Approximation Algorithms
CSCI 432

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How do you show a problem is in the set P?

Solve it in polynomial time.

P — $\boldsymbol{P}$ is the set of problems that are solvable by an algorithm whose running time is polynomial time.

NP — Set of problems that are verifiable in polynomial time.

$SUBSET - SUM$

Claim: $SUBSET - SUM = \{\langle S, t \rangle : S = \{x_1, \dots, x_n\}$, and there exists some $\{y_1, \dots, y_m\} \subseteq S$ such that $\sum y_i = t\} \in NP$.

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Example:

$\langle\{4, 11, 16, 21, 27\}, 25\rangle$

Is there a subset of $\{4, 11, 16, 21, 27\}$ that sums to 25?

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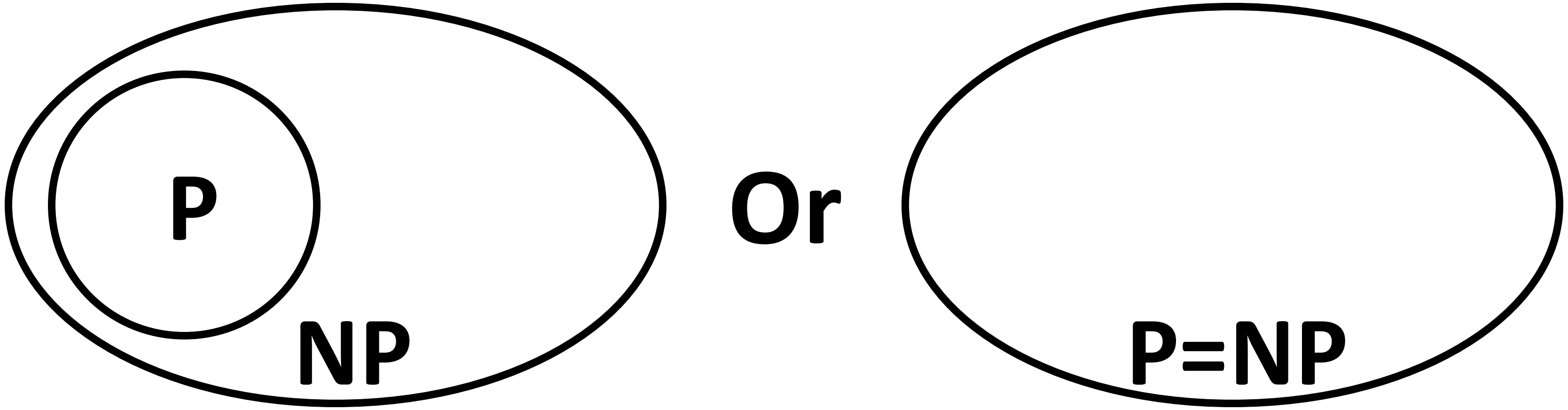
Yes, $4 + 21 = 25$.

How do we verify $SUBSET - SUM$ answers?

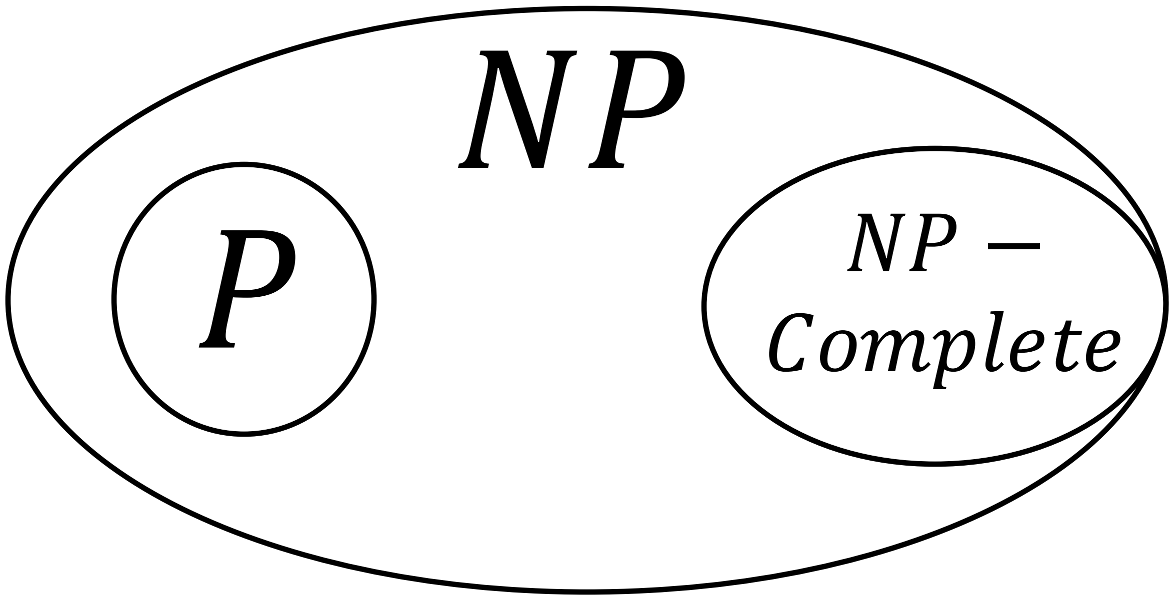
P — $\boldsymbol{P}$ is the set of problems that are solvable by an algorithm whose running time is polynomial time.

