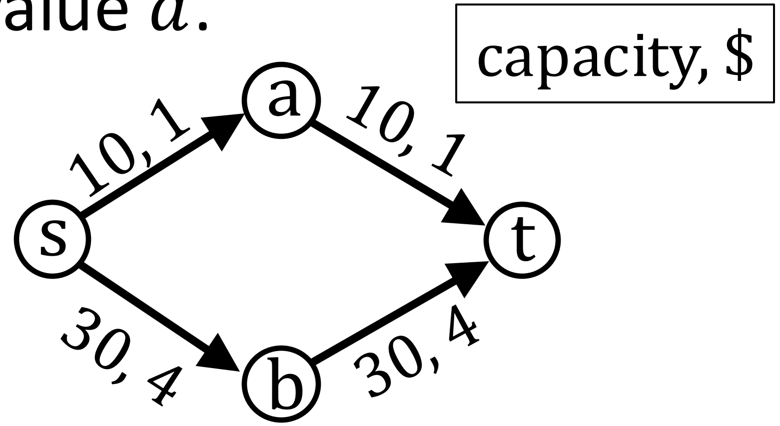

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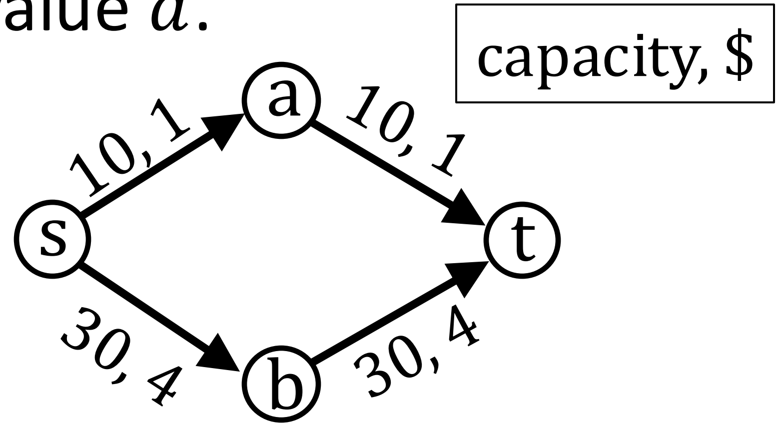

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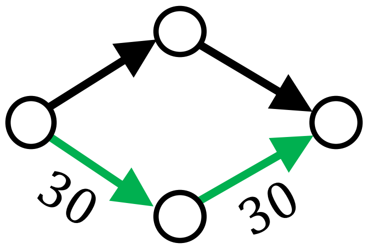

$\boxed{\boldsymbol{y_e = 1}}$

$4 + 4 = 8$

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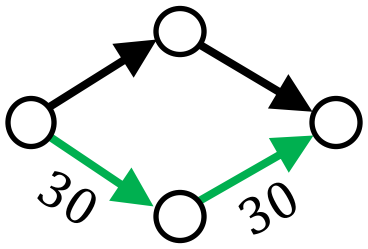

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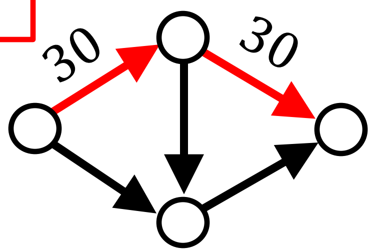

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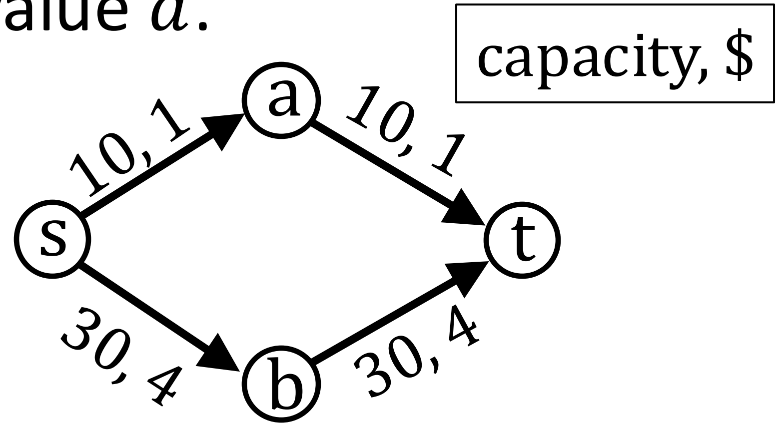

