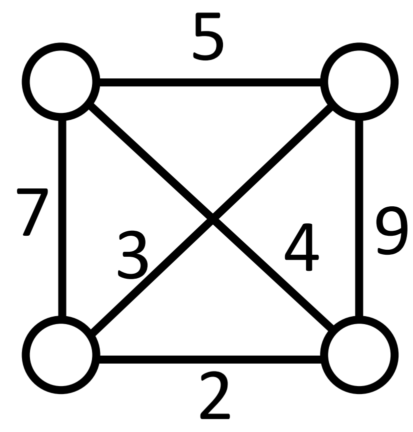
Travelling Salesman Problem
CSCI 432

Travelling Salesman Problem

TSP: Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city once and returns to the origin city?

Example:



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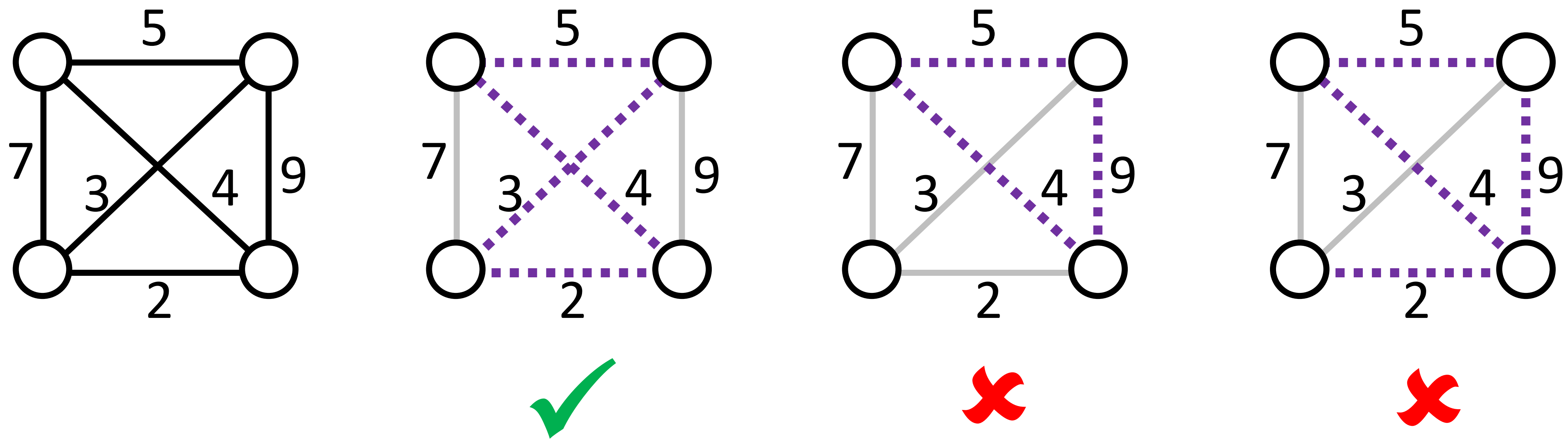

Hamiltonian Cycle Problem

Hamiltonian Cycle: Given a graph, find a cycle that visits each vertex exactly once.

∈ NP-Complete

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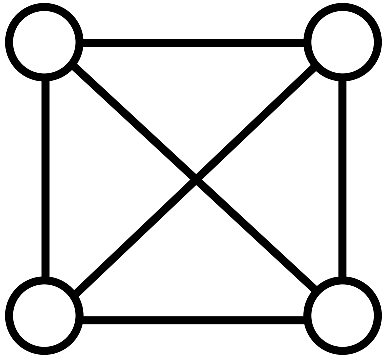

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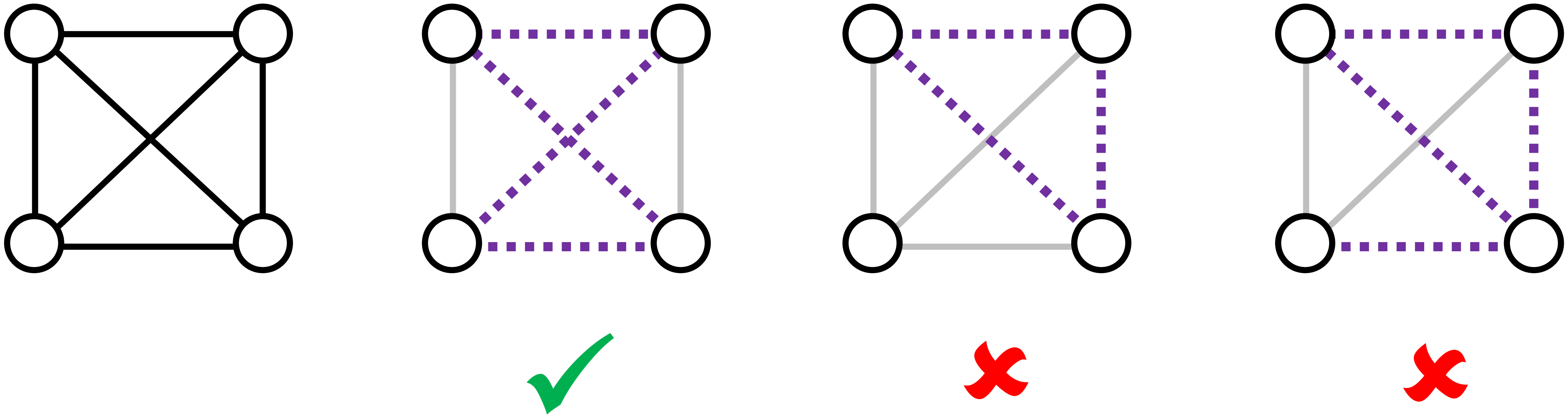

TSP ∈ NP-Complete

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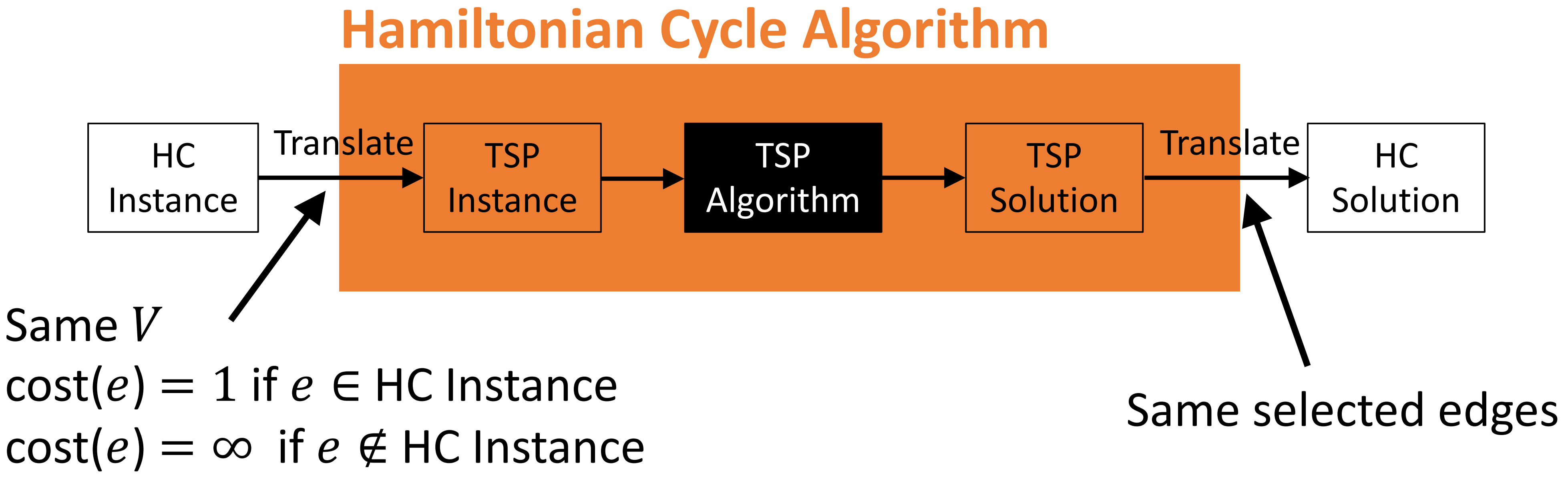

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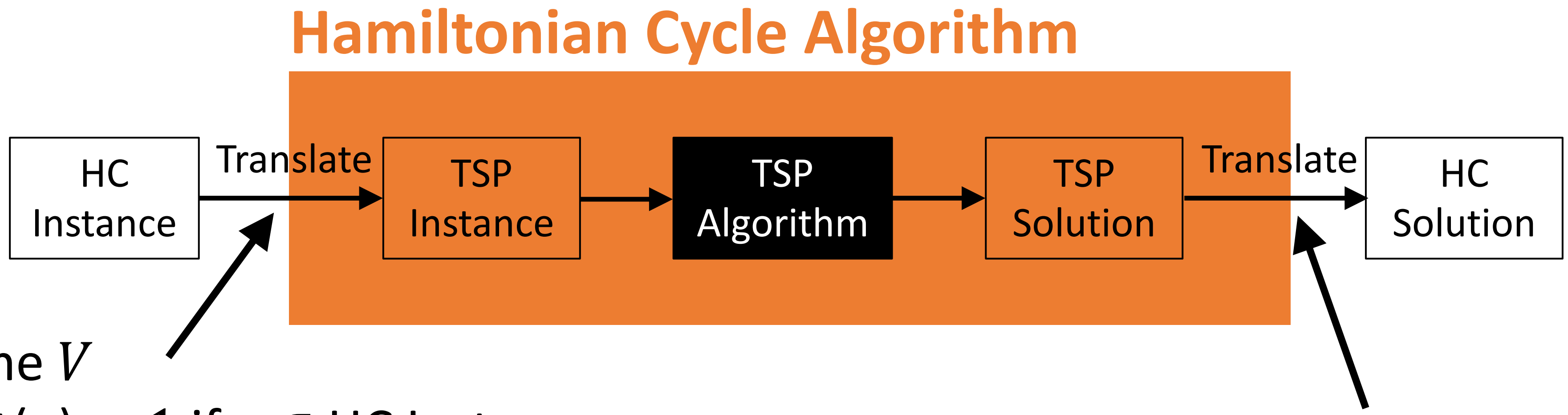

Same V
$\text{cost}(e) = 1$ if $e \in$ HC Instance
$\text{cost}(e) = 1 + \beta, \beta > 1$ if $e \notin$ HC Instance

Same selected edges

TSP Approximation Algorithm

Translation: cost(e) $= 1$ if e ∈ HC
cost(e) $= 1 + \beta, \beta > 1$ if e ∉ HC

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If $G = (V, E)$ has a Hamiltonian Cycle, $\text{OPT}_{\text{TSP}} =$?