NP — Set of problems that are verifiable in polynomial time.

How does P relate to NP?



NP $Complete$ — Set of problems in NP whose algorithms can solve any other problem in NP with polynomial extra time.



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Interesting Properties:

- A polynomial time algorithm for ***any*** $NP - Complete$ problem gives a polynomial time algorithm for ***every*** problem in NP (i.e., $P = NP$)…

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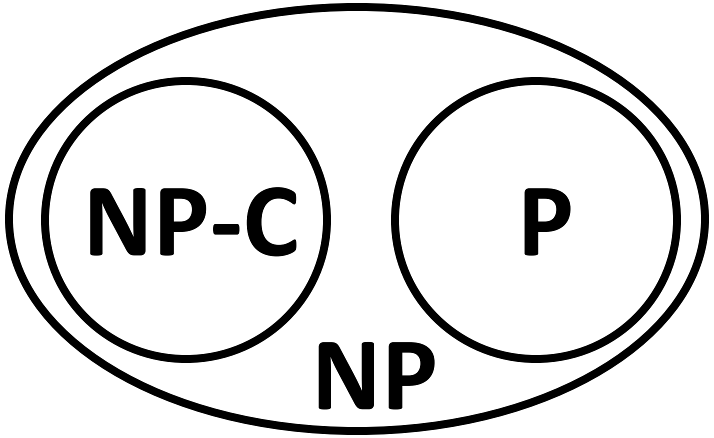
Interesting Properties:

- A polynomial time algorithm for ***any*** $NP - Complete$ problem gives a polynomial time algorithm for ***every*** problem in NP (i.e., $P = NP$)…
- … including all the other $NP - Complete$ problems.
- The thing that makes one NP-C problem (possibly) unsolvable in polynomial time is the exact same thing that makes every other NP-C problem (possibly) unsolvable in polynomial time.

Handling NP-Completeness



Techniques to handle NP-Complete problems:

1. Brute Force (i.e. Exponential Time).
2. Heuristics.
3. Approximation Algorithms.
4. Fixed-parameter Tractable Algorithms.

Approximation Algorithms

For a minimization problem.

$$\text{ALG} \leq \alpha \, \text{OPT}$$

Cost (size) of algorithm's solution.

Approximation Ratio

Cost (size) of optimal solution.

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Example: If my CheapestPizzaInBozeman algorithm is a 1.25-approximation algorithm, the cost of the pizza it finds is at most 1.25 times the optimal cost.

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Note: if problem is a maximization problem, $ALG \geq \frac{1}{\alpha} OPT$

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Example:

- Suppose I know my algorithm is a 1.12-approximation algorithm.

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- Suppose I know my algorithm is a 1.12-approximation algorithm.
- Suppose my algorithm returns a solution of cost (size) 746.125.

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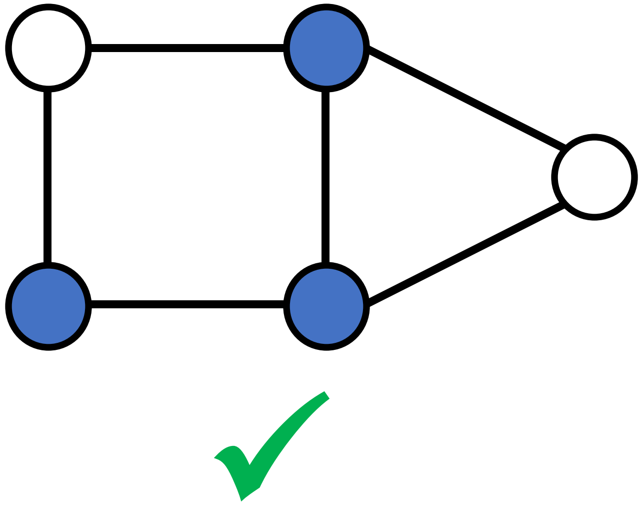
- Suppose my algorithm returns a solution of cost (size) 746.125.

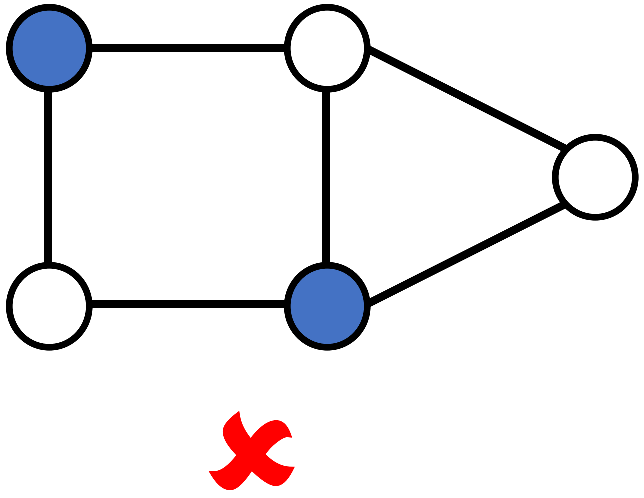
Then, I know that $746.125 \leq 1.12\ \text{OPT}$

$$\Rightarrow \frac{746.125}{1.12} = 666.183 \leq \text{OPT} \leq 746.125$$

Vertex Cover

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.