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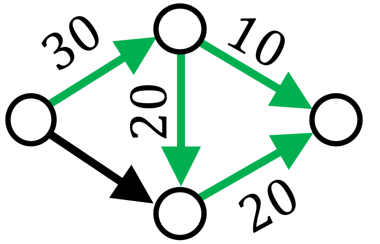

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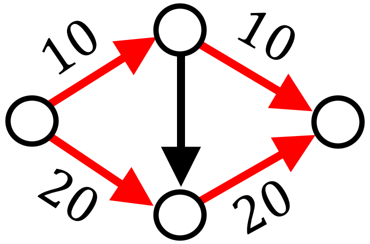

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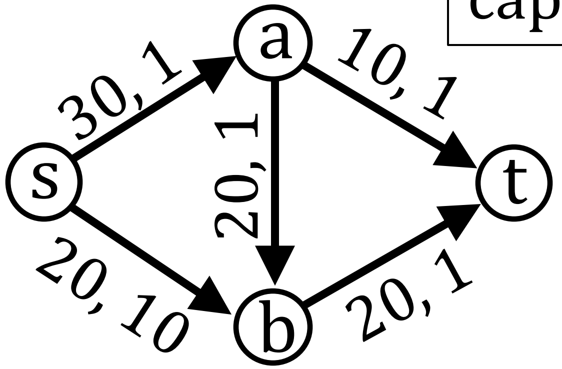

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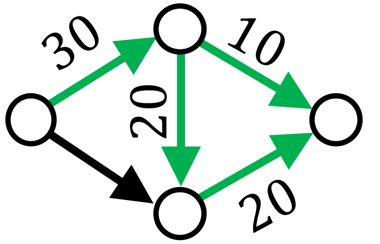

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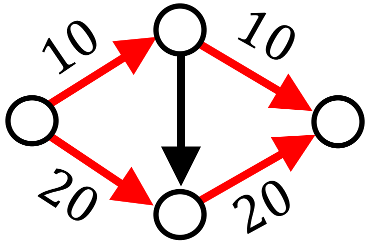

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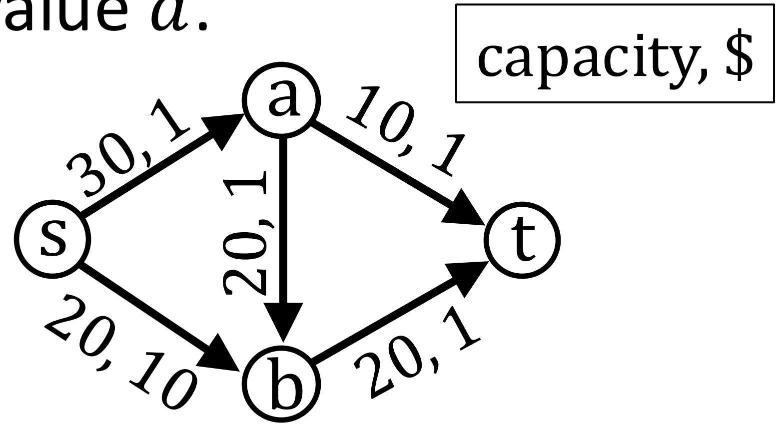

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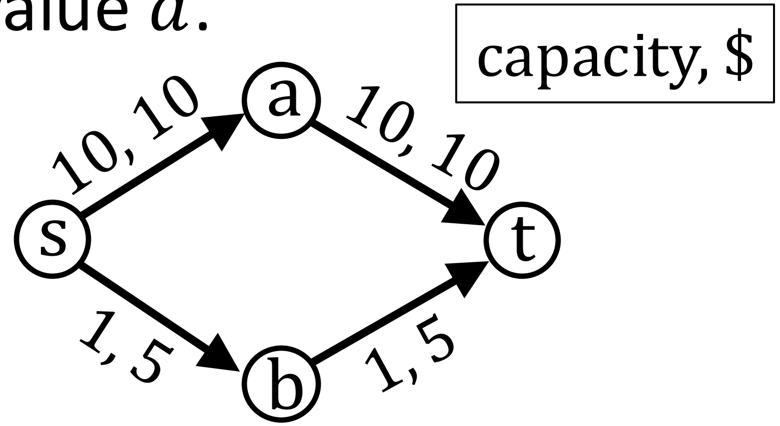

