Set Cover
CSCI 532

Approximation Algorithms

$$ALG \leq \alpha \, OPT$$

Cost (size) of algorithm's solution.

Approximation Ratio

Cost (size) of optimal solution.

Set Cover

Set Cover: Given a set of elements (the universe), and sets containing those elements, find the smallest number of sets so that every element of the universe is included.

Example:

Set Cover

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Example:

$$U = \{1, 4, 7, 8, 10\}$$

$$S = \{\{1, 7, 8\}, \{1, 4, 7\}, \{7, 8\}, \{4, 8, 10\}\}$$

Set Cover

Set Cover: Given a set of elements (the universe), and sets containing those elements, find the smallest number of sets so that every element of the universe is included.

Example:

$$U = \{1, 4, 7, 8, 10\}$$

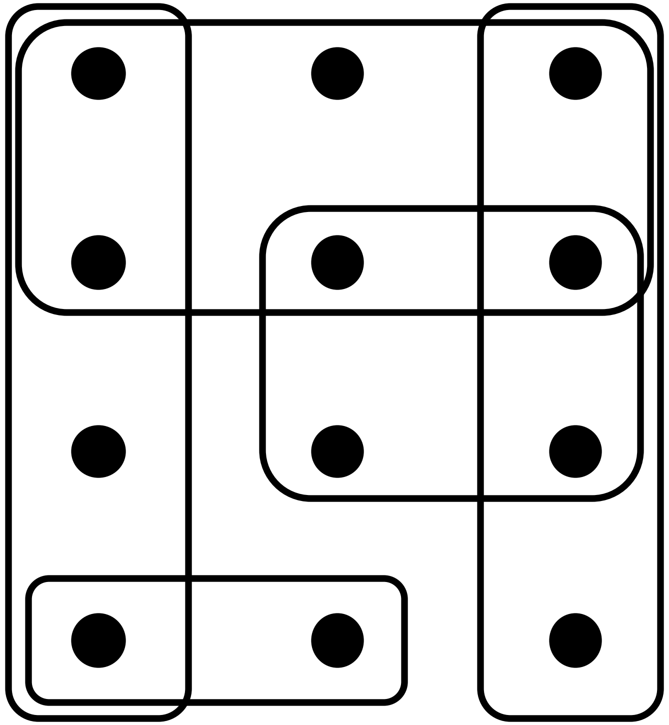
$$S = \{\{1, 7, 8\}, \{1, 4, 7\}, \{7, 8\}, \{4, 8, 10\}\}$$

$\{\{1, 7, 8\}, \{4, 8, 10\}\}$ ✓ $\{\{1, 4, 7\}, \{7, 8\}\}$ ✗

Set Cover

Set Cover: Given a set of elements (the universe), and sets containing those elements, find the smallest number of sets so that every element of the universe is included.

Example:



Set Cover

Set Cover: Given a set of elements (the universe), and sets containing those elements, find the smallest number of sets so that every element of the universe is included.

Algorithm:

?

Set Cover

Set Cover: Given a set of elements (the universe), and sets containing those elements, find the smallest number of sets so that every element of the universe is included.

Greedy Algorithm:

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while element of universe not included
  select Si with largest number of excluded elements.
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Set Cover

Suppose the universe contains n elements.

Set Cover

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OPT = # sets in an optimal solution to cover all n elements.

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Game Plan:

Bound the maximum number of sets in ALG by…

Bounding the maximum number of iterations of the algorithm by...

Bounding the size of each set added by the algorithm.

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Suppose the universe contains n elements.

What can we say about the first set selected?

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Suppose the universe contains n elements.

What can we say about the first set selected?

It’s the biggest!

At each iteration, we cover the largest number of uncovered elements, and all the elements are uncovered in the first iteration.

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? $\leq$ |Biggest Set| $\leq$?

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Which do I care about?

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? $\leq$ |Biggest Set| $\leq$?

Guarantee we do at least this good.

Guarantee we don't do better than this.