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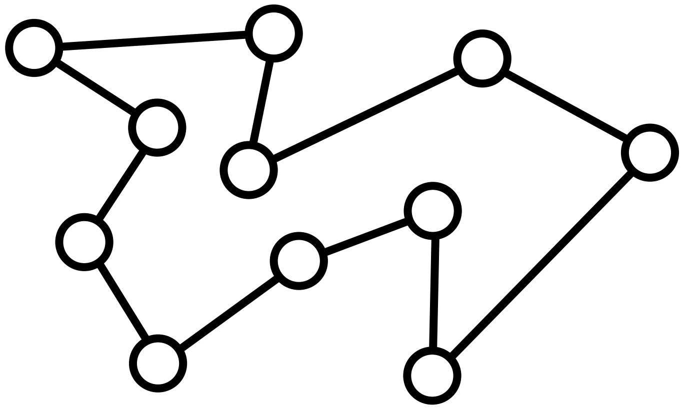

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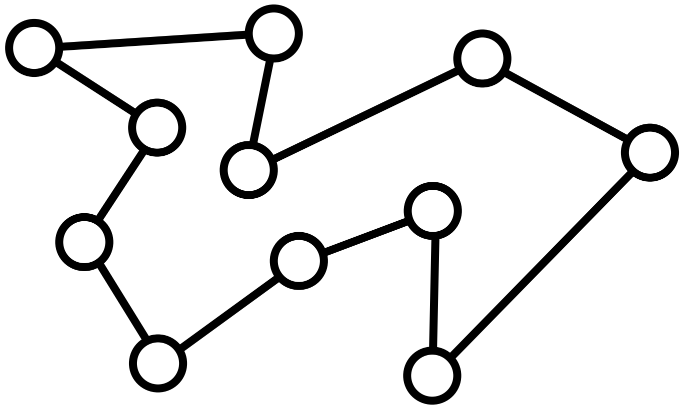

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If $G = (V, E)$ has a Hamiltonian Cycle, $\text{OPT}_{\text{TSP}} = |V|$
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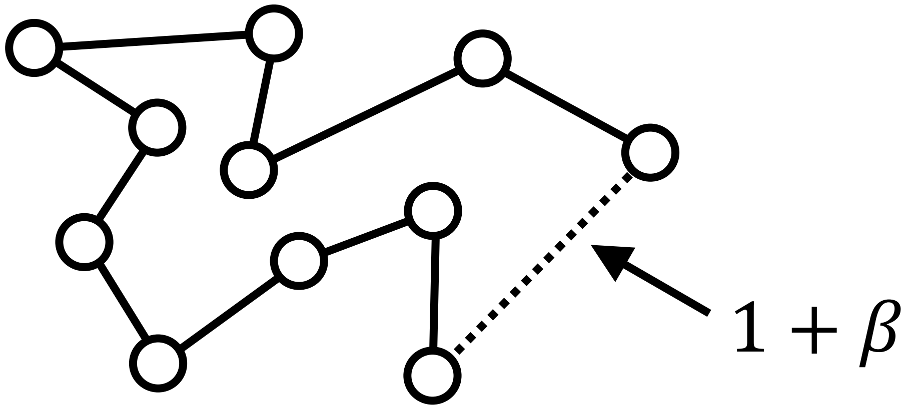

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Let A be an α-approximation algorithm for TSP (i.e. ALG $\leq \alpha$ OPT)
Let $G = (V, E)$ be input to Hamiltonian Cycle.
Let $G', \beta = \alpha\, |V|$ be input to TSP.

TSP Approximation Algorithm

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What happens when A runs on G', β...

If G has a Hamiltonian Cycle?

$\text{ALG}_A \leq$?

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TSP Approximation Algorithm

Let $G', \beta = \alpha\,|V|$ be input to TSP and run A:

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HamiltonianCycleExists(G)
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Yes! Any approximation algorithm for TSP will solve the NP-Complete Hamiltonian Cycle problem!

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$\therefore$?

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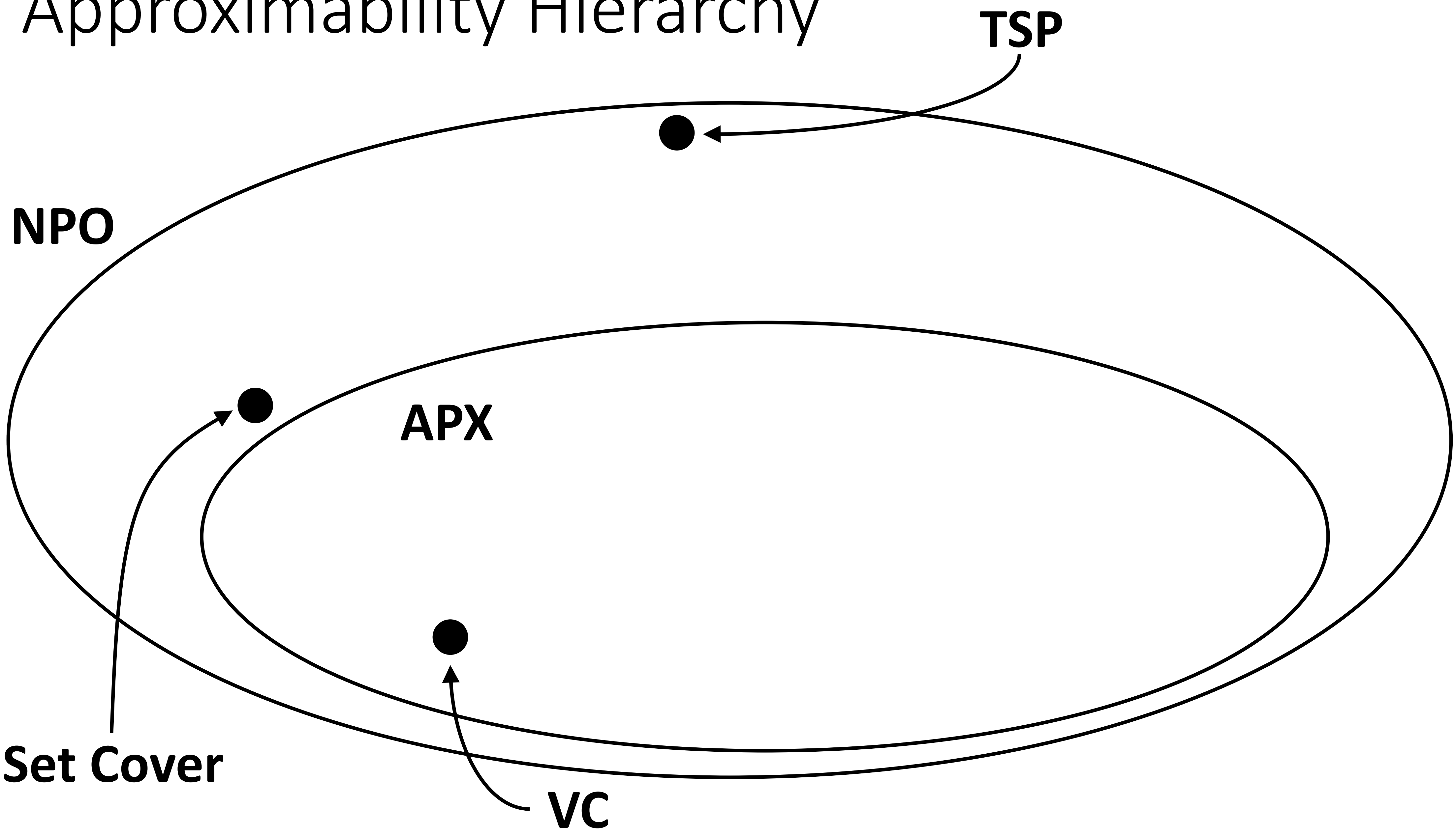
$\therefore \nexists$ poly time approx alg for TSP, unless P = NP