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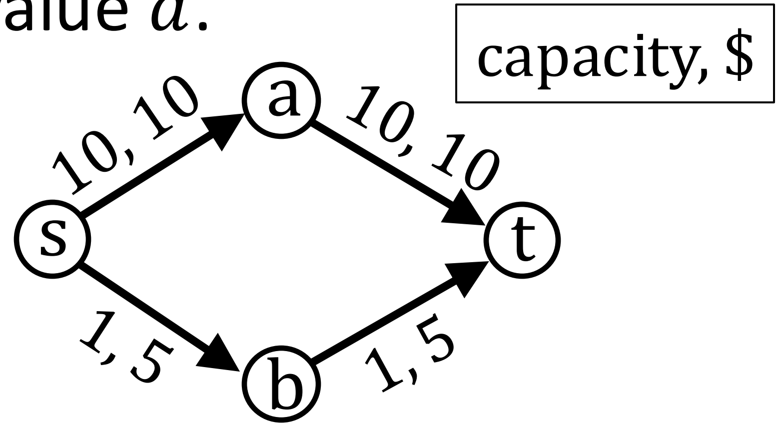
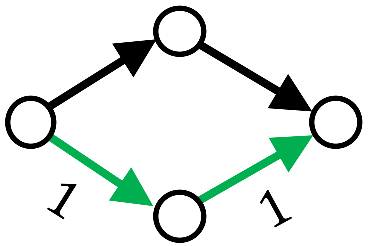


Let $d = 1$

$\boxed{\boldsymbol{y_e = 1}}$



Would this still work?

No. It could decrease capacity (and under pay) on “cheap” edges.

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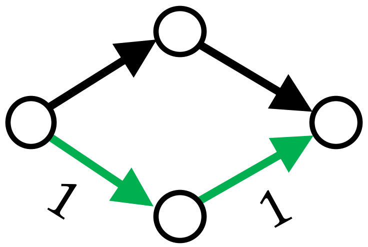
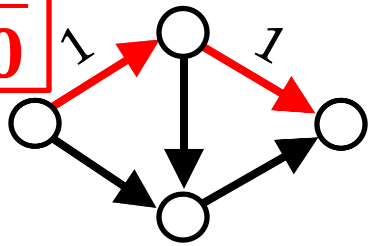


$\boxed{\boldsymbol{y_e = 1}}$

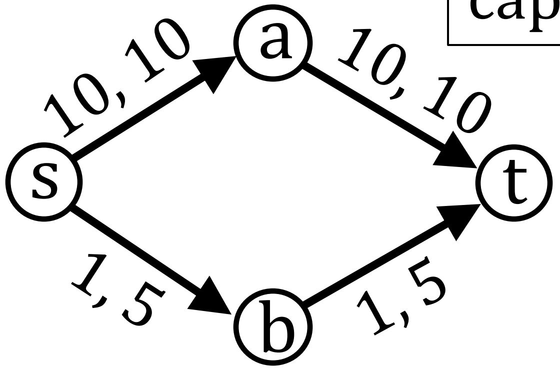
$5 + 5 = 10$

$\boxed{\boldsymbol{y_e = \frac{1}{10}}}$



$1 + 1 = 2$

capacity, \$



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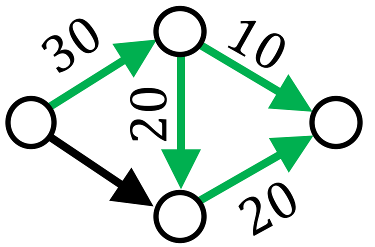
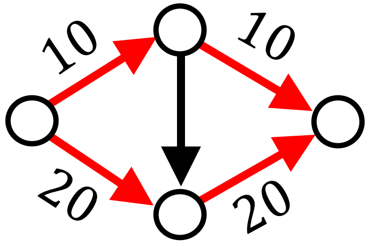
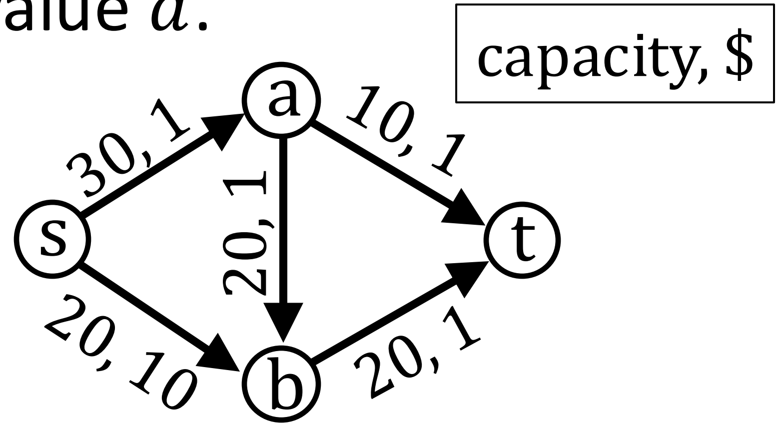


$1+1+1+1=4$



$1+10+1+1=13$



What happens here?