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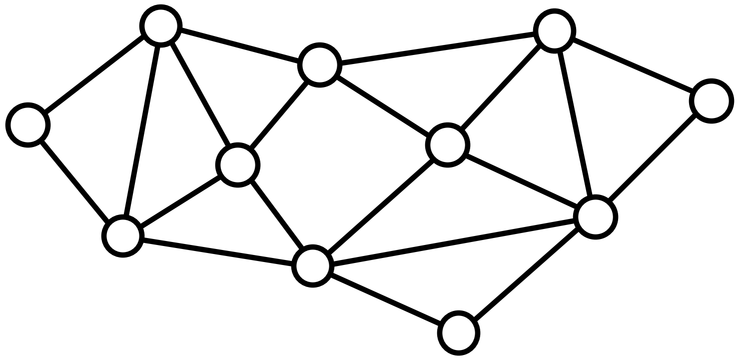

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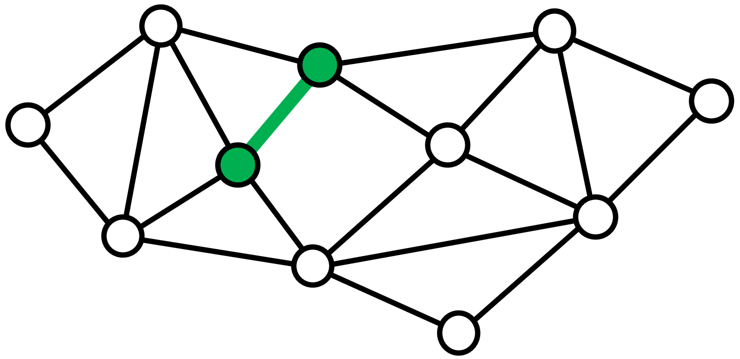

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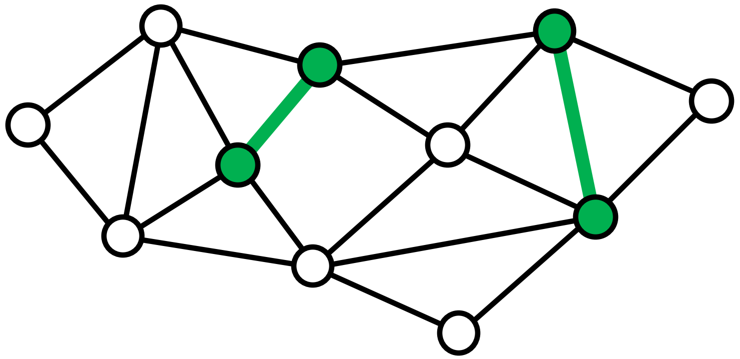

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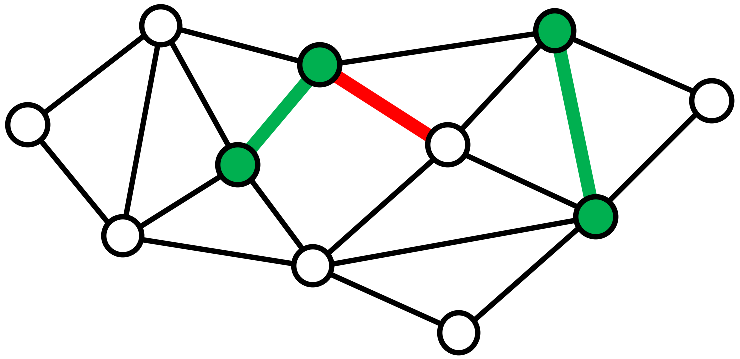

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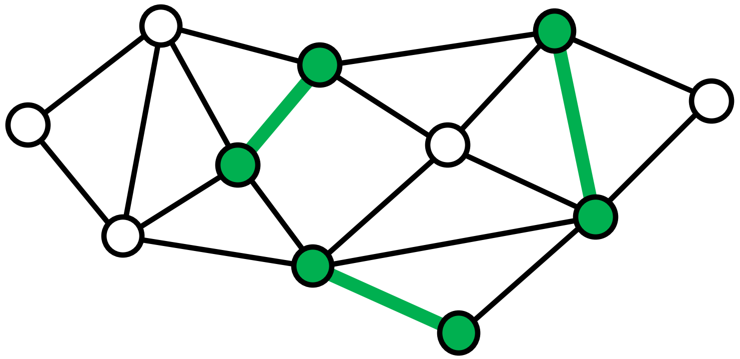