Approximability Hierarchy



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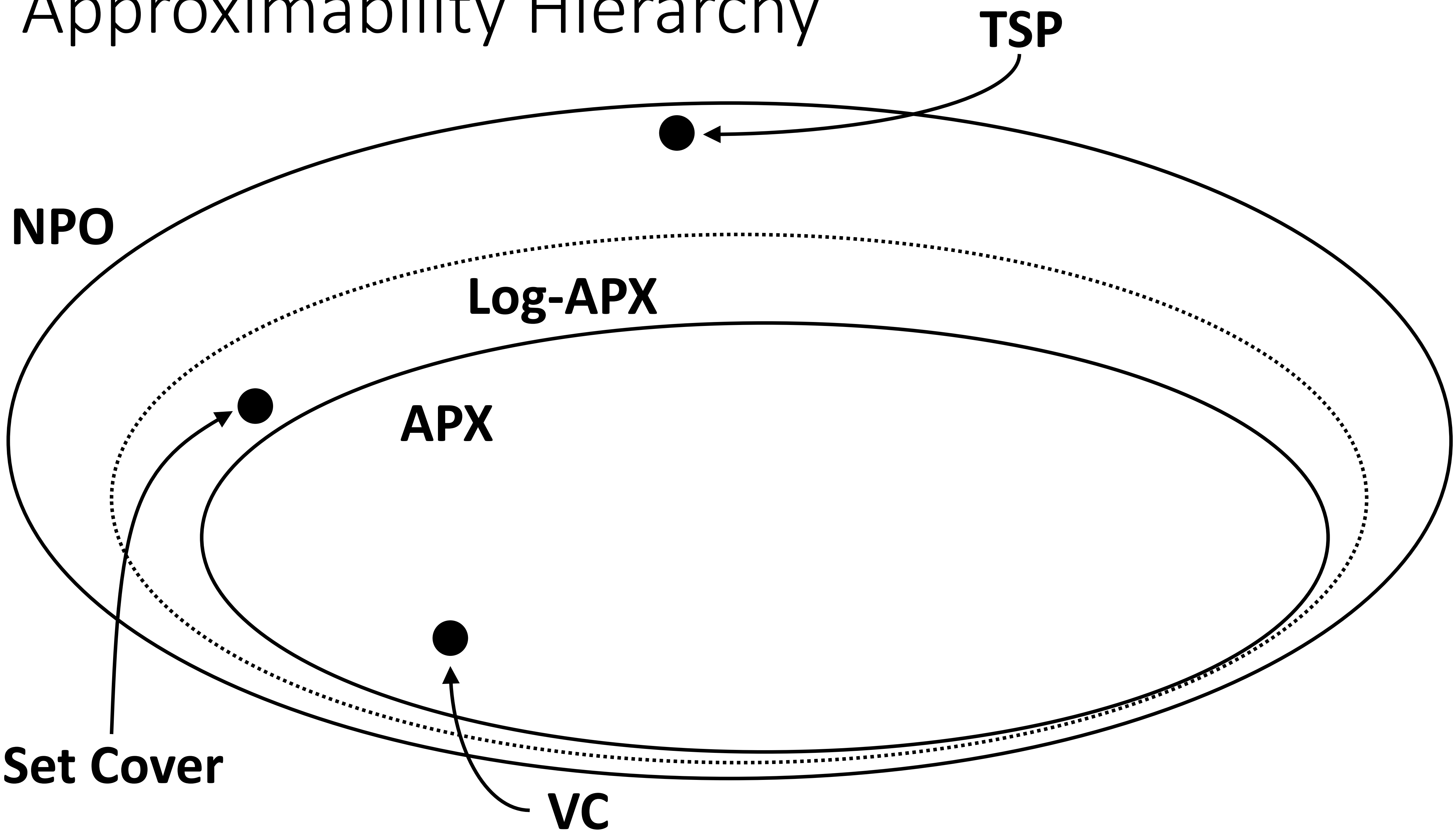

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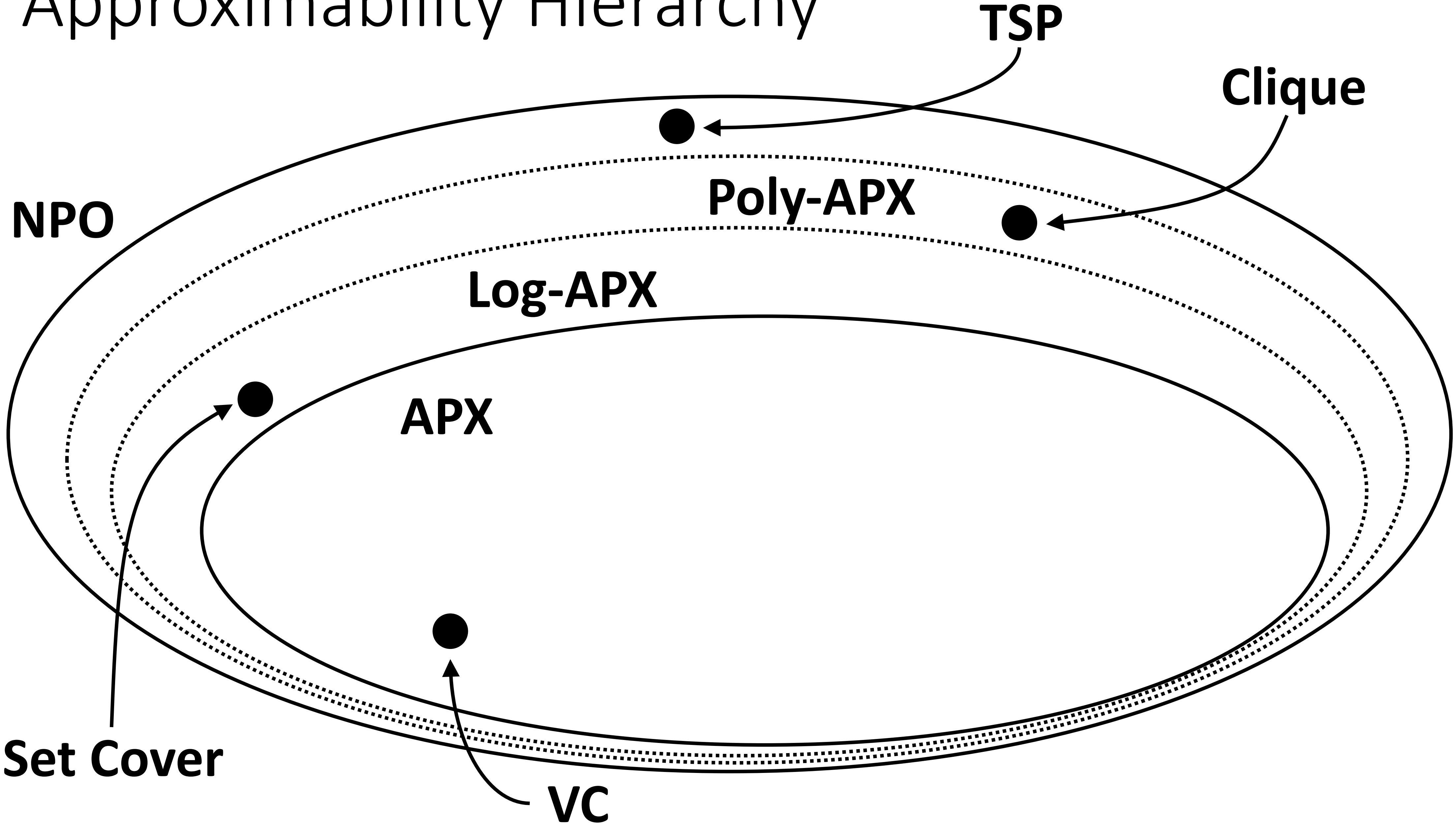

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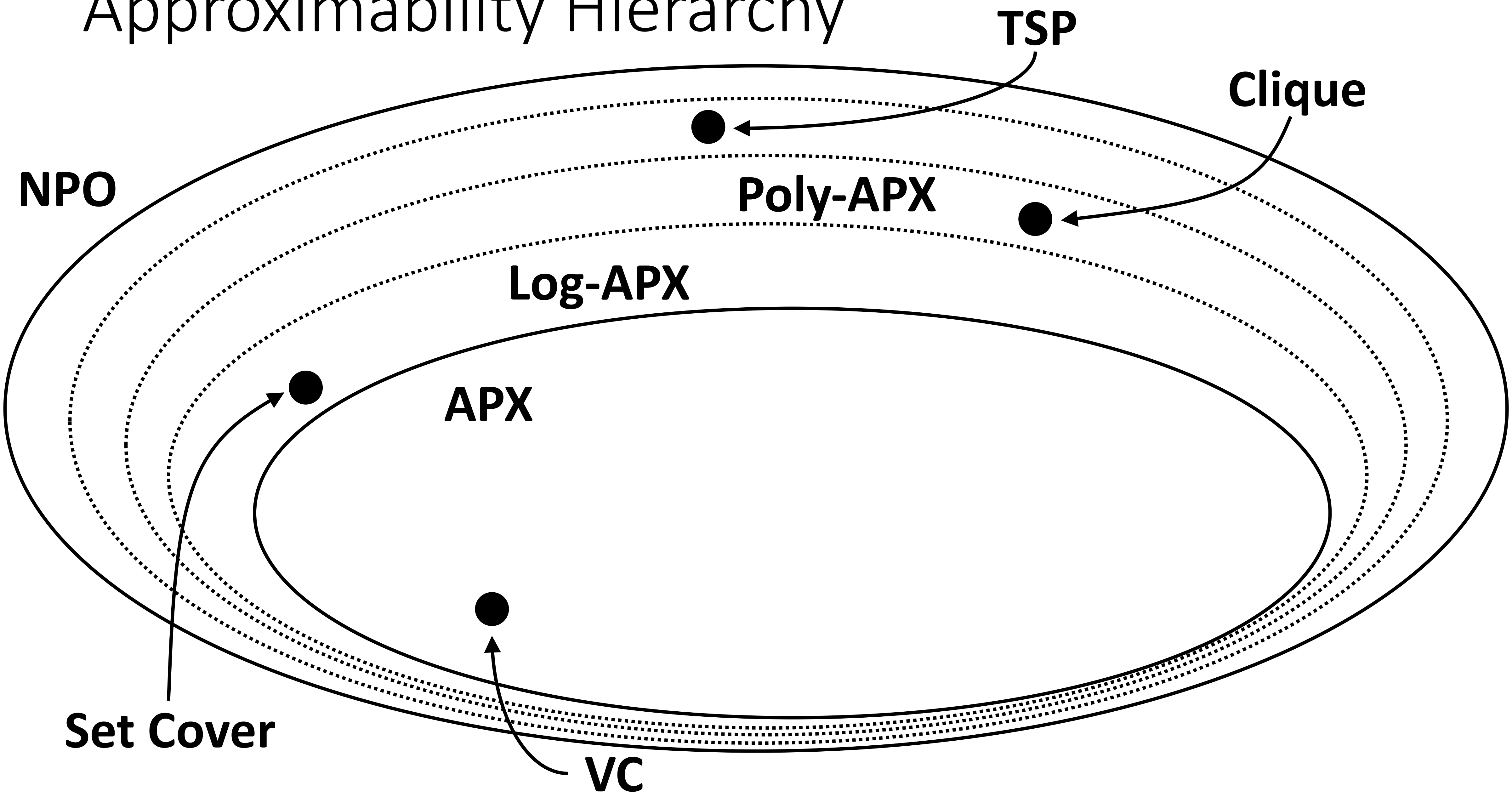