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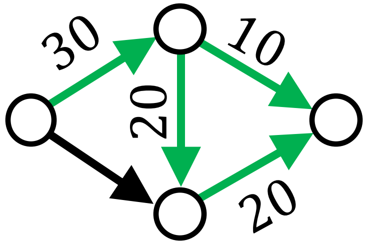

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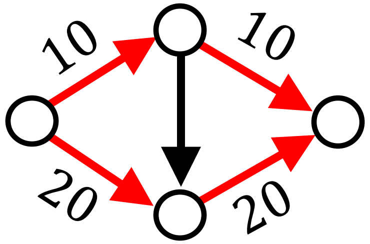

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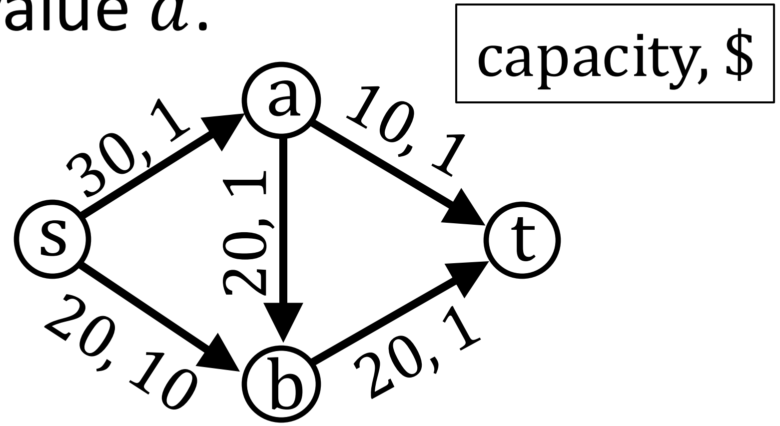

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Shortest s–t path over the unit cost ($/unit flow) edge weights.

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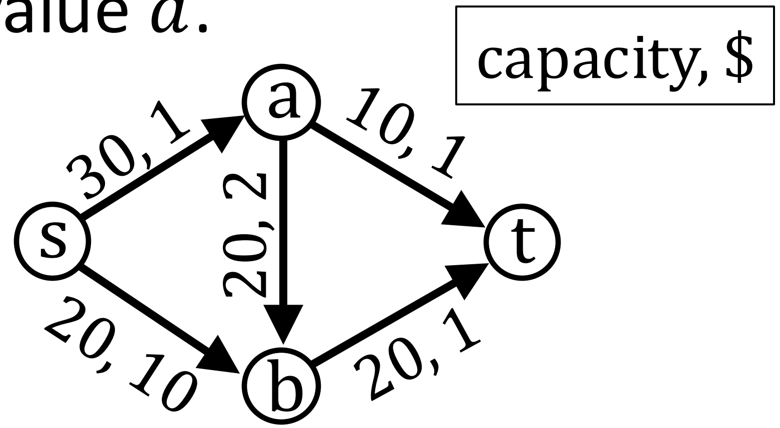

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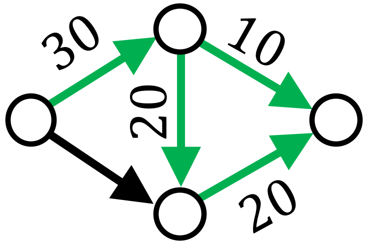
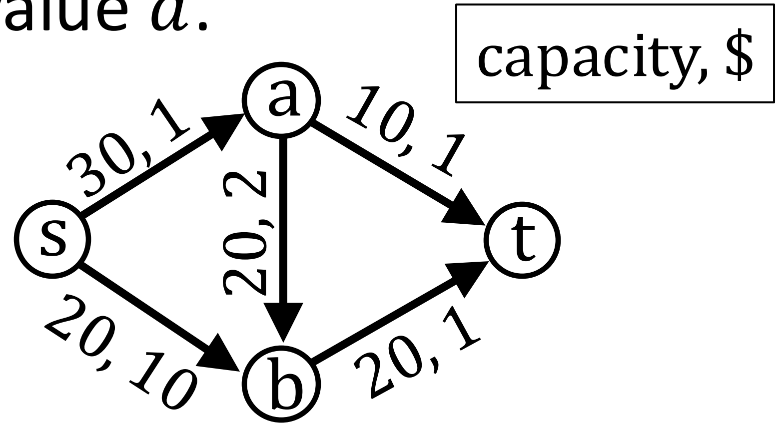