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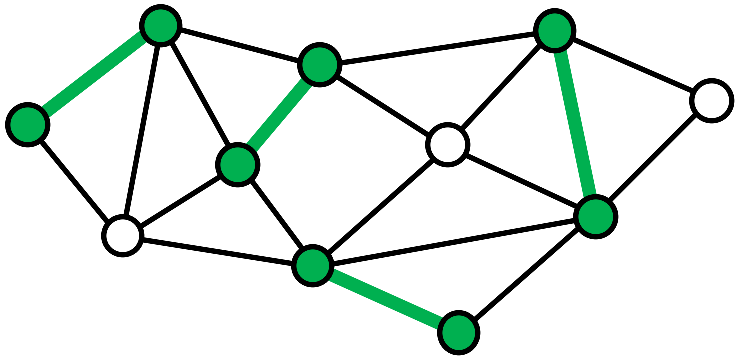

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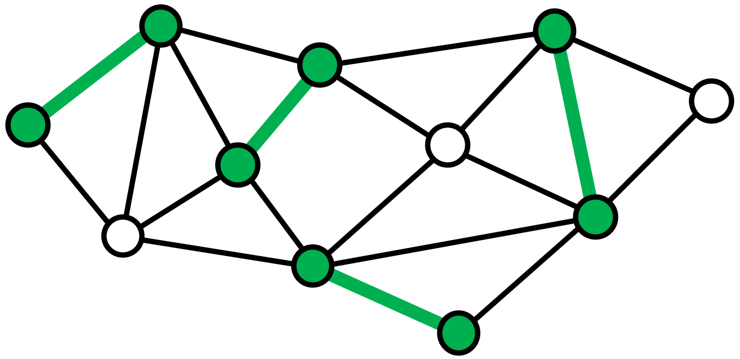
Consider a set of edges, $E' \subset E$, that do not share vertices.



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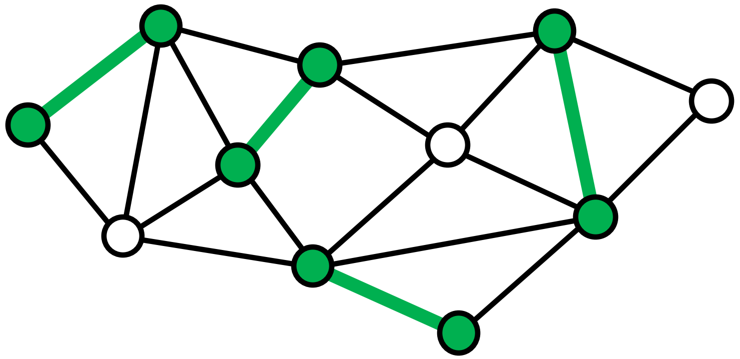
Consider a set of edges, $E' \subset E$, that do not share vertices. Is there a relationship between the minimum vertex cover and $|E'|$?



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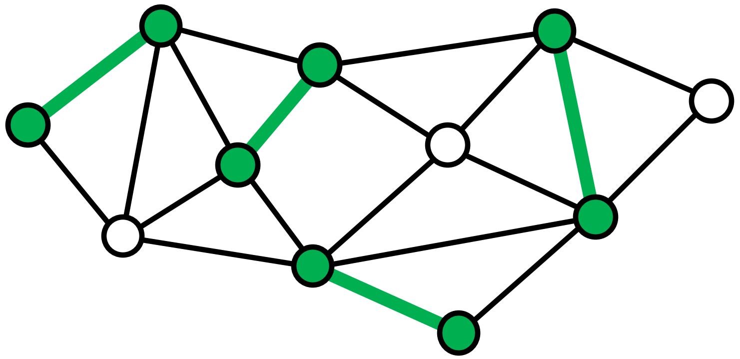
$$|E'| \leq \text{OPT}$$

Size of actual smallest vertex cover.



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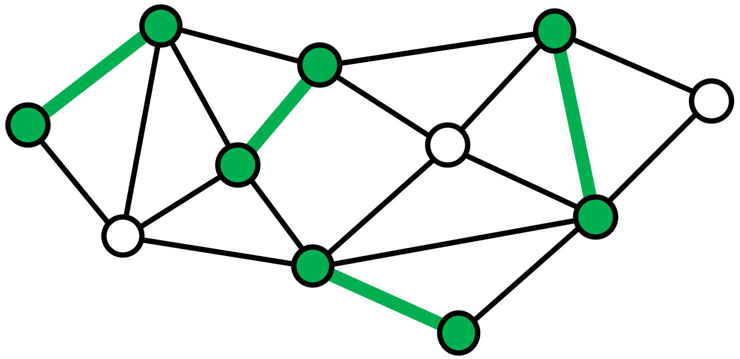
If we selected fewer than one vertex per edge, we would not have a vertex cover, because that edge would not be covered!



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We cannot find optimal vertex covers in poly time unless $P = NP$, but this algorithm is at worst 2-times optimal.

