Dynamic Programming
CSCI 432

Rod Cutting

length	1	2	3	4
profit	$1	$5	$8	$9

Given a rod of length n and a table of prices,
determine the maximum profit obtainable
by cutting the rod up and selling the pieces.

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Rod Cutting

Assume there is a profit ≥ 0 for each possible length.

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Rod Cutting

Can we use dynamic programming?

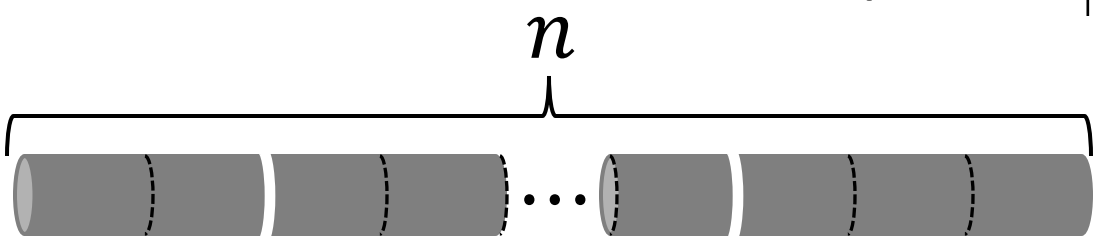
Rod Cutting