$1 + 2 + 1 + 1 = 5$



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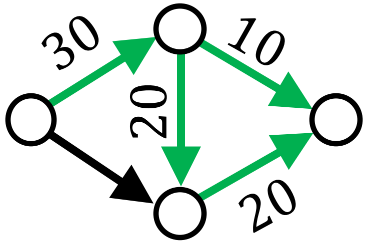

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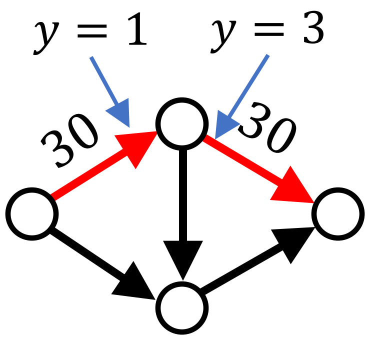
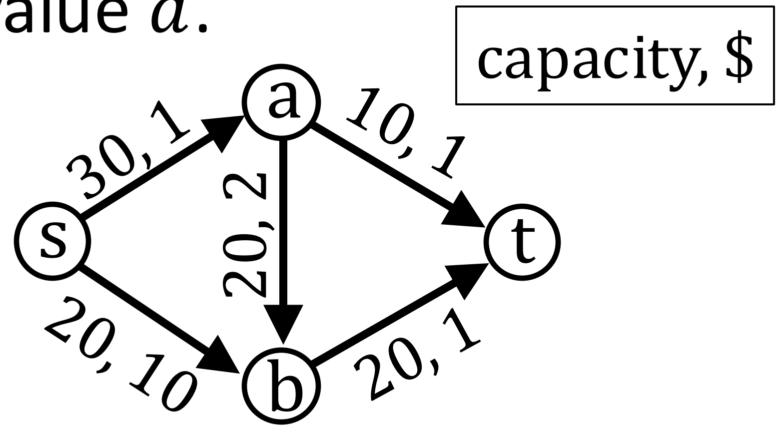


$1+3=4$



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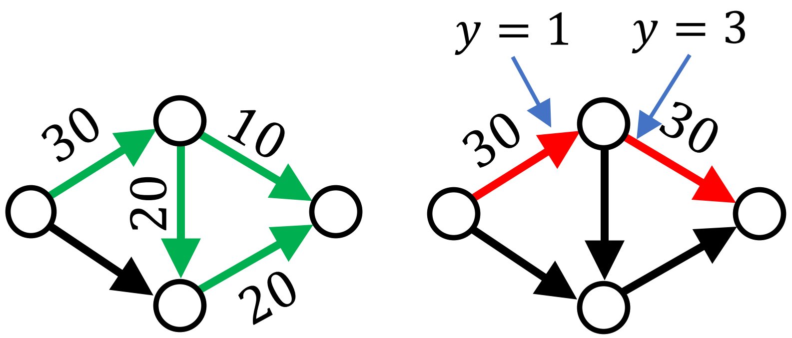
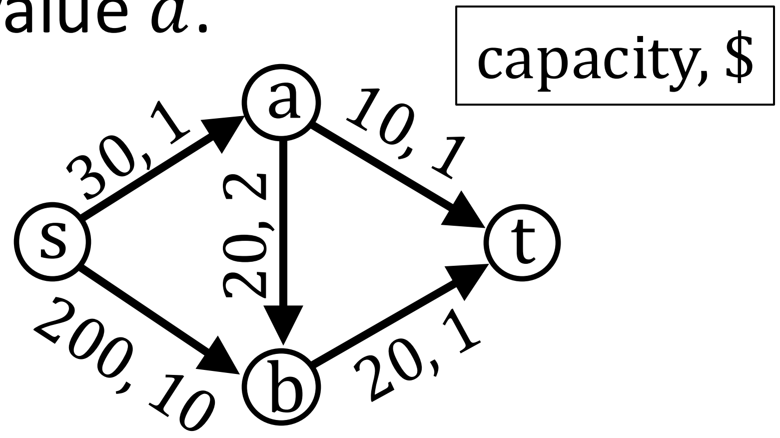


