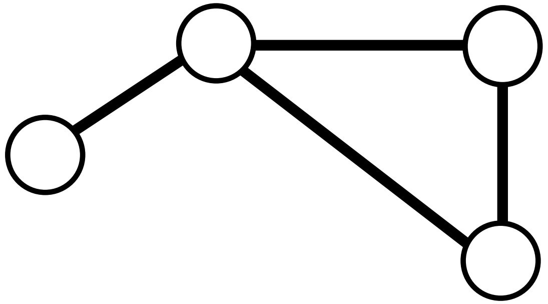
Minimum Spanning Trees

CSCI 432

Minimum Spanning Tree (MST)



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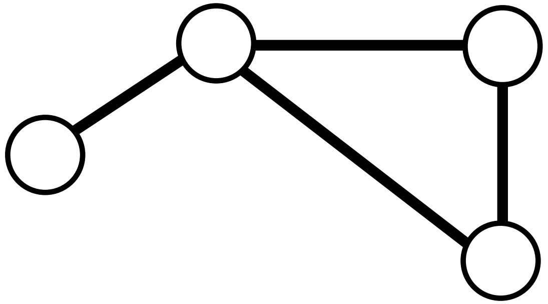

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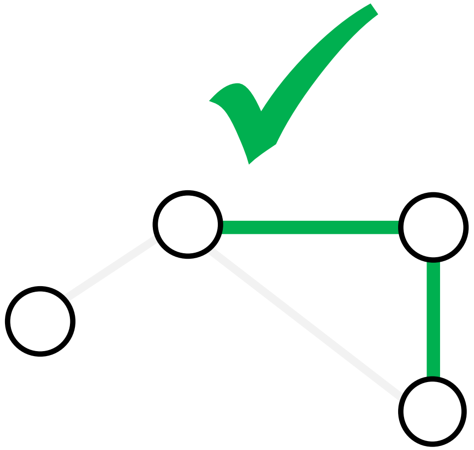
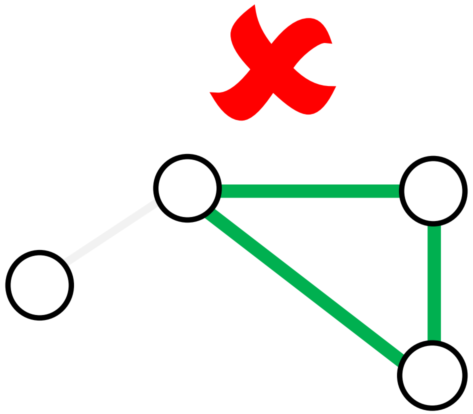
Tree if it is connected and acyclic.





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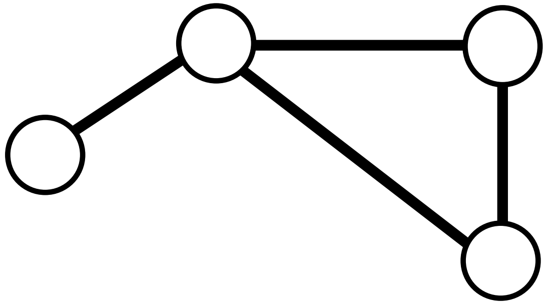

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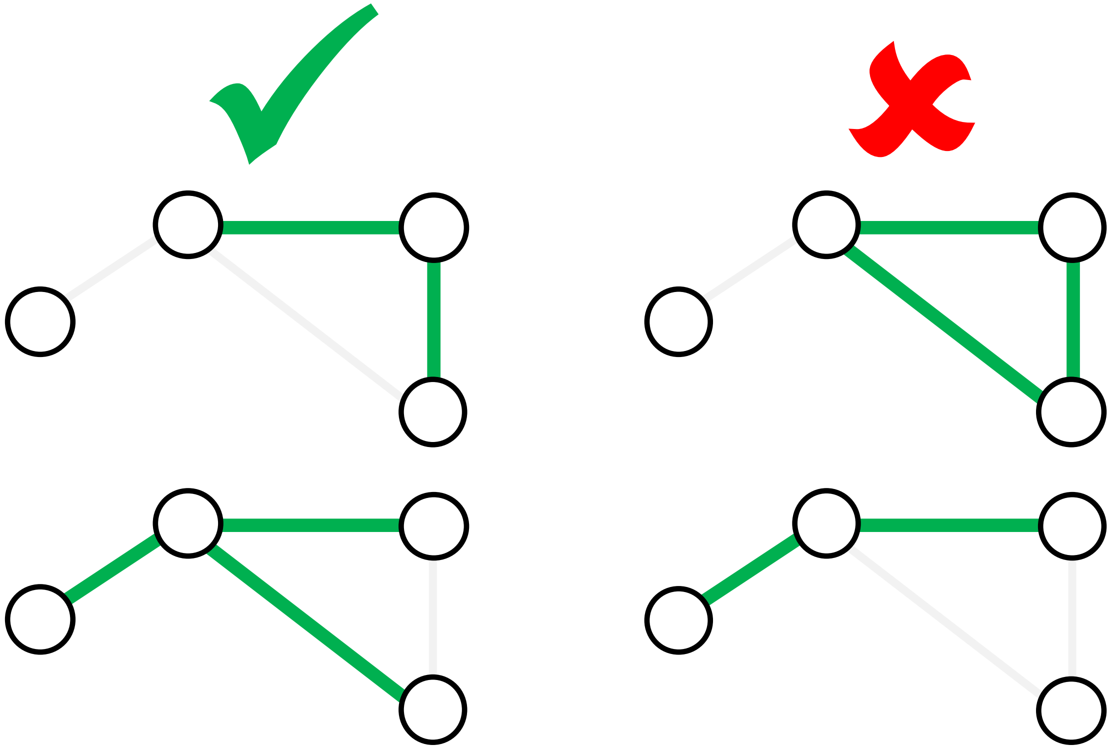