Does t

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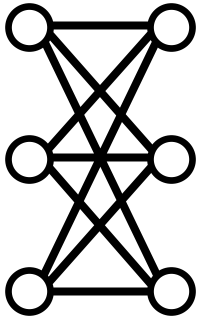
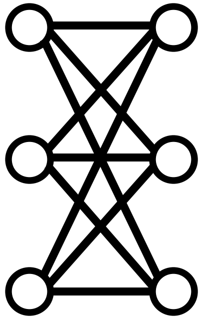
Complete
Bipartite Graph



ALG



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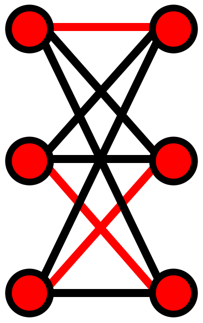

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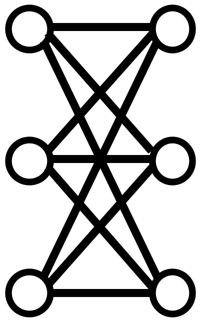

OPT

$|\text{ALG}| = n$: If v is not selected, all neighbors are $\Rightarrow \frac{n}{2}$ edges are selected $\Rightarrow$ all n vertices are selected.

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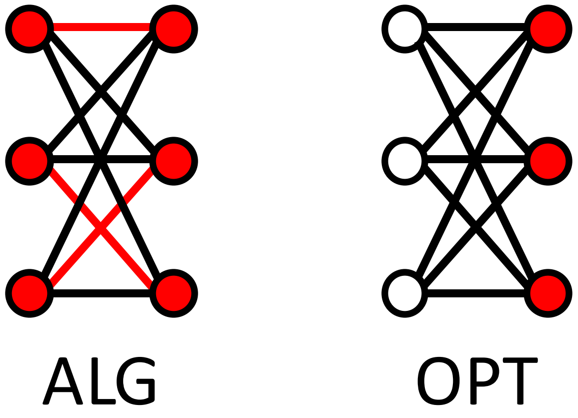

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$|\text{OPT}| = \frac{n}{2}$: Fewer than $\frac{n}{2}$ nodes selected $\Rightarrow$ $\exists$ unselected edge.

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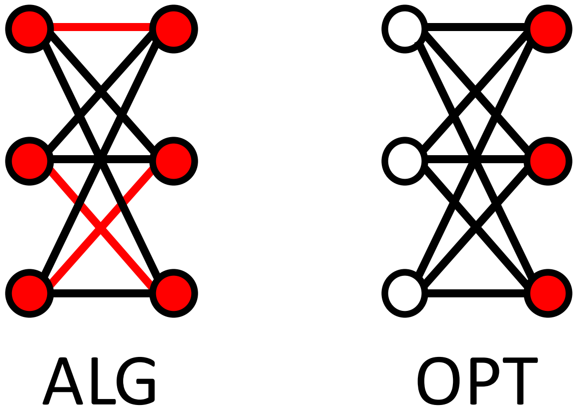

$\therefore$ The best Vertex Cover can be approximated is within a factor of 2

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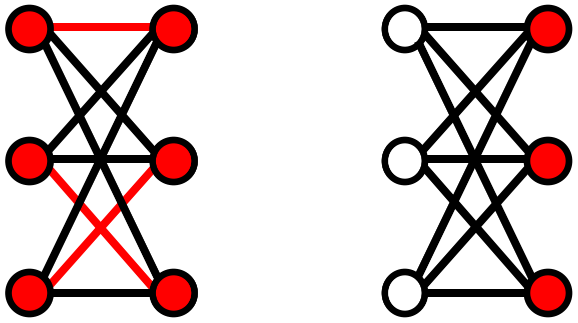

ALG OPT

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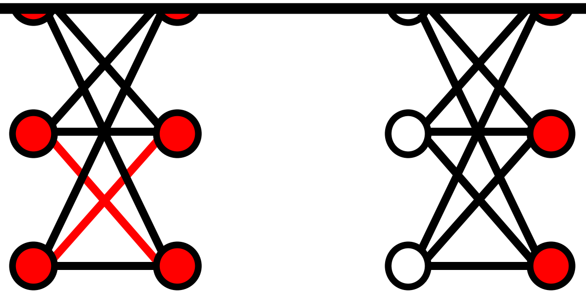
Is th

Vertex Cover is approximable within the bound $2 - \frac{\log\log|V|}{2\log|V|}$ and inapproximable within the bound 1.3606.