$1 + 2 + 1 + 1 = 5$ $\quad$ $1 + 3 = 4$



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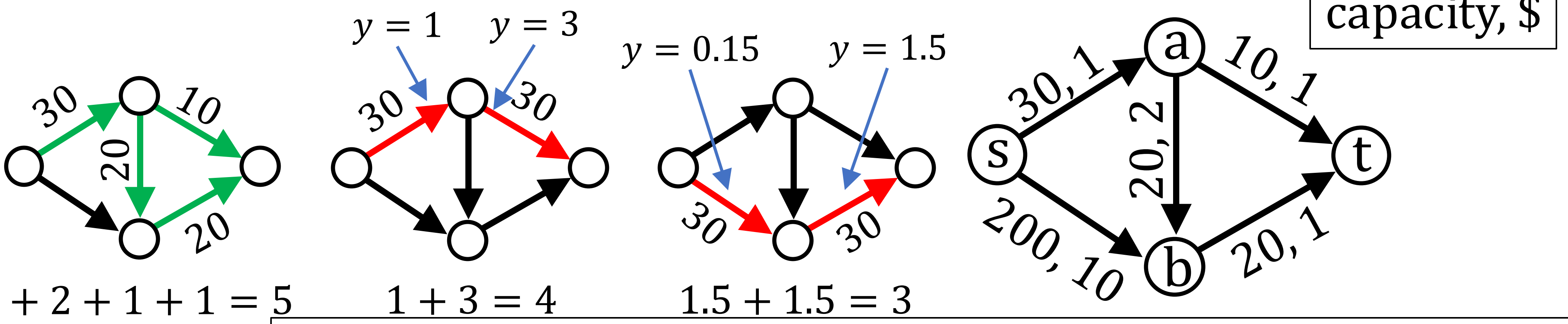

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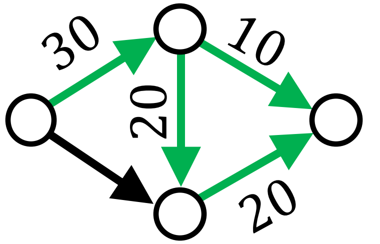

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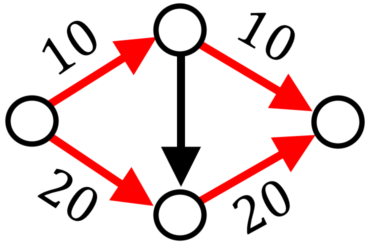

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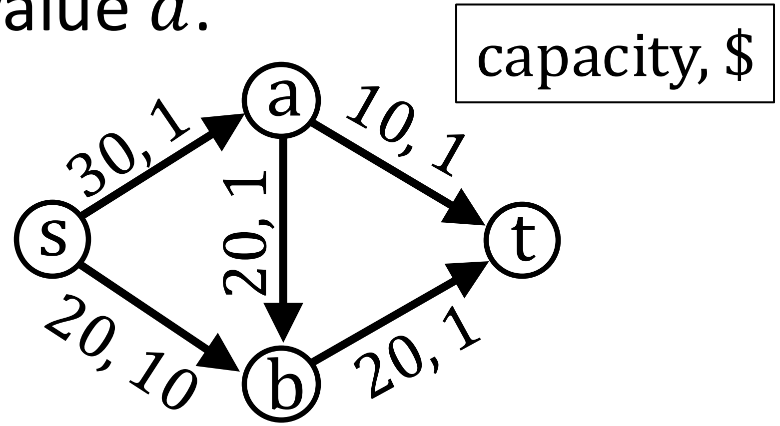

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The minimum fixed-cost flow problem is NP-Hard*, so there is likely no LP for it.

*not proven in class today!