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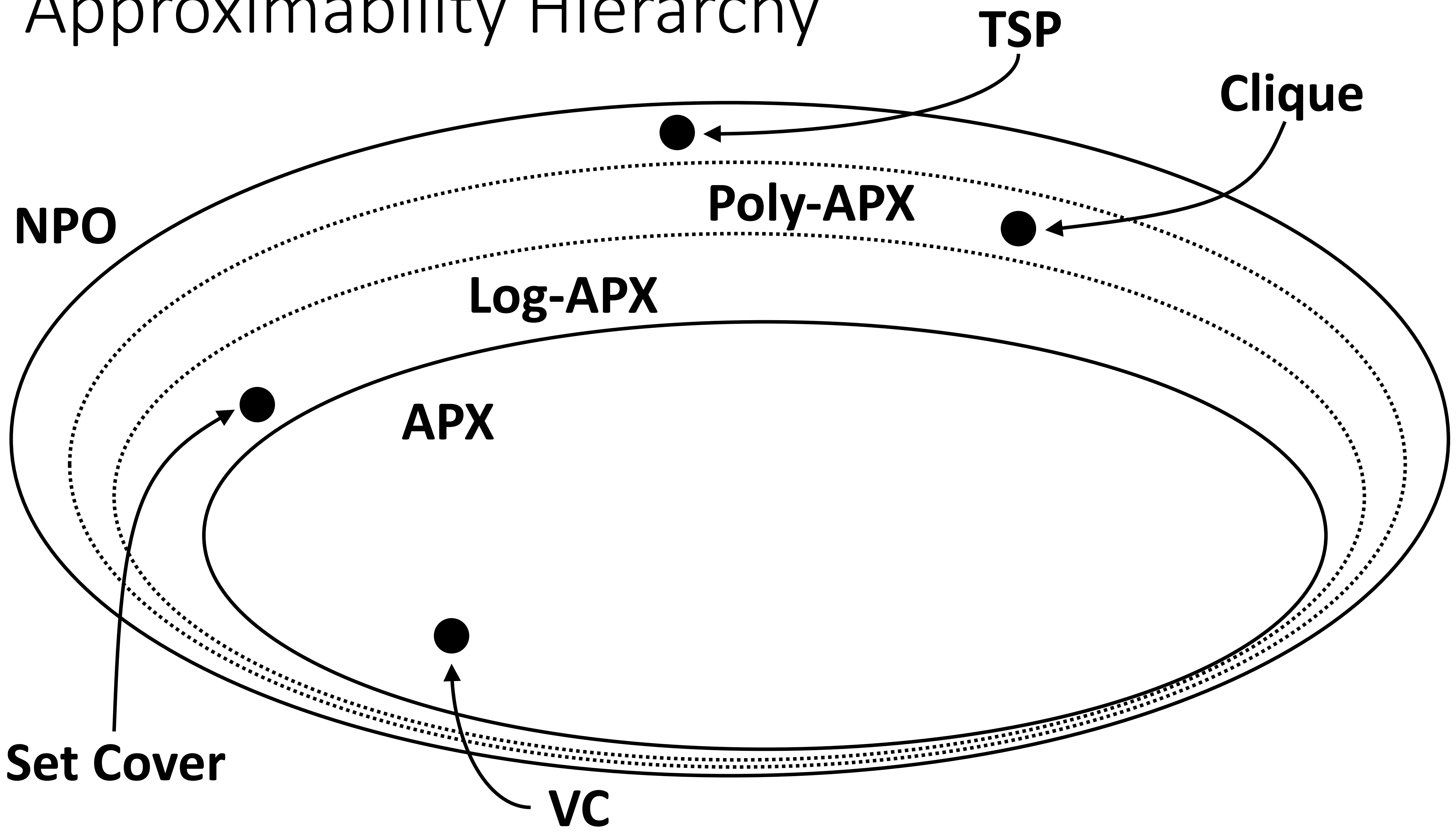


Special Case - Metric TSP

TSP: Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city once and returns to the origin city?

Special Case - Metric TSP

TSP: Given a list of cities and the distances between each pair of cities **(satisfying the triangle inequality)**, what is the shortest possible route that visits each city once and returns to the origin city?

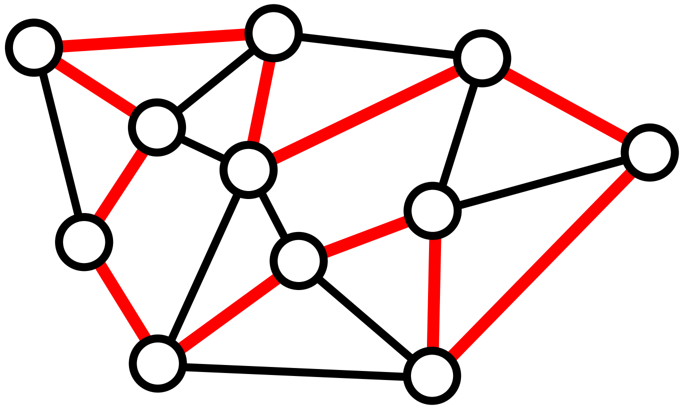
$$\text{dist}(u, v) \leq \text{dist}(u, w) + \text{dist}(w, v)$$

Special Case - Metric TSP

Find some structure that is:

1. Easy to compute.
2. Related to TSP.
3. Lower bound on OPT.

Special Case - Metric TSP



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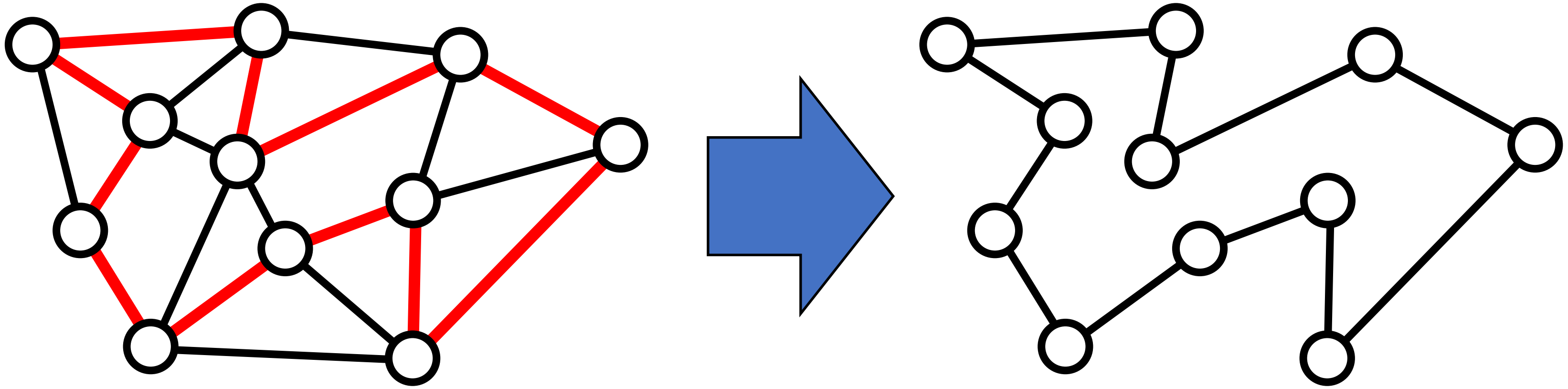