Complete Bipartite Graph



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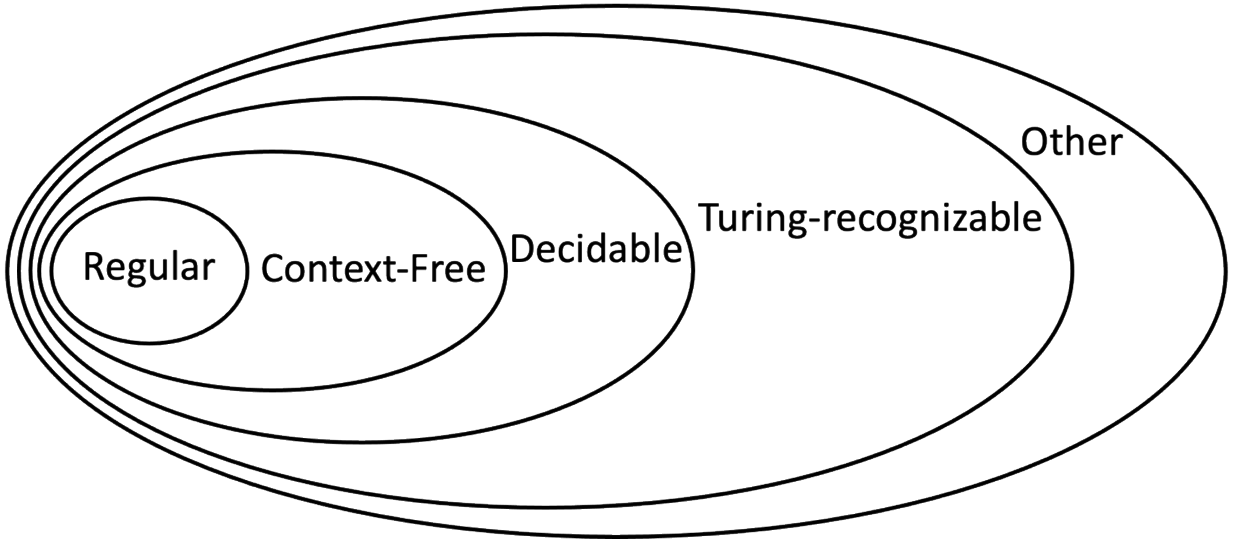
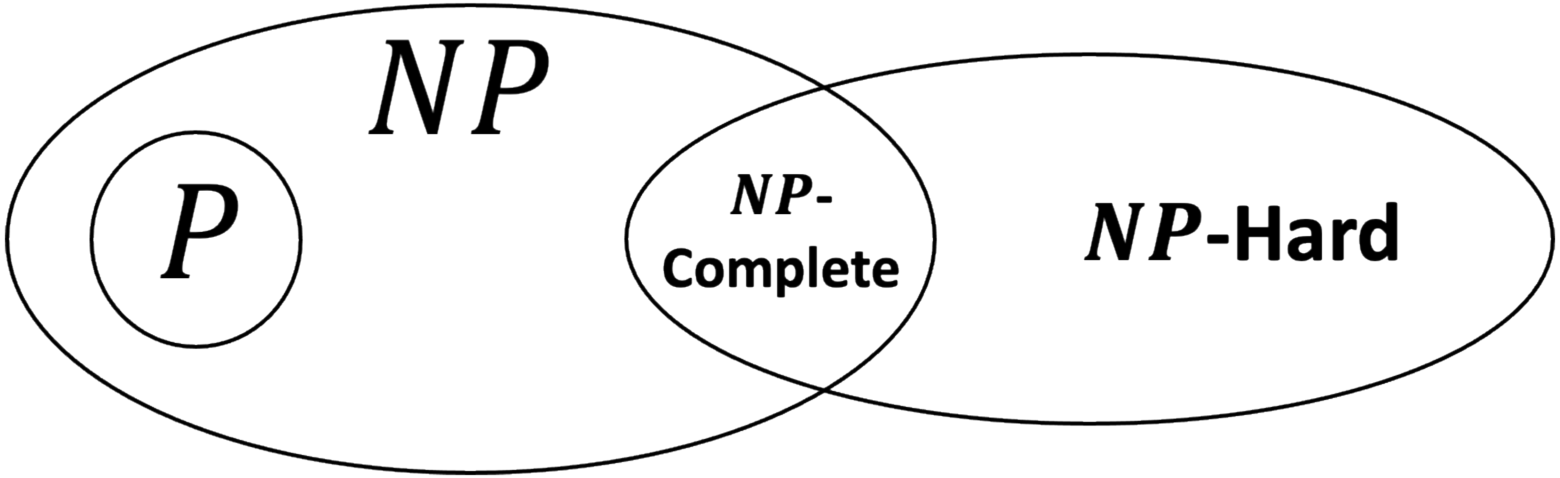
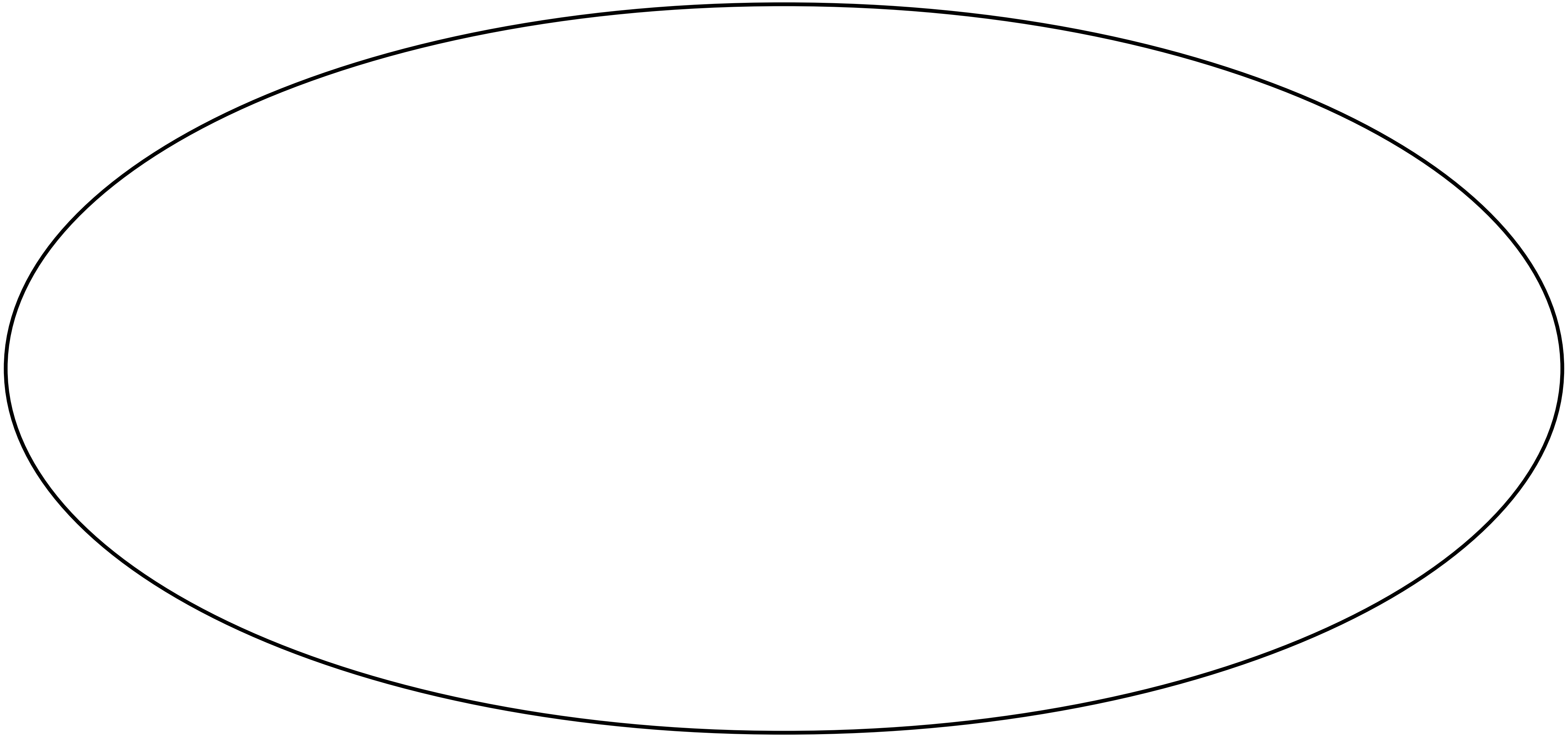