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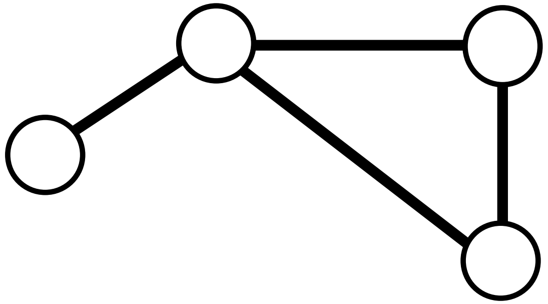

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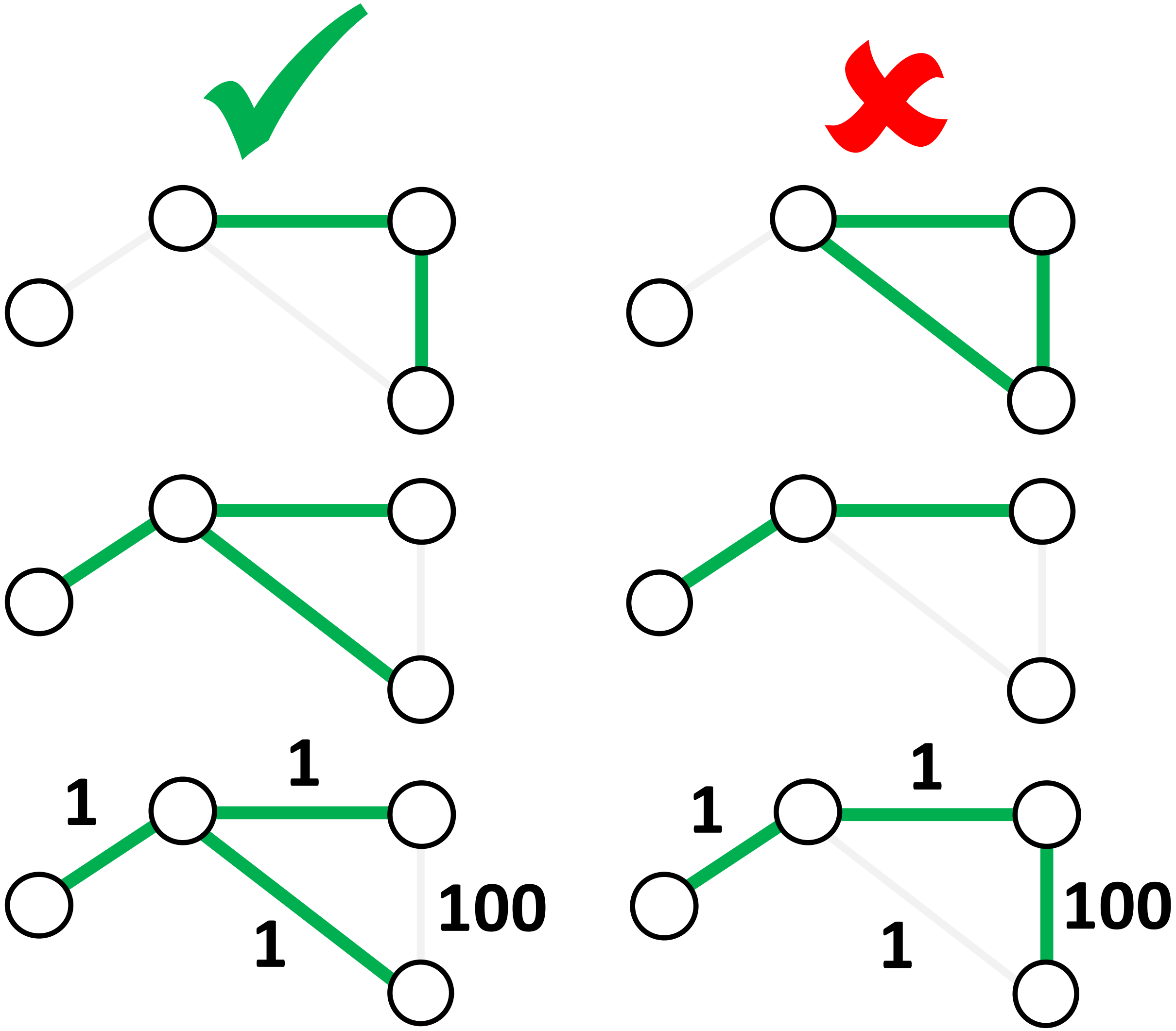
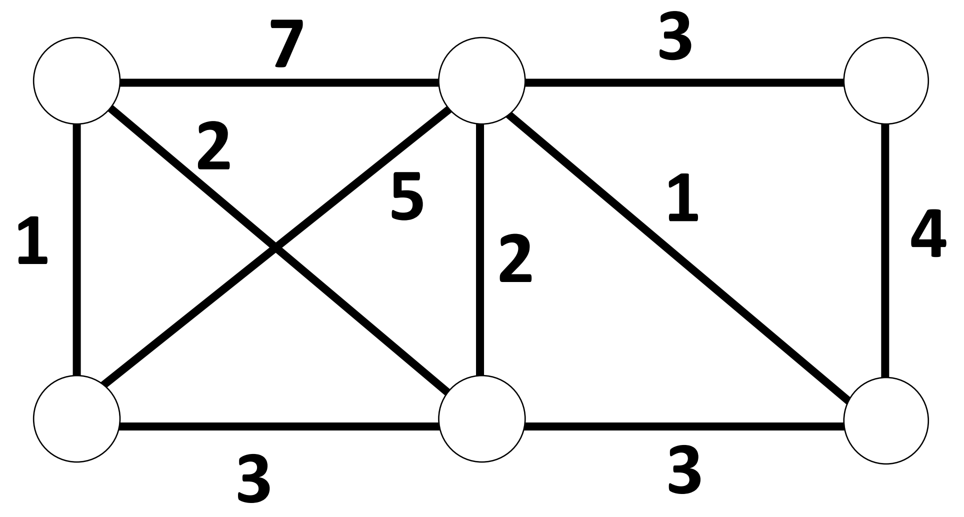