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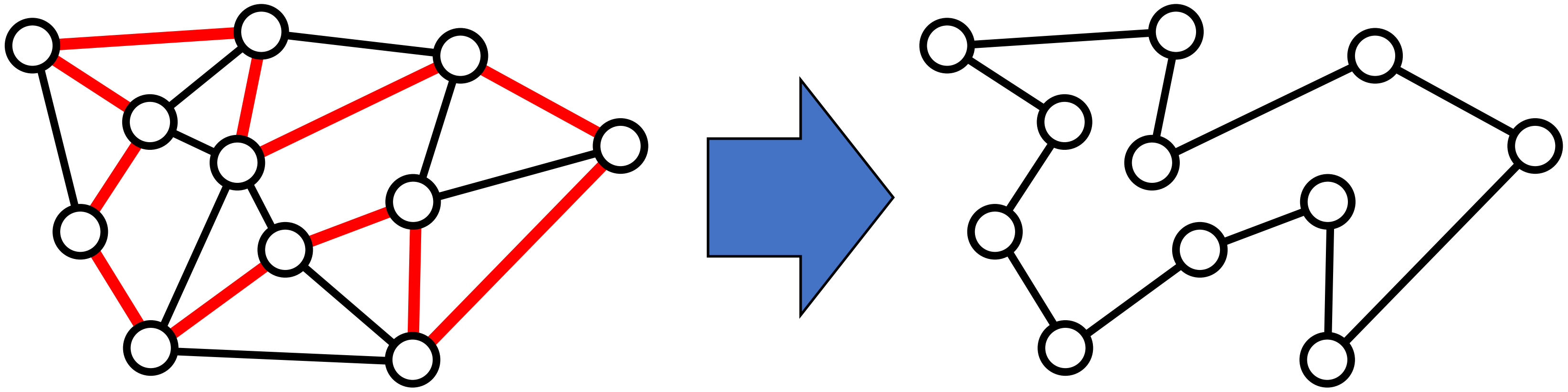

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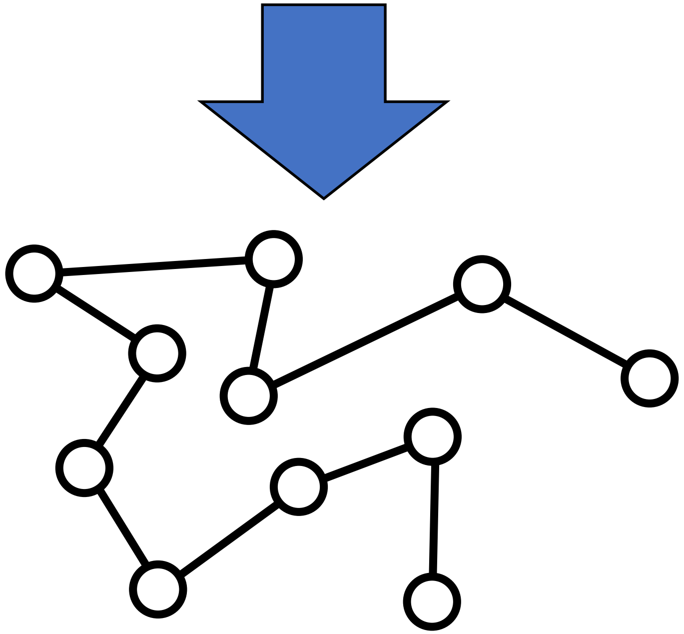

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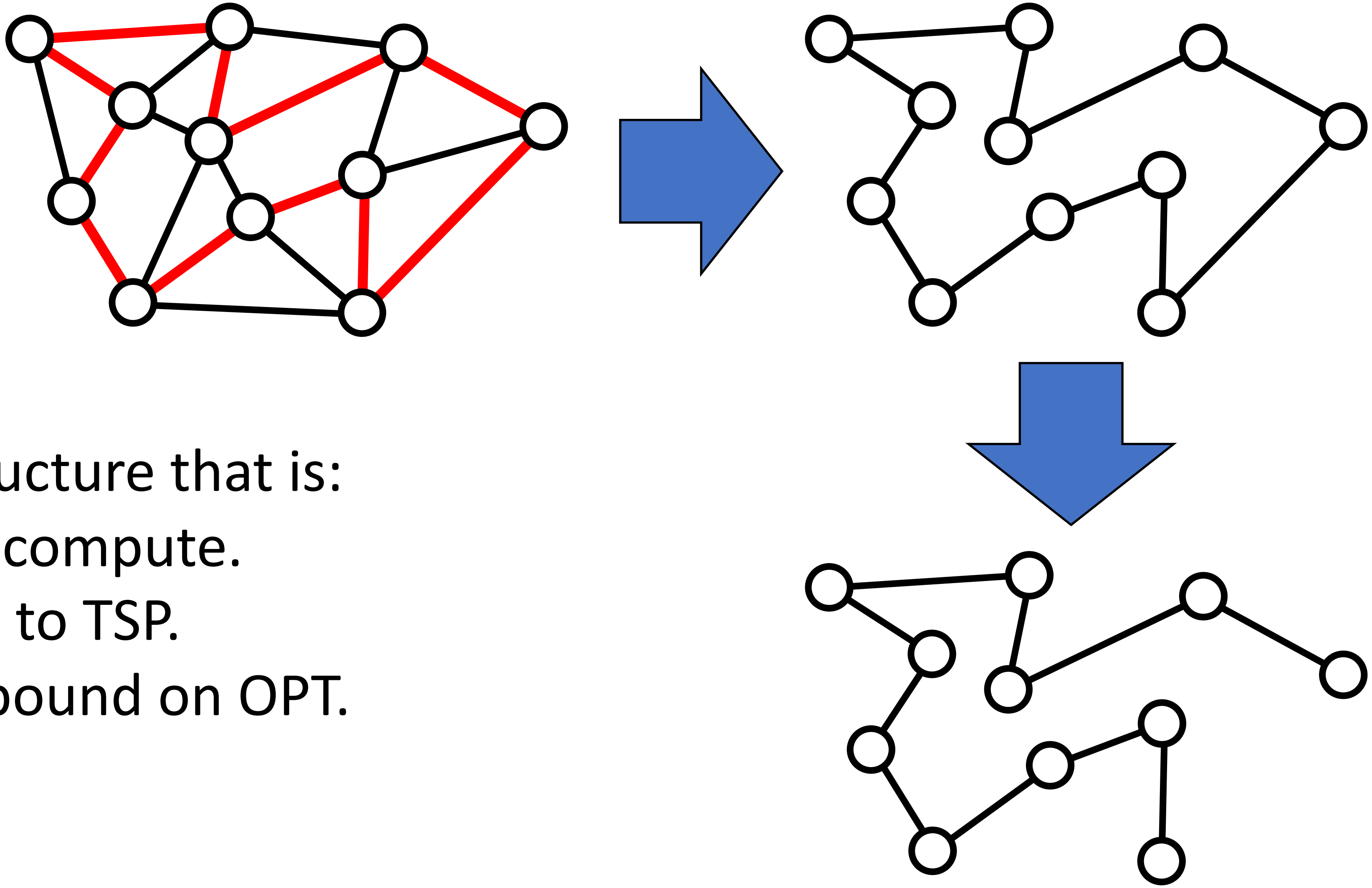

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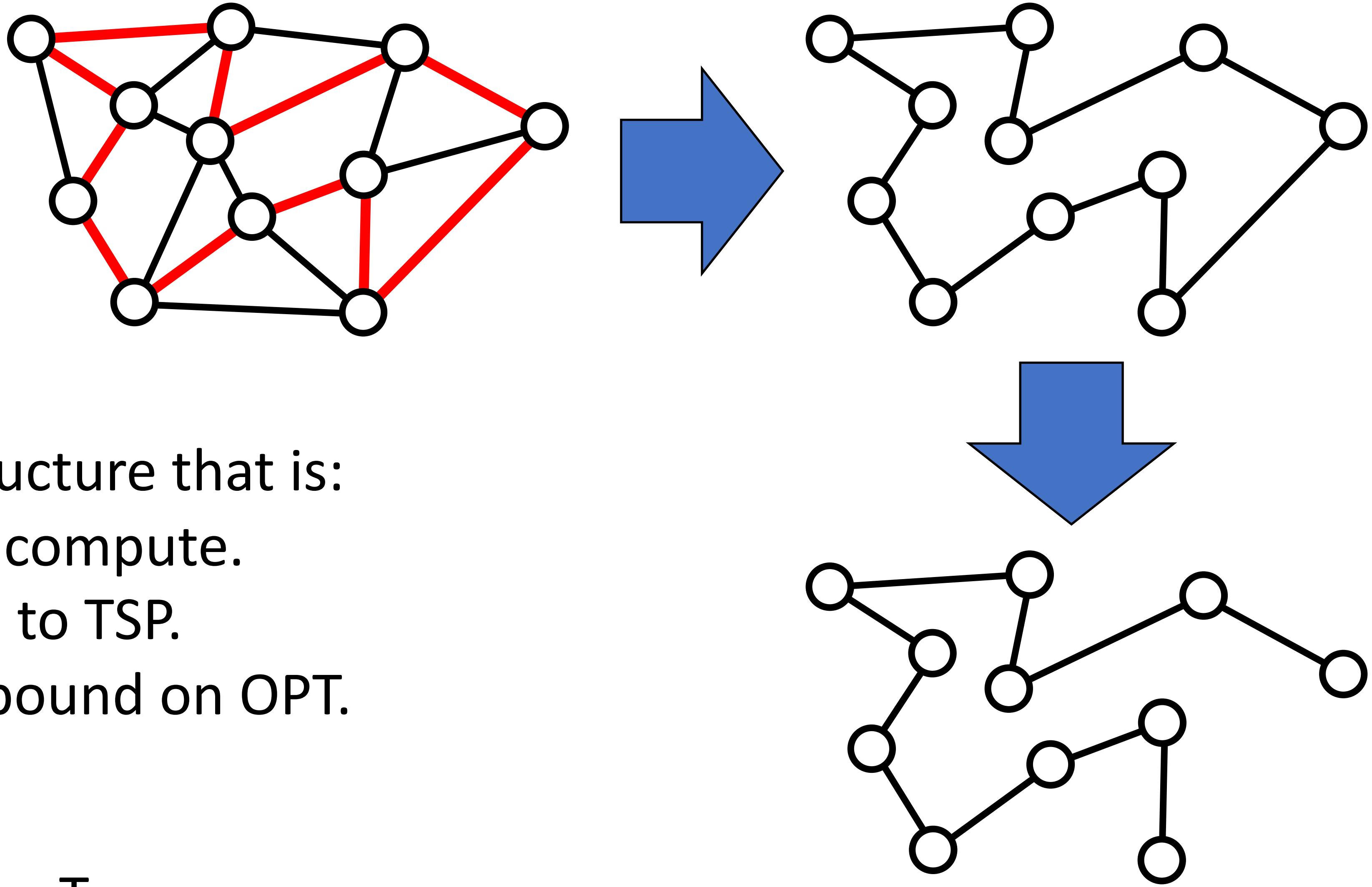
What is this?