Computability Hierarchy



Complexity Hierarchy

Approximability Hierarchy



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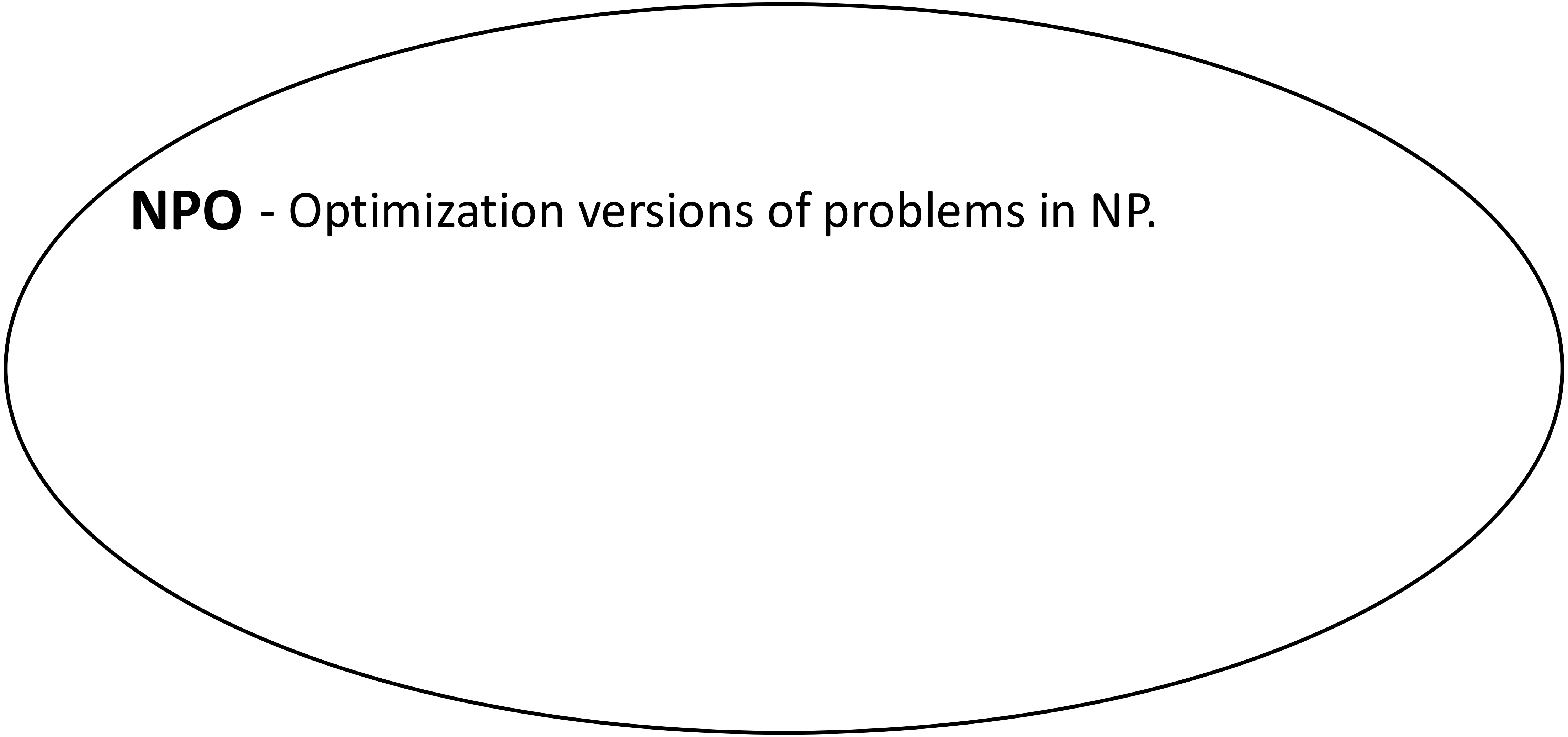