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Suppose the universe contains n elements.

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It's the biggest!

? $\leq$ **|Biggest Set|**

Guarantee we do at least this good.

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Suppose the universe contains n elements.

What can we say about the first set selected?

It's the biggest!

$$\frac{\boldsymbol{n}}{\boldsymbol{OPT}} \leq |\textbf{Biggest Set}|$$

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What if each set had fewer than $\frac{n}{OPT}$ elements in it?

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Then OPT of those sets would cover fewer than n elements:

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covered elements $\leq$ Size of sets $\times$ # of sets

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What can we say about the first set selected?

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What if each set had fewer than $\frac{n}{OPT}$ elements in it?

Then OPT of those sets would cover fewer than n elements:

$$\text{\# covered elements} \leq \text{Size of sets} \times \text{\# of sets} < \frac{n}{OPT} OPT = n.$$

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What if each set had fewer than $\frac{n}{OPT}$ elements in it?

Then OPT of those sets would cover fewer than n elements:

$$\text{\# covered elements} \leq \text{Size of sets} \times \text{\# of sets} < \frac{n}{OPT} OPT = n.$$

Not a valid solution!

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ALG = # sets selected by the algorithm to cover all n elements.

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Suppose the universe contains n elements.
The first set selected will have $\geq \frac{n}{OPT}$ elements.

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Suppose the universe contains n elements.
The first set selected will have $\geq \frac{n}{OPT}$ elements.
Then, the number of elements remaining after the first iteration is:

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Original # elements $-$ **# elements removed**

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ALG = # sets selected by the algorithm to cover all n elements.

OPT = # sets in an optimal solution to cover all n elements.

Suppose the universe contains n elements.
The first set selected will have $\geq \frac{n}{OPT}$ elements.
Then, the number of elements remaining after the first iteration is:

?

Original # elements $-$ **# elements removed**

↓

$\boldsymbol{n}$

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OPT = # sets in an optimal solution to cover all n elements.

Suppose the universe contains n elements.
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Then, the number of elements remaining after the first iteration is:

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Original # elements − **# elements removed**

$\boldsymbol{n}$ $\quad$ $\geq \boldsymbol{\frac{n}{OPT}}$

Set Cover

ALG = # sets selected by the algorithm to cover all n elements.

OPT = # sets in an optimal solution to cover all n elements.

Suppose the universe contains n elements.

The first set selected will have $\geq \frac{n}{OPT}$ elements.

Then, the number of elements remaining after the first iteration is:

$$n_1 \leq n - \frac{n}{OPT}$$

The first set could be $> \frac{n}{OPT}$, which would leave fewer elements remaining.

Original # elements − **# elements removed**

$$\boldsymbol{n} \qquad \geq \frac{\boldsymbol{n}}{\boldsymbol{OPT}}$$

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Suppose the universe contains n elements.
The first set selected will have $\geq \frac{n}{OPT}$ elements.
Then, the number of elements remaining after the first iteration is:

$$n_1 \leq n - \frac{n}{OPT} = n\left(1 - \frac{1}{OPT}\right)$$

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Suppose the universe contains n elements.
The first set selected will have $\geq \frac{n}{OPT}$ elements.
Then, the number of elements remaining after the first iteration is:

$$n_1 \leq n - \frac{n}{OPT} = n\left(1 - \frac{1}{OPT}\right)$$

What can we say about the second set selected?

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$$n_1 \leq n - \frac{n}{OPT} = n\left(1 - \frac{1}{OPT}\right)$$

What can we say about the second set selected? ← **It covers the most uncovered elements.**

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The first set selected will have $\geq \frac{n}{OPT}$ elements.
Then, the number of elements remaining after the first iteration is:

$$n_1 \leq n - \frac{n}{OPT} = n\left(1 - \frac{1}{OPT}\right)$$

What can we say about the second set selected?

Suppose the first set was in the optimal solution.

Suppose the first set was <u>not</u> in the optimal solution.