Special Case - Metric TSP



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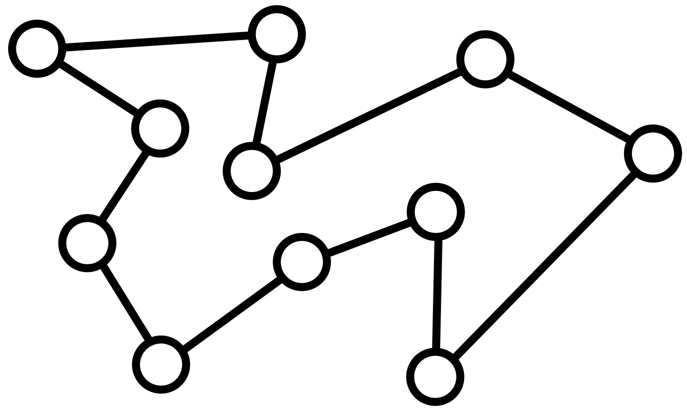
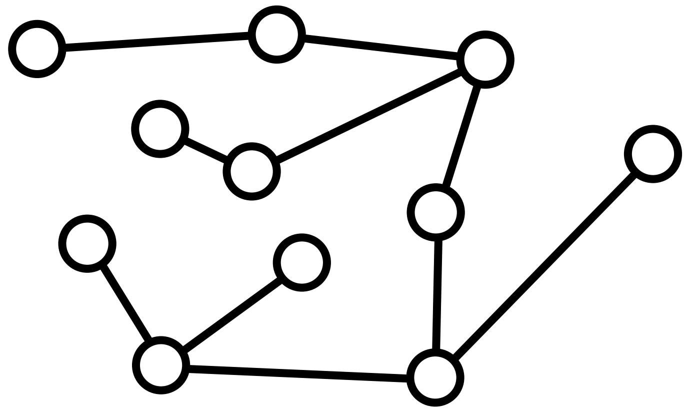
What is this?

Spanning Tree

Special Case - Metric TSP

Relationship between OPT and cost of MST?

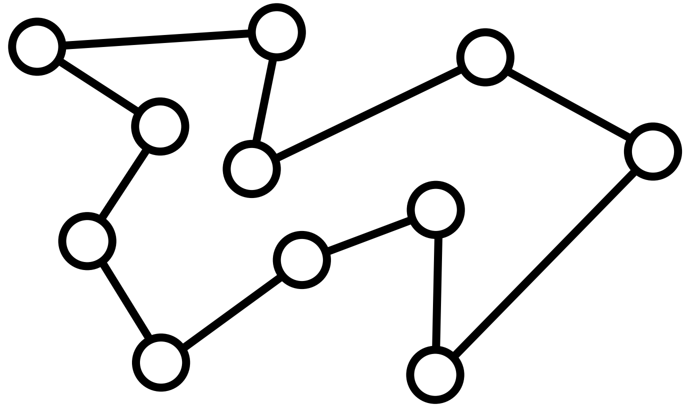


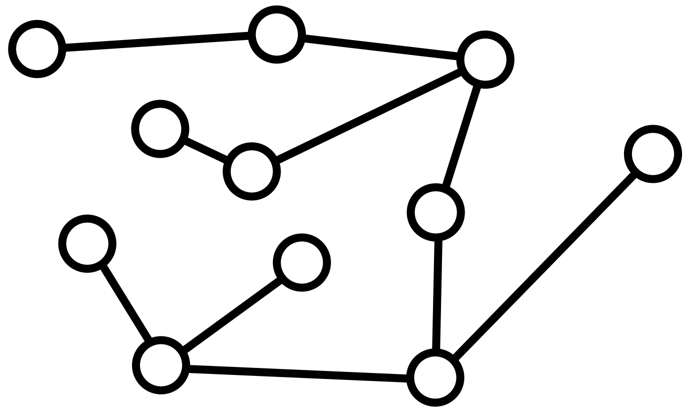


Special Case - Metric TSP

Relationship between OPT and cost of MST?

$\text{OPT} \geq \text{cost(MST)}$



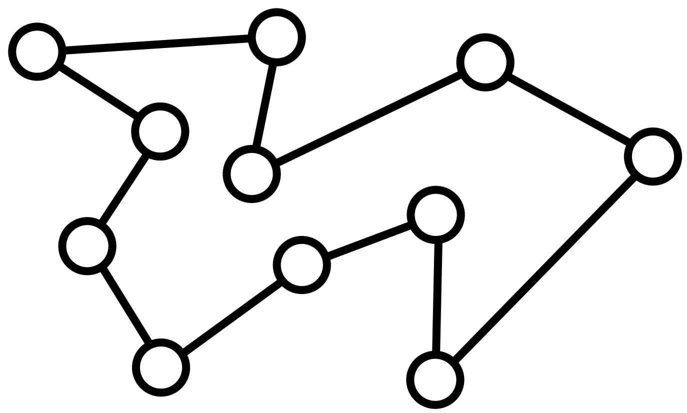


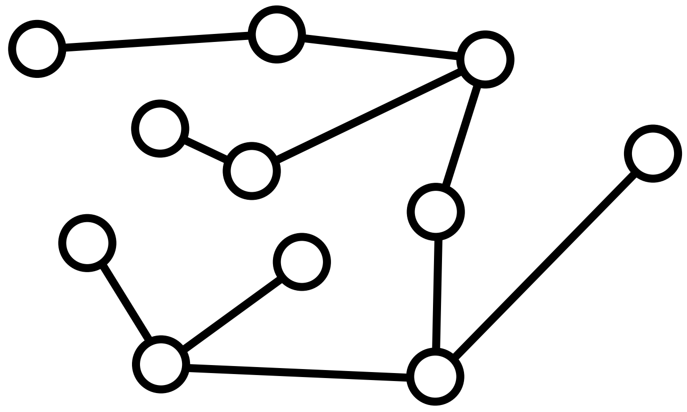
Special Case - Metric TSP

Relationship between OPT and cost of MST?

$OPT \geq cost(MST)$

How to turn MST into tour of cities?





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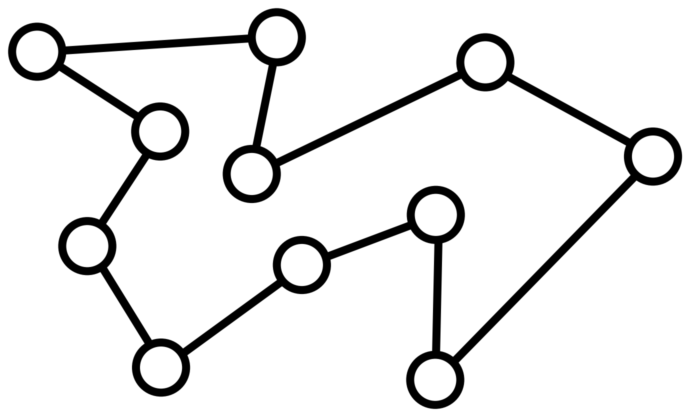
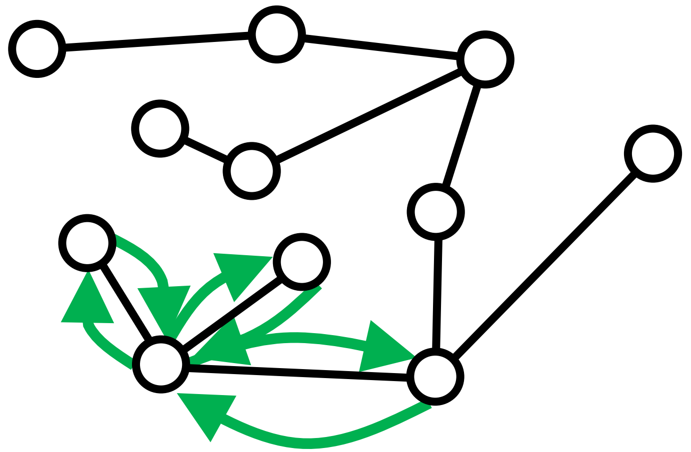