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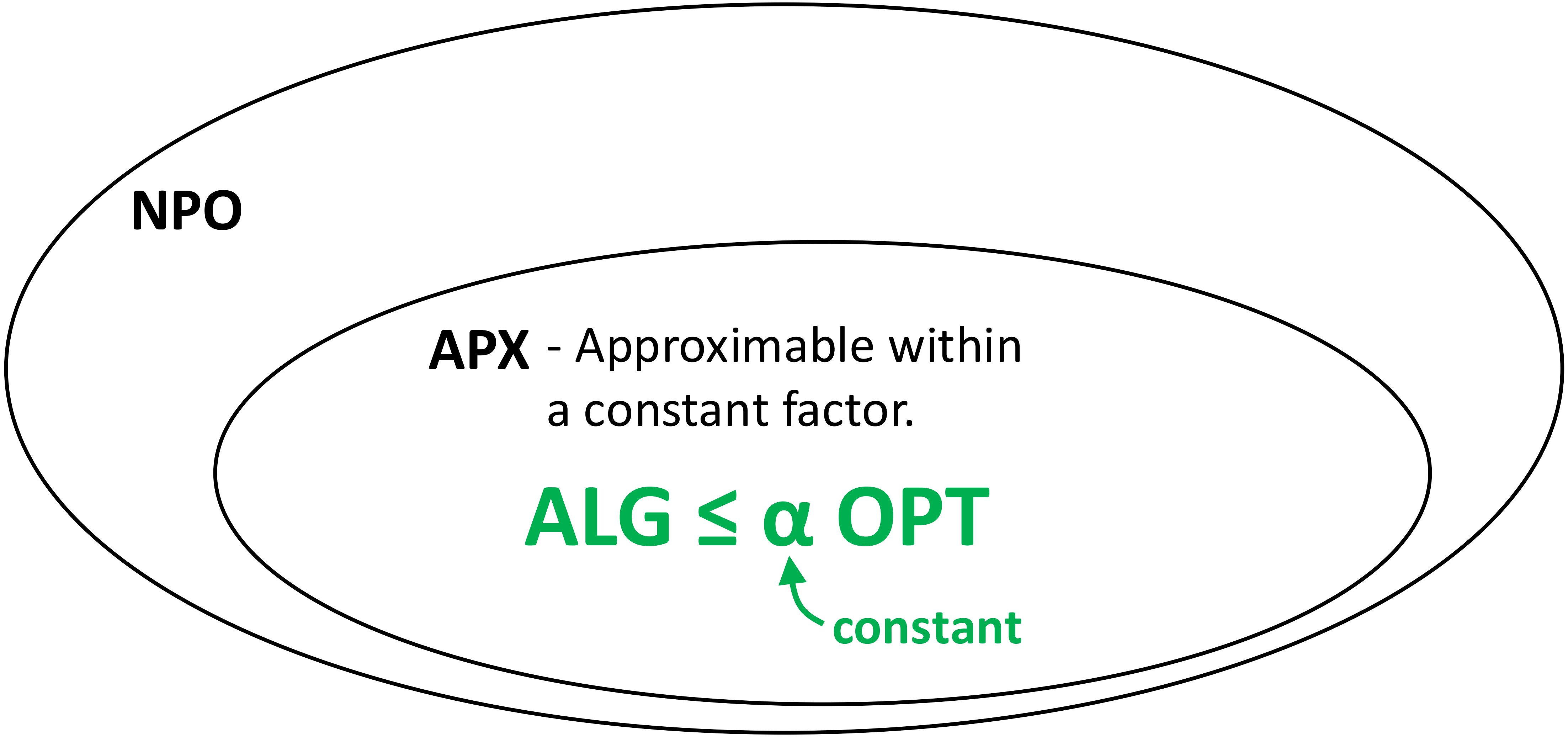

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