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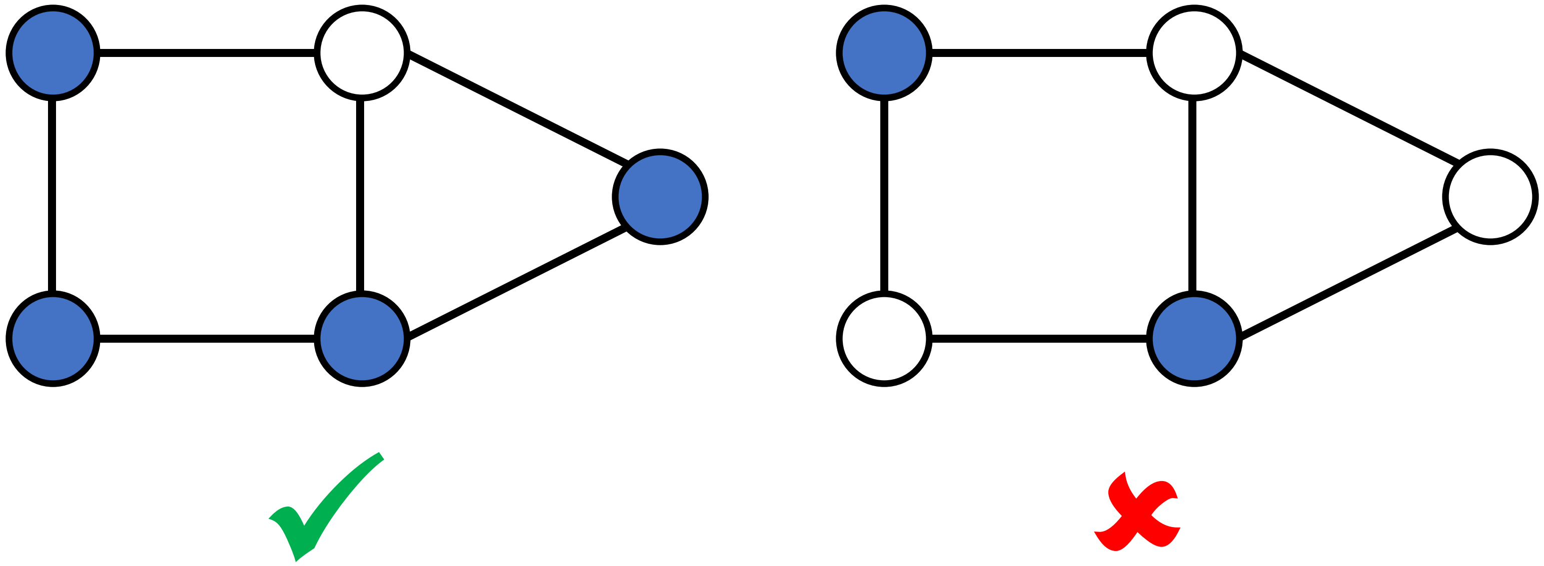

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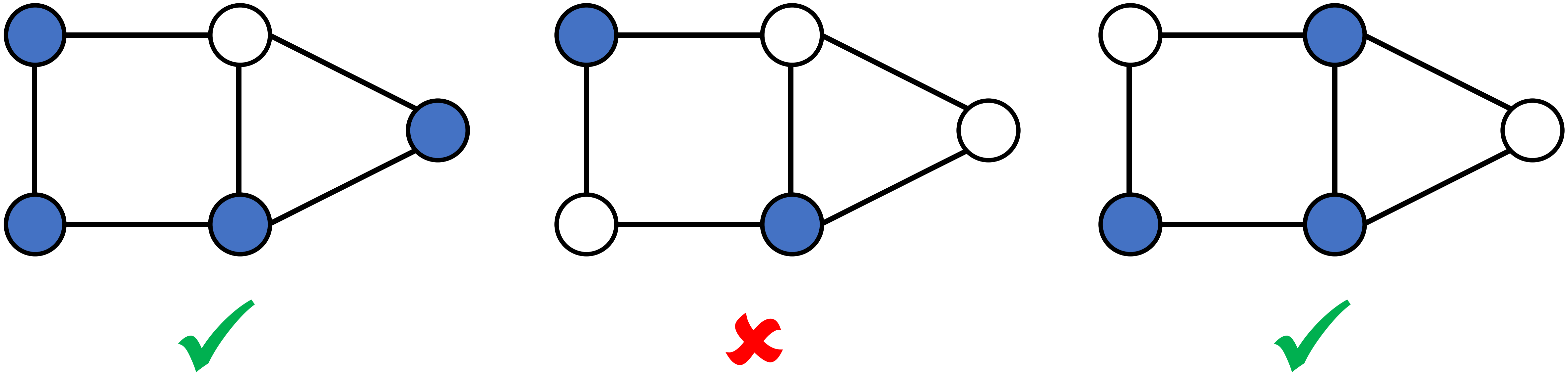

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What if we make an LP for Vertex Cover?

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What if we make an LP for Vertex Cover?