Tree if it is connected and acyclic.

Spanning tree if it is a tree and includes all vertices in the graph.

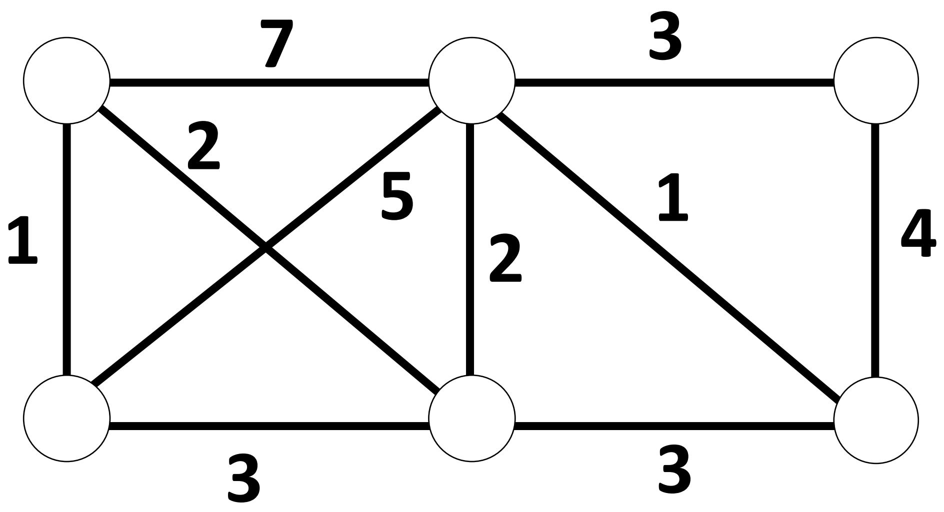
Minimum spanning tree if it is a spanning tree whose sum of edge costs is the minimum possible value.

MST Problem



<u>Goal:</u> Given a connected, edge weighted graph, find its Minimum Spanning Tree.

MST Problem

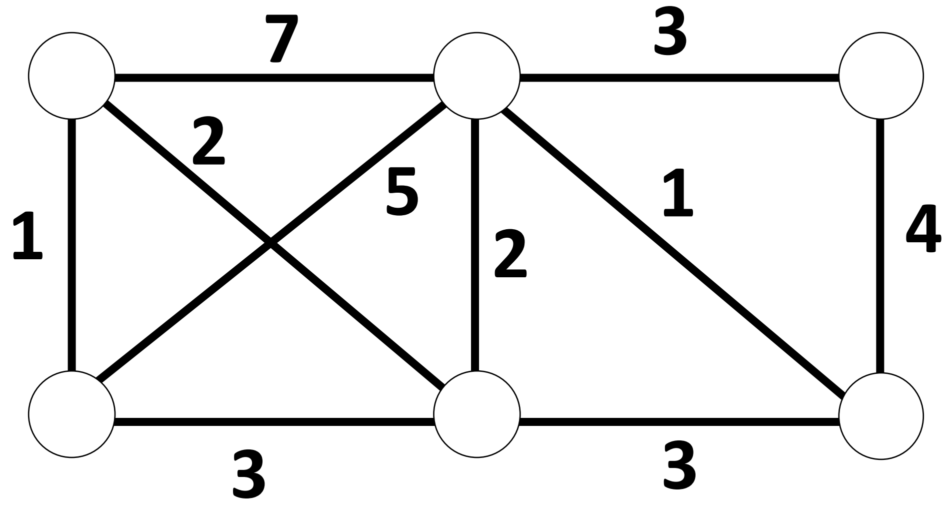


Greedy Algorithms:

- Make the choice that best helps some objective.
- Do not look ahead, plan, or revisit past decisions.
- Hope that optimal local choices lead to optimal global solutions.

Kruskal’s MST Algorithm

Algorithm: Add the edge with smallest weight, that does not create a cycle.



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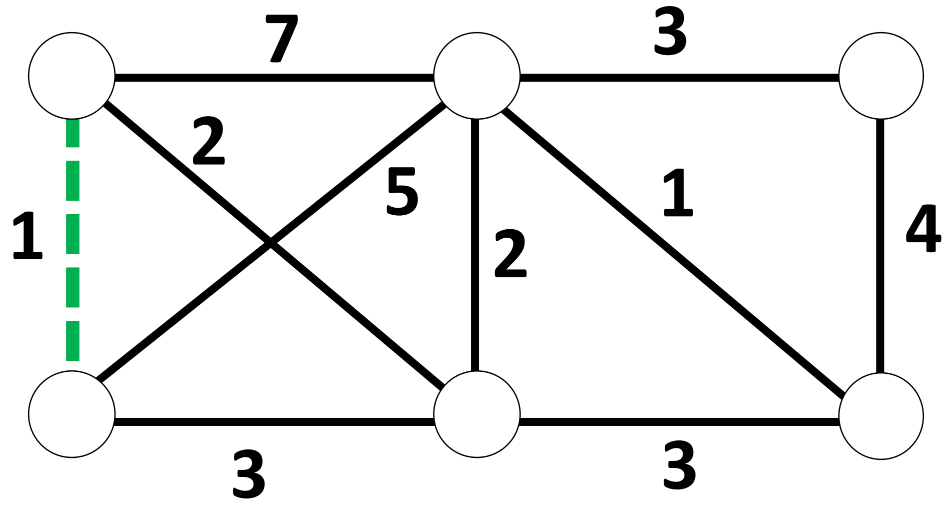

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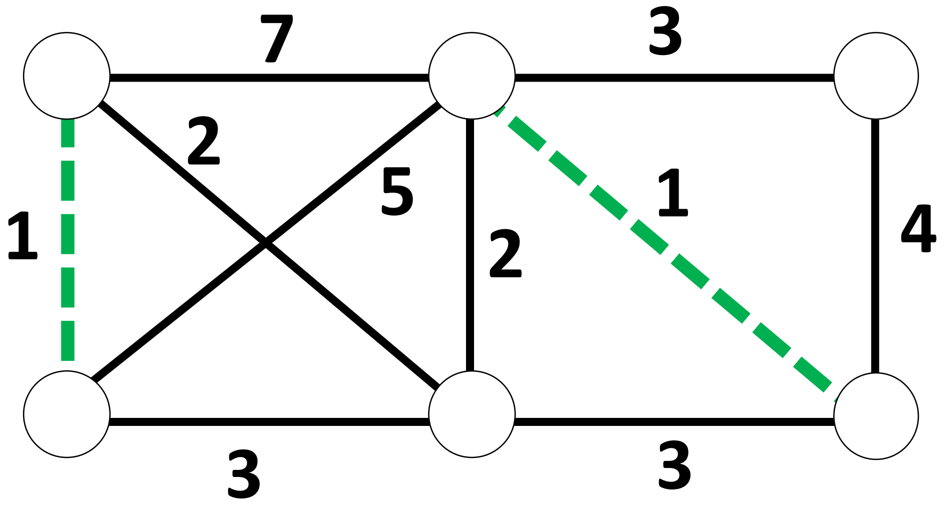