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Suppose the universe contains n elements.

The first set selected will have $\geq \frac{n}{OPT}$ elements.

Then, the number of elements remaining after the first iteration is:

$$n_1 \leq n - \frac{n}{OPT} = n\left(1 - \frac{1}{OPT}\right)$$

What can we say about the second set selected?

Suppose the first set was in the optimal solution.

Then, a remaining set must have at least $\frac{n_1}{OPT-1}$ uncovered elements.

Suppose the first set was <u>not</u> in the optimal solution.

If not, how do the remaining $OPT - 1$ optimal sets cover the remaining n_1 elements?

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Suppose the universe contains n elements.
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Then, the number of elements remaining after the first iteration is:

$$n_1 \leq n - \frac{n}{OPT} = n\left(1 - \frac{1}{OPT}\right)$$

What can we say about the second set selected?

Suppose the first set was in the optimal solution.

Then, a remaining set must have at least $\frac{n_1}{OPT-1}$ uncovered elements.

$$\Rightarrow \frac{n_1}{OPT-1} \leq |\text{Second Set}|$$

Suppose the first set was <u>not</u> in the optimal solution.

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Suppose the universe contains n elements.

The first set selected will have $\geq \frac{n}{OPT}$ elements.

Then, the number of elements remaining after the first iteration is:

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Suppose the first set was <u>not</u> in the optimal solution.

Then, the n_1 elements must still be covered by OPT.

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Suppose the universe contains n elements.

The first set selected will have $\geq \frac{n}{OPT}$ elements.

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Suppose the first set was <u>not</u> in the optimal solution.

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$$\Rightarrow \frac{n_1}{OPT} \leq |\text{Second Set}|$$

If not, how do the OPT optimal sets cover the remaining n_1 elements?

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Suppose the universe contains n elements.
The first set selected will have $\geq \frac{n}{OPT}$ elements.
Then, the number of elements remaining after the first iteration is:

$$n_1 \leq n - \frac{n}{OPT} = n\left(1 - \frac{1}{OPT}\right)$$

What can we say about the second set selected?

Suppose the first set was in the optimal solution.

Then, a remaining set must have at least $\frac{n_1}{OPT-1}$ uncovered elements.

$$\Rightarrow \frac{n_1}{OPT} < \frac{n_1}{OPT-1} \leq |\text{Second Set}|$$

Suppose the first set was <u>not</u> in the optimal solution.

Then, the n_1 elements must still be covered by OPT.

$$\Rightarrow \frac{n_1}{OPT} \leq |\text{Second Set}|$$

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Suppose the universe contains n elements.

The first set selected will have $\geq \frac{n}{OPT}$ elements.

Then, the number of elements remaining after the first iteration is:

$$n_1 \leq n - \frac{n}{OPT} = n\left(1 - \frac{1}{OPT}\right)$$

Some remaining set has at least $\frac{n_1}{OPT}$ uncovered elements.

Suppose the first set was in the optimal solution.

Then, a remaining set must have at least $\frac{n_1}{OPT-1}$ uncovered elements.

$$\Rightarrow \frac{n_1}{OPT} < \frac{n_1}{OPT-1} \leq |\text{Second Set}|$$

Suppose the first set was not in the optimal solution.

Then, the n_1 elements must still be covered by OPT.

$$\Rightarrow \frac{n_1}{OPT} \leq |\text{Second Set}|$$

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Then, the number of elements remaining after the second iteration is:

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Suppose the universe contains n elements.
The first set selected will have $\geq \frac{n}{OPT}$ elements.
Then, the number of elements remaining after the first iteration is:

$$n_1 \leq n - \frac{n}{OPT} = n\left(1 - \frac{1}{OPT}\right)$$

Some remaining set has at least $\frac{n_1}{OPT}$ uncovered elements.
Then, the number of elements remaining after the second iteration is:

$$n_2 \leq n_1 - \frac{n_1}{OPT}$$

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Some remaining set has at least $\frac{n_1}{OPT}$ uncovered elements.
Then, the number of elements remaining after the second iteration is:

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Some remaining set has at least $\frac{n_1}{OPT}$ uncovered elements.