Then it would be solvable in polynomial time…

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ILP?

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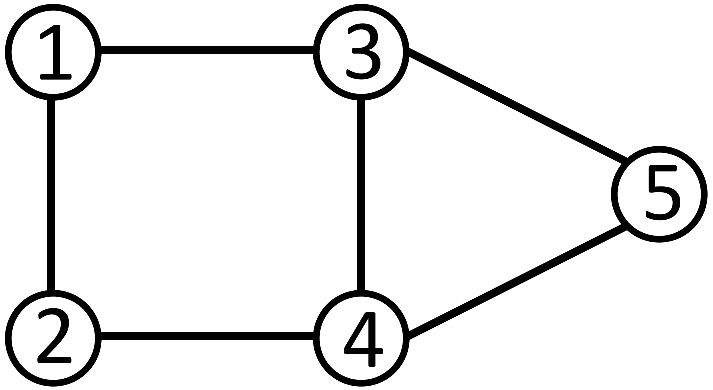
<u>Example:</u>

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:

$$
\begin{aligned}
x_1 + x_2 &\geq 1 \\
x_1 + x_3 &\geq 1 \\
x_2 + x_4 &\geq 1 \\
x_3 + x_4 &\geq 1 \\
x_3 + x_5 &\geq 1 \\
x_4 + x_5 &\geq 1
\end{aligned}
$$



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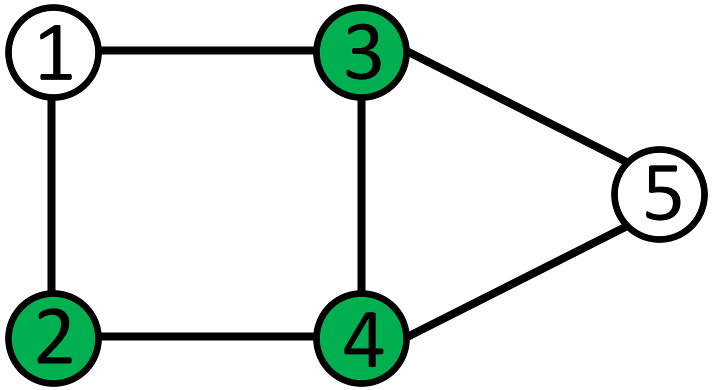

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<u>Example:</u>

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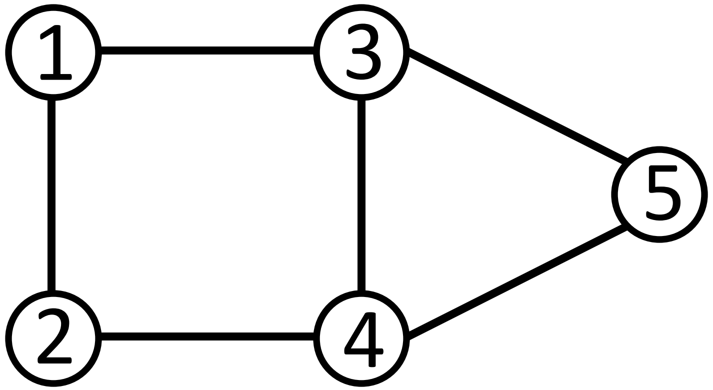

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What happens?

$x_1 = 0.5$
$x_2 = 0.5$
$x_3 = 0.5$
$x_4 = 0.5$
$x_5 = 0.5$

Example:

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:

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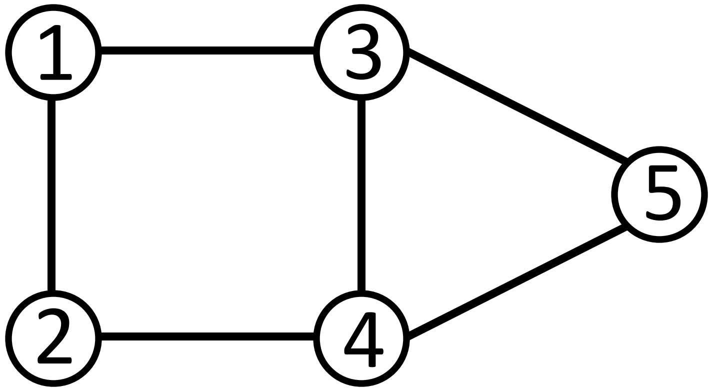

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Since Vertex Cover is in NP-Complete, and ILPs can solve VC, solving ILPs is also NP-Complete.