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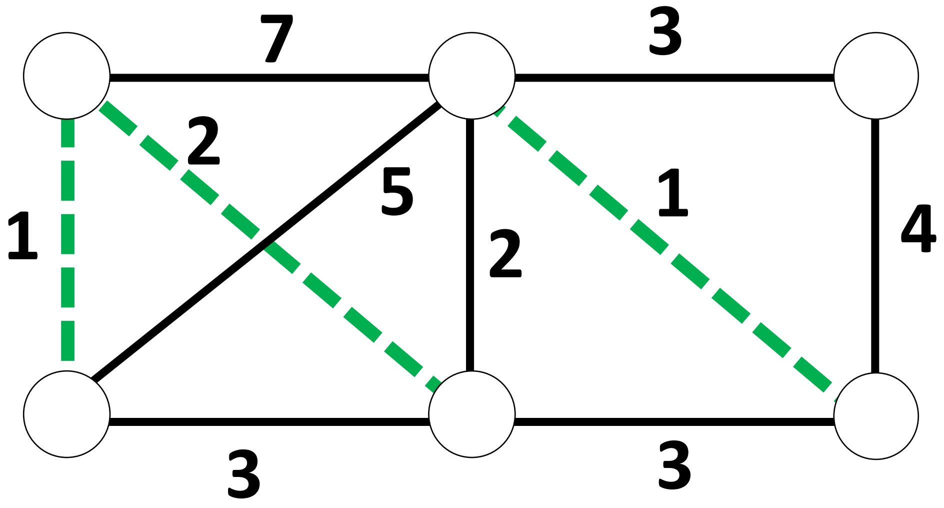

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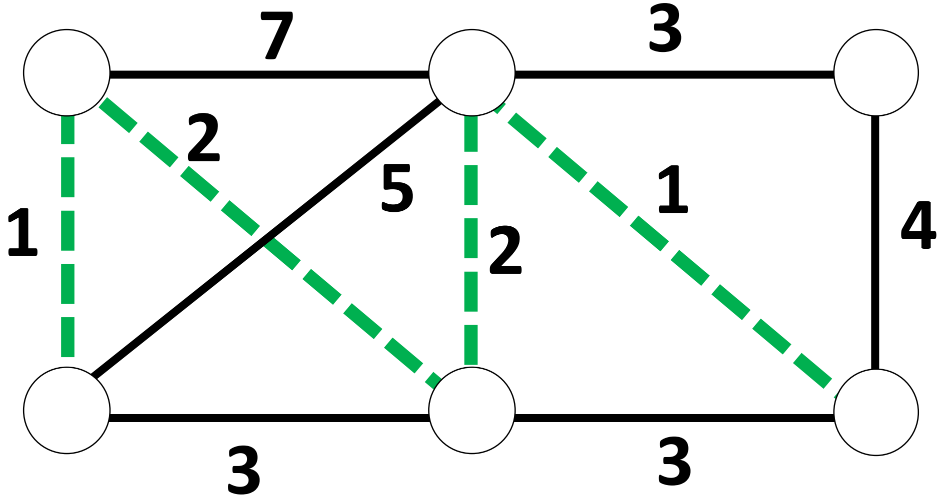

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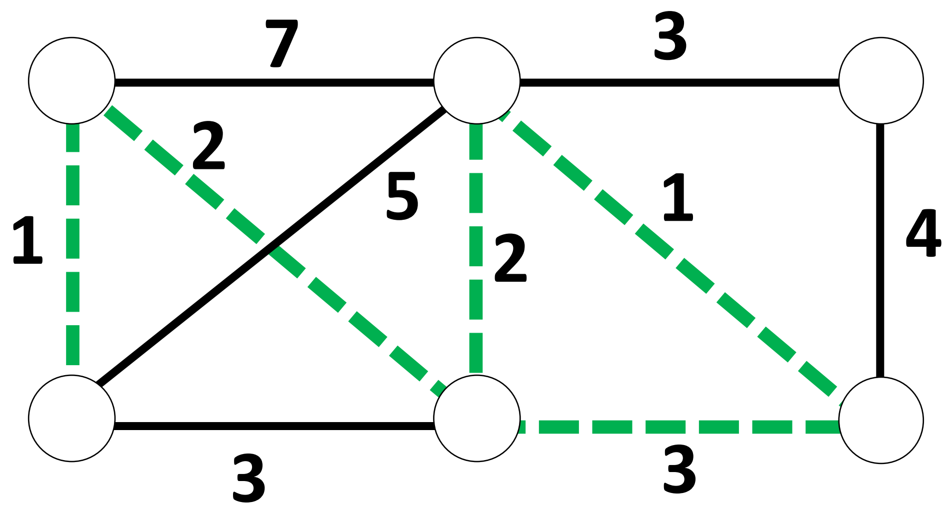

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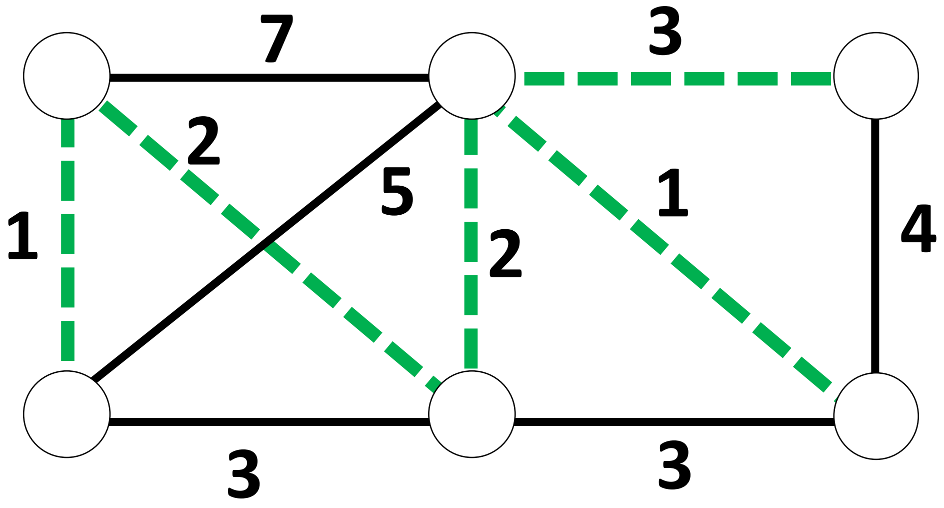