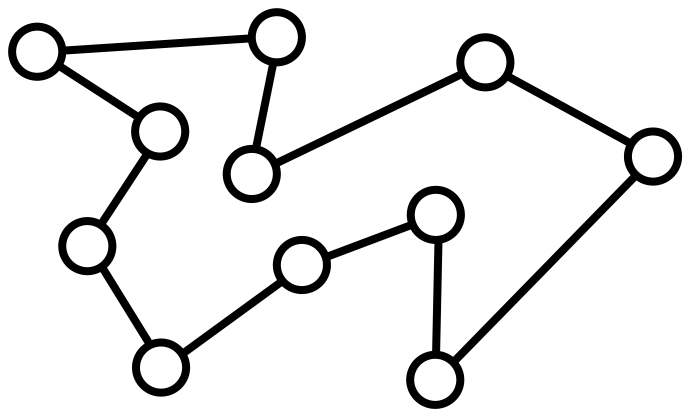
Special Case - Metric TSP

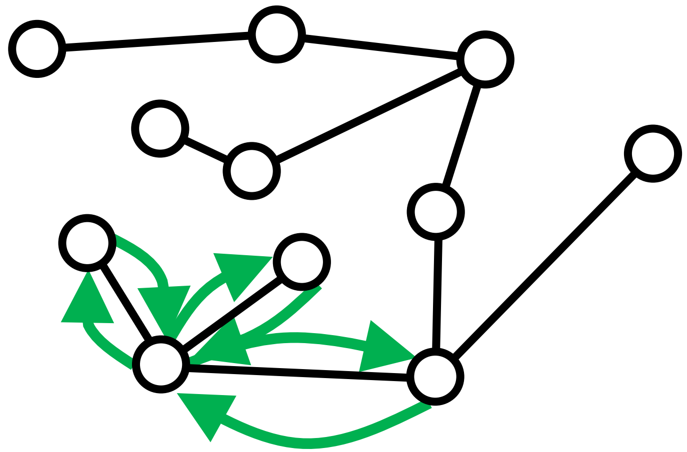
Relationship between OPT and cost of MST?

$OPT \geq cost(MST)$

How to turn MST into tour of cities?

What is the cost of this tour?





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Relationship between OPT and cost of MST?

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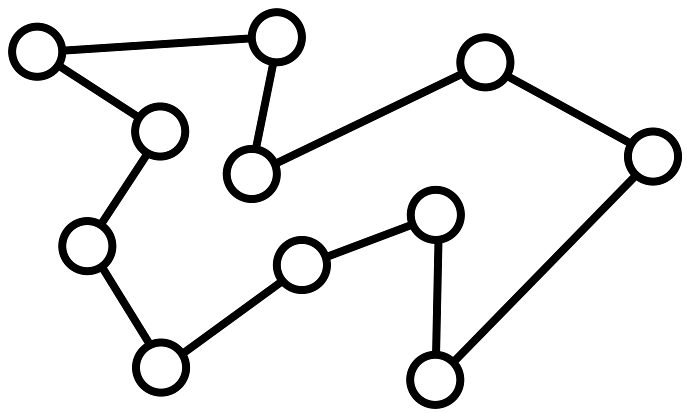
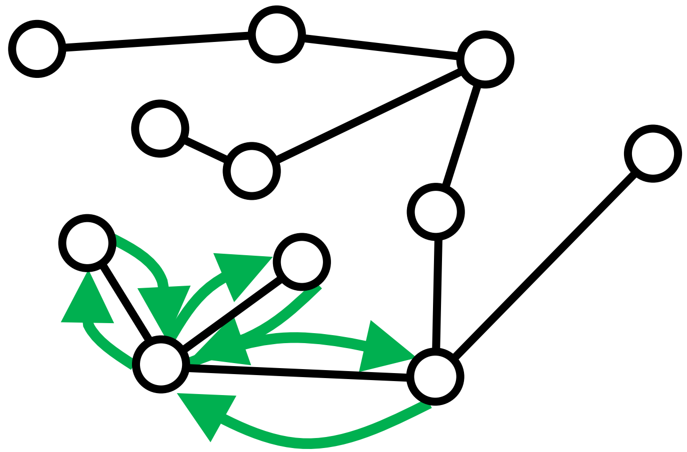


How to turn MST into tour of cities?

What is the cost of this tour?

$$\text{ALG} = 2\ \text{cost(MST)}$$



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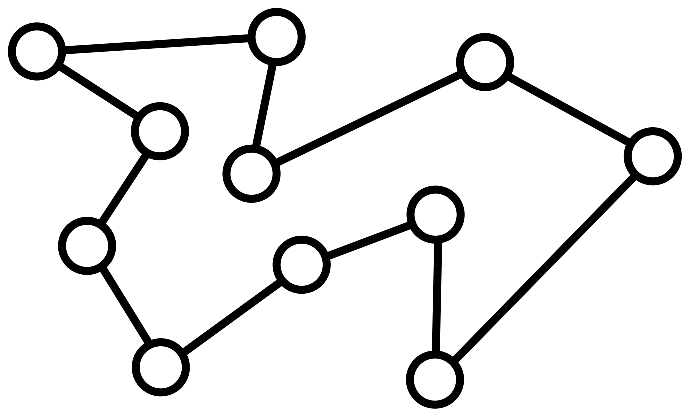
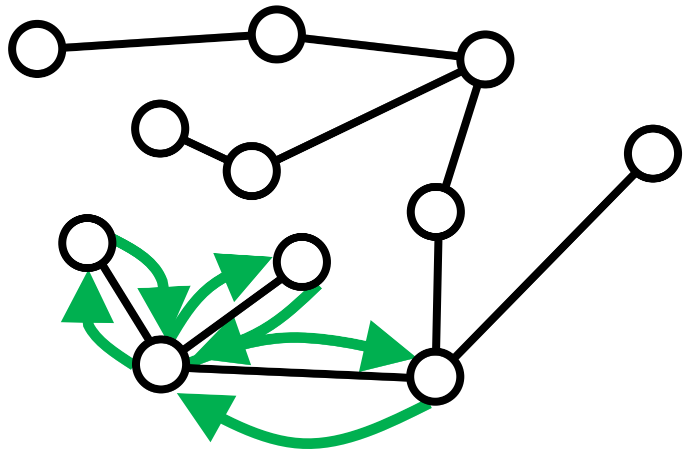
Relationship between ALG and OPT?





Special Case - Metric TSP

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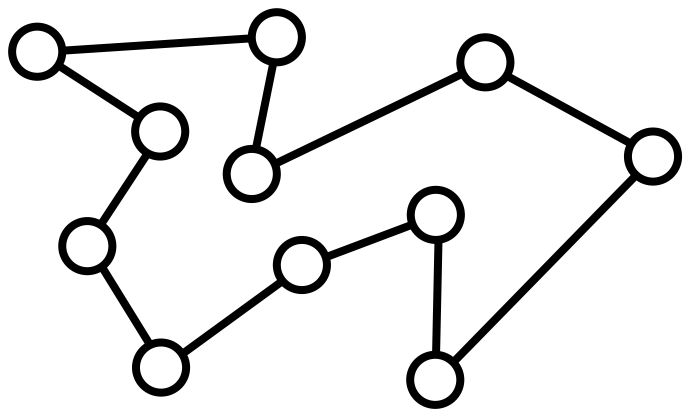

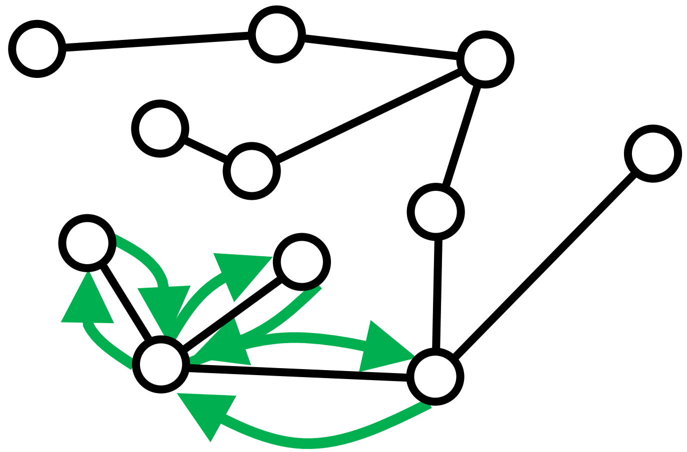

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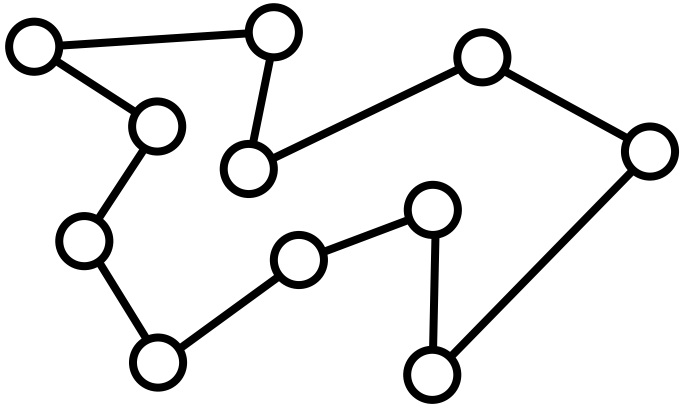