Then, the number of elements remaining after the second iteration is:

$$n_2 \leq n_1 - \frac{n_1}{OPT} = n_1\left(1 - \frac{1}{OPT}\right) \leq n\left(1 - \frac{1}{OPT}\right)^2$$

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$$n_2 \leq n_1 - \frac{n_1}{OPT} = n_1\left(1 - \frac{1}{OPT}\right) \leq n\left(1 - \frac{1}{OPT}\right)^2$$

In general, after t iterations:

$$n_t \leq n_{t-1} - \frac{n_{t-1}}{OPT} = n_{t-1}\left(1 - \frac{1}{OPT}\right) \leq \cdots \leq n\left(1 - \frac{1}{OPT}\right)^t$$

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Suppose the universe contains n elements.
Before the t^{th} iteration, some remaining set has at least $\frac{n_{t-1}}{OPT}$ uncovered elements and the number of elements remaining after the t^{th} iteration is:

$$n_t \leq n_{t-1} - \frac{n_{t-1}}{OPT} = n_{t-1}\left(1 - \frac{1}{OPT}\right) \leq n\left(1 - \frac{1}{OPT}\right)^t$$

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Big picture:

How many sets are added each iteration?

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Big picture:

How many sets are added each iteration? **1**

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Big picture:

How many sets are added each iteration? **1**

ALG = **?**

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Big picture:

How many sets are added each iteration? **1**

ALG = **# iterations**

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Big picture:

How many sets are added each iteration? **1**

ALG = **# iterations**

When does the algorithm terminate?

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ALG = # sets selected by the algorithm to cover all n elements.

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$$n_t \leq n_{t-1} - \frac{n_{t-1}}{OPT} = n_{t-1}\left(1 - \frac{1}{OPT}\right) \leq n\left(1 - \frac{1}{OPT}\right)^t$$

Big picture:

How many sets are added each iteration? **1**

ALG = **# iterations**

When does the algorithm terminate? **When $n_t < 1$**

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ALG = # sets selected by the algorithm to cover all n elements.

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Suppose the universe contains n elements.
Before the t^{th} iteration, some remaining set has at least $\frac{n_{t-1}}{OPT}$ uncovered elements and the number of elements remaining after the t^{th} iteration is:

$$n_t \leq n_{t-1} - \frac{n_{t-1}}{OPT} = n_{t-1}\left(1 - \frac{1}{OPT}\right) \leq n\left(1 - \frac{1}{OPT}\right)^t$$

Big picture:

How many sets are added each iteration? **1**

ALG = **# iterations**

When does the algorithm terminate? **When $n_t < 1$**

What value of t makes $n_t < 1$

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Suppose the universe contains n elements.
Before the t^{th} iteration, some remaining set has at least $\frac{n_{t-1}}{OPT}$ uncovered elements and the number of elements remaining after the t^{th} iteration is:

$$n_t \leq n_{t-1} - \frac{n_{t-1}}{OPT} = n_{t-1}\left(1 - \frac{1}{OPT}\right) \leq n\left(1 - \frac{1}{OPT}\right)^t$$

Trust that $1 - x < e^{-x}$ for all $x \neq 0$.

Set Cover

ALG = # sets selected by the algorithm to cover all n elements.

OPT = # sets in an optimal solution to cover all n elements.

Suppose the universe contains n elements.
Before the t^{th} iteration, some remaining set has at least $\frac{n_{t-1}}{OPT}$ uncovered elements and the number of elements remaining after the t^{th} iteration is:

$$n_t \leq n_{t-1} - \frac{n_{t-1}}{OPT} = n_{t-1}\left(1 - \frac{1}{OPT}\right) \leq n\left(1 - \frac{1}{OPT}\right)^t$$

Trust that $1 - x < e^{-x}$ for all $x \neq 0$.

$$n_t \leq n\left(1 - \frac{1}{OPT}\right)^t$$

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Suppose the universe contains n elements.
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$$n_t \leq n_{t-1} - \frac{n_{t-1}}{OPT} = n_{t-1}\left(1 - \frac{1}{OPT}\right) \leq n\left(1 - \frac{1}{OPT}\right)^t$$

Trust that $1 - x < e^{-x}$ for all $x \neq 0$.

$$n_t \leq n\left(1 - \frac{1}{OPT}\right)^t < n\left(e^{-\frac{1}{OPT}}\right)^t$$

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$$n_t \leq n_{t-1} - \frac{n_{t-1}}{OPT} = n_{t-1}\left(1 - \frac{1}{OPT}\right) \leq n\left(1 - \frac{1}{OPT}\right)^t$$

Trust that $1 - x < e^{-x}$ for all $x \neq 0$.

$$n_t \leq n\left(1 - \frac{1}{OPT}\right)^t < n\left(e^{-\frac{1}{OPT}}\right)^t = ne^{-\frac{t}{OPT}}$$

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Suppose the universe contains n elements.

Before the t^{th} iteration, some remaining set has at least $\frac{n_{t-1}}{OPT}$ uncovered elements and the number of elements remaining after the t^{th} iteration is:

$$n_t \leq n_{t-1} - \frac{n_{t-1}}{OPT} = n_{t-1}\left(1 - \frac{1}{OPT}\right) \leq n\left(1 - \frac{1}{OPT}\right)^t$$

Trust that $1 - x < e^{-x}$ for all $x \neq 0$.

$$n_t \leq n\left(1 - \frac{1}{OPT}\right)^t < n\left(e^{-\frac{1}{OPT}}\right)^t = ne^{-\frac{t}{OPT}}$$

What value of t makes $n_t < 1$

Set Cover

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$$\Longrightarrow \boldsymbol{ALG \leq \ln n \, OPT}$$

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Find an instance of 16 elements where the optimal solution is 2, but the algorithm will find a solution of 4.

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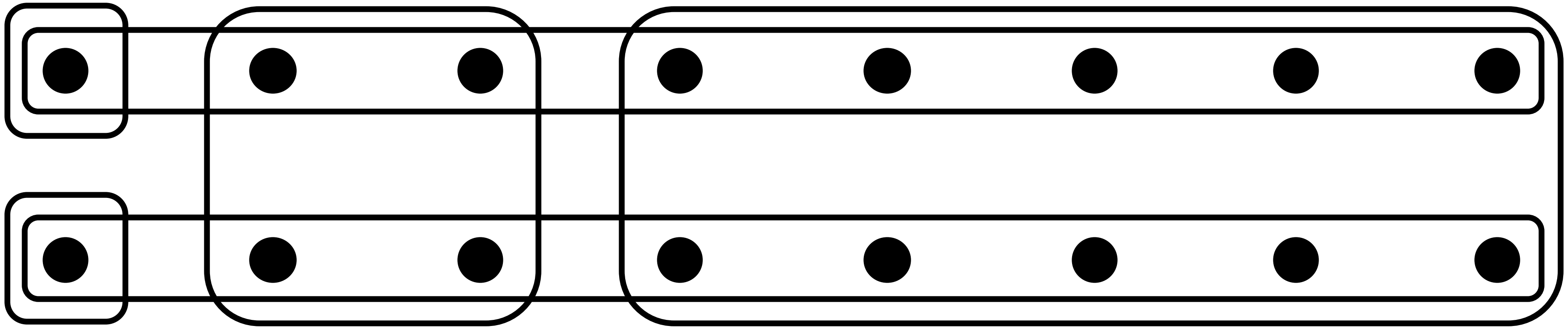

Set Cover

It turns out that Set Cover cannot be approximated within the bound of $\left(1 - o(1)\right) \ln n$, unless P = NP.

Approximability Hierarchy