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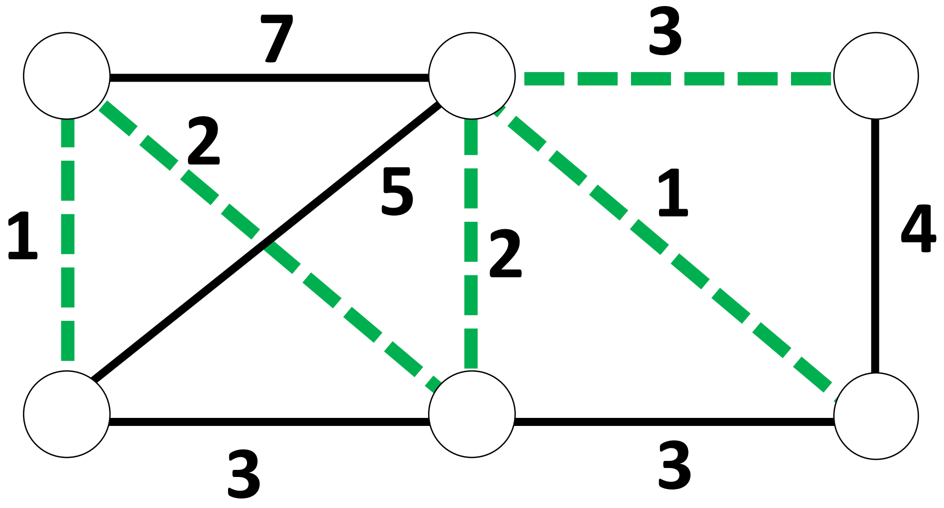

Basic algorithm analysis questions:

1. Is the solution valid? (Does it actually find a spanning tree?)
2. Is the solution optimal? (Does it find the best spanning tree?)
3. What is the running time?

Kruskal’s MST Algorithm

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Proof of validity: ?

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2. T spans G.

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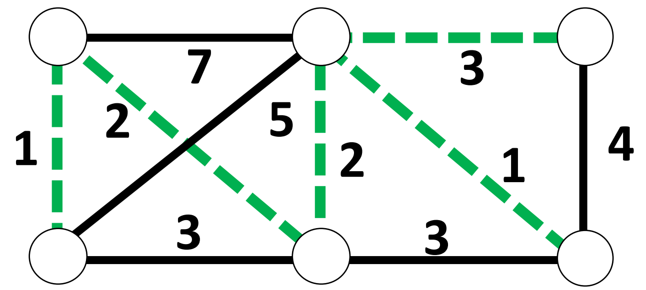

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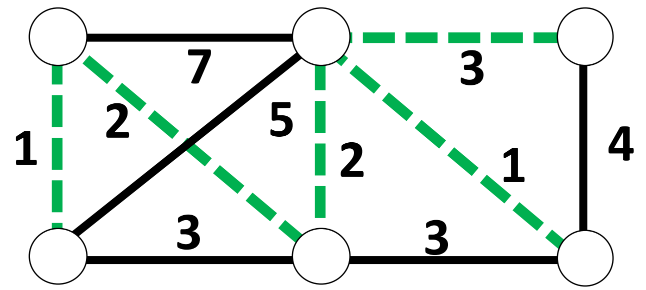

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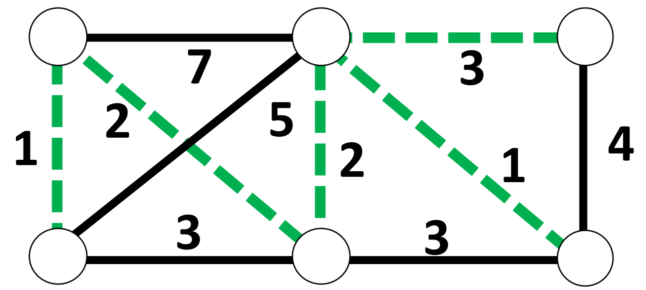
$\therefore T$ is a spanning tree of G



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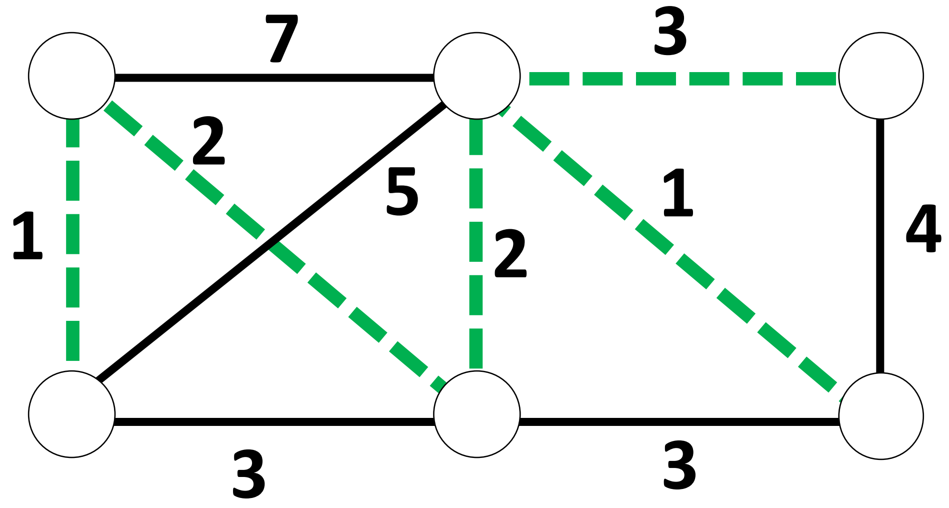

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Thus, the spanning tree constructed *is* the MST.

Spanning Trees all have $|V| -$ 1 edges.

MST Cut Property

Assume unique edge costs.

<u>Lemma:</u> Suppose that S is a subset of nodes from $G = (V, E)$. Then, the cheapest edge e between S and $V \setminus S$ is part of the MST.

<u>Proof:</u>

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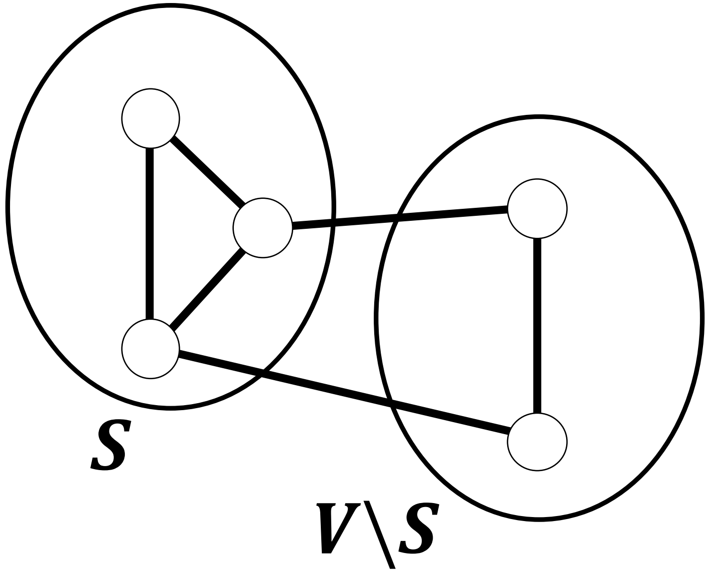

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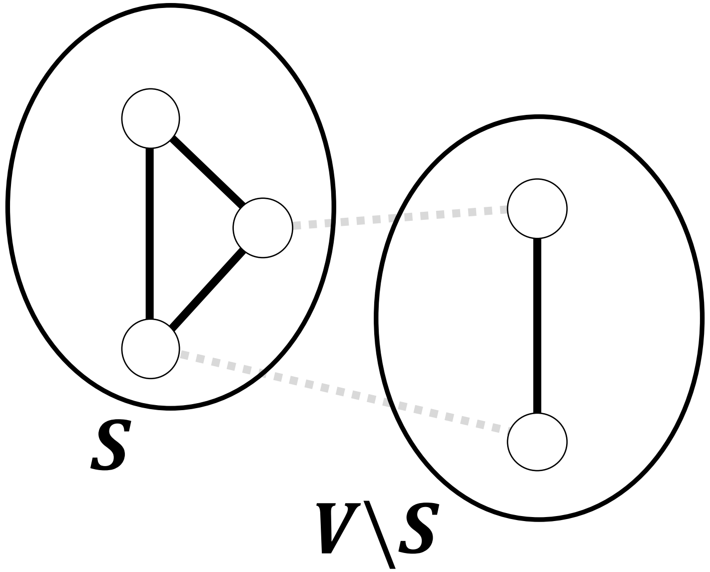

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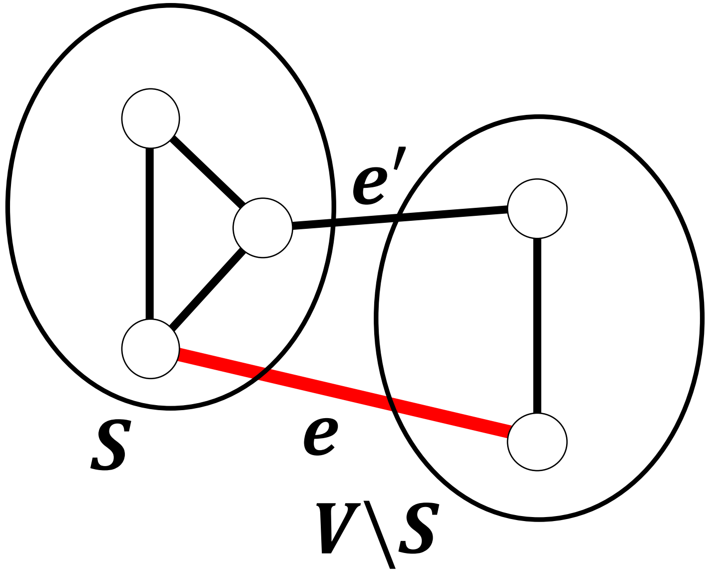

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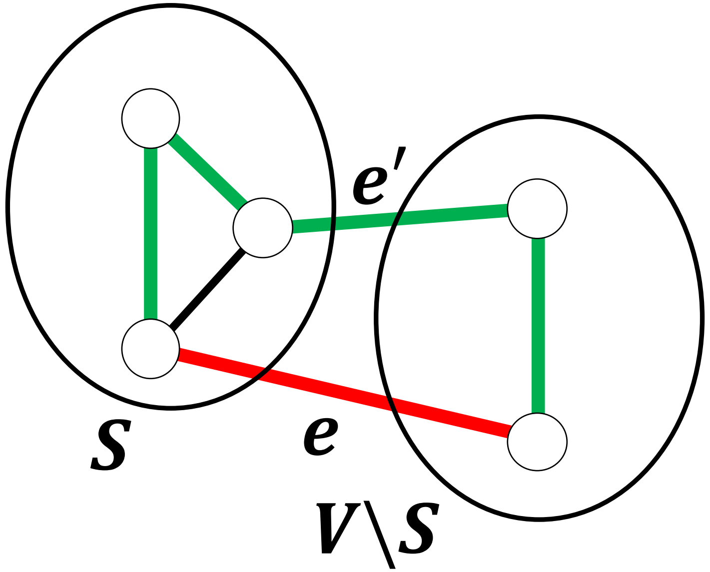

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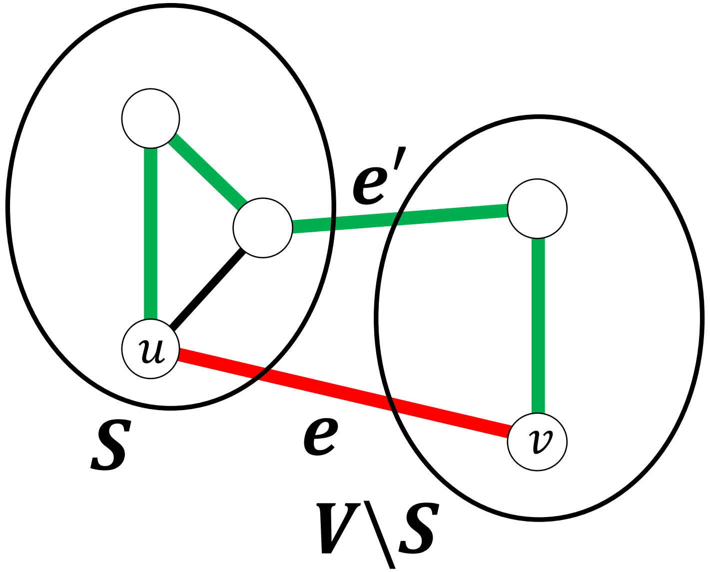

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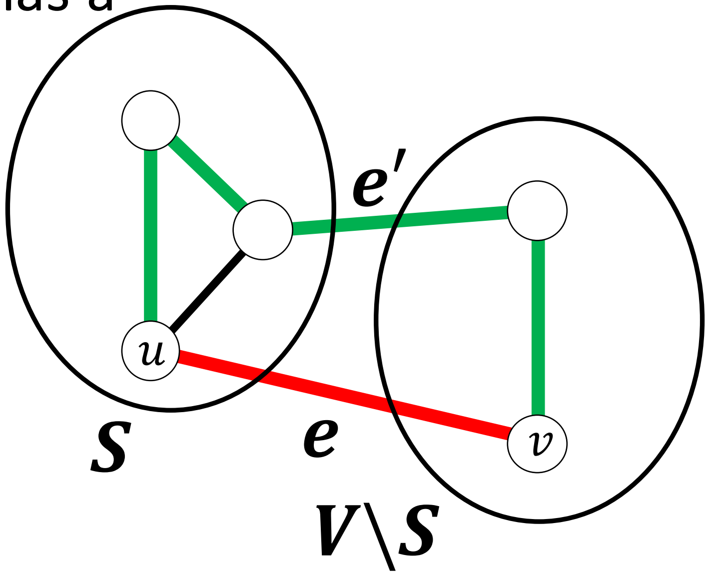

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