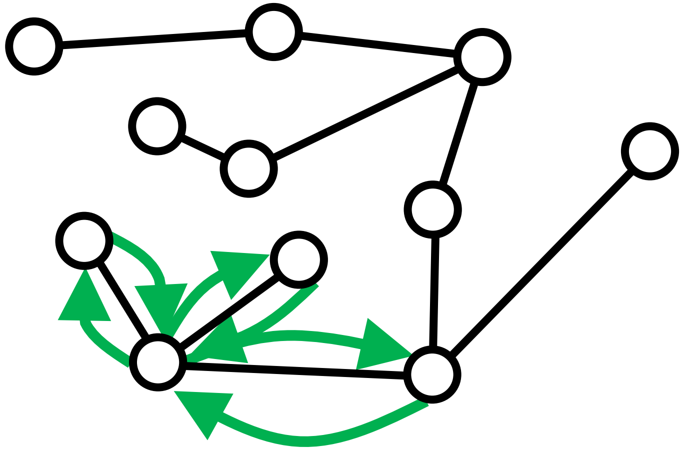

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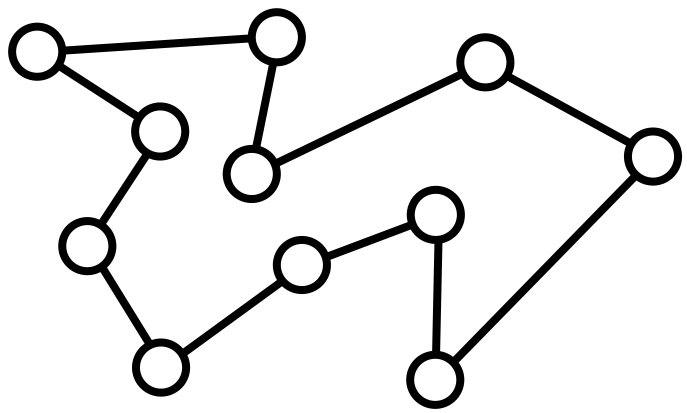

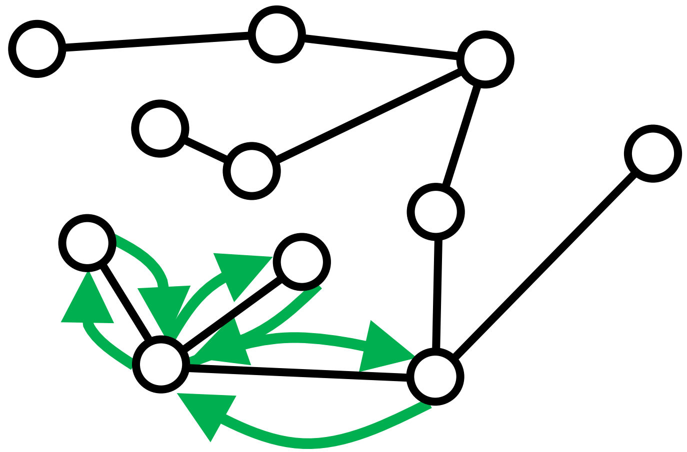

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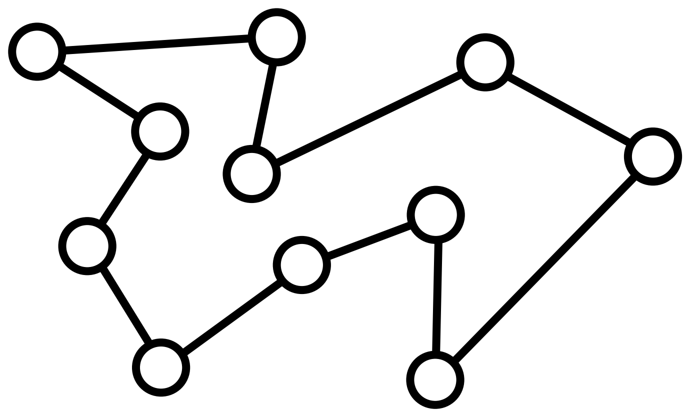

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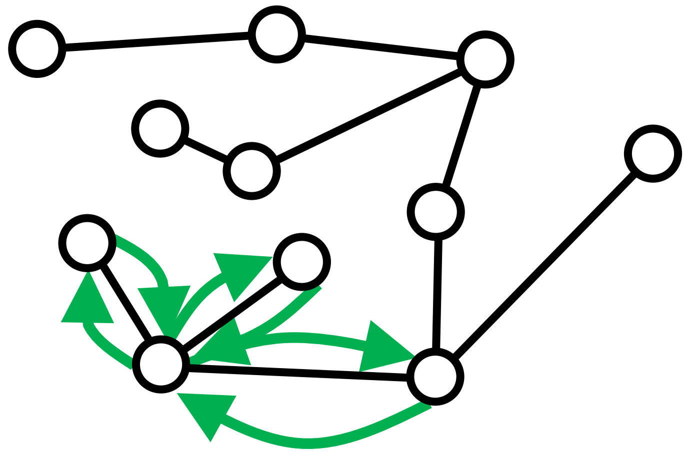
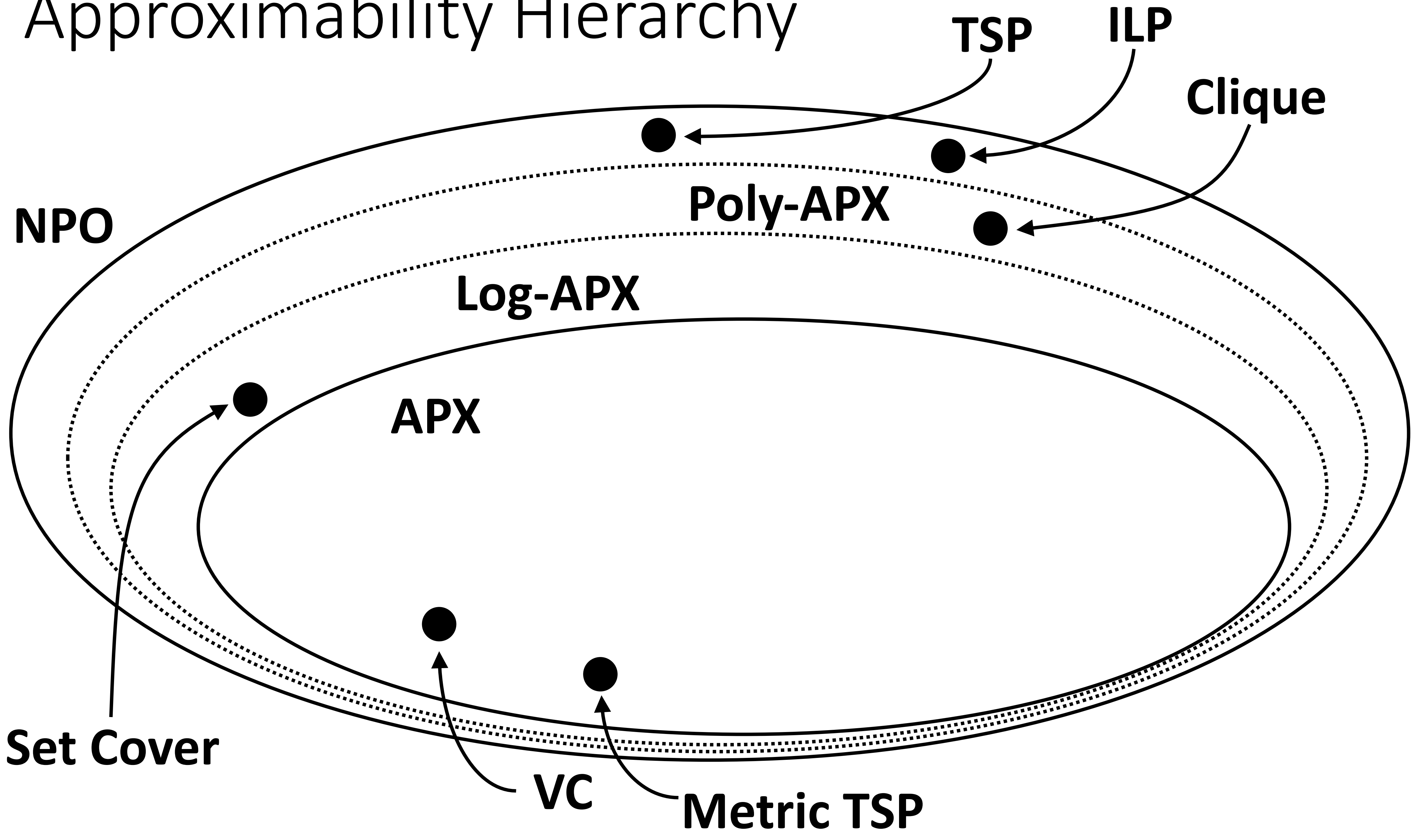


How can we eliminate double visits (without messing up the cost)?

Skip to next unvisited vertex. Can only decrease cost (triangle inequality).

$\text{dist}(u, v) \leq \text{dist}(u, w) + \text{dist}(w, v)$

Approximability Hierarchy



TSP ILP

Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city once and returns to the origin city?

If we can build an ILP for TSP, then approximating general ILPs would approximate TSP, which would optimally solve Hamiltonian Cycle…

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So, if we can build an ILP for TSP, solving general ILPs is also inapproximable.

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Somehow we need to make sure all these deployed edges actually form a cycle.

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Doesn't matter where we start.
Every city is on the route.

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Violated by any cycle that does not return to city 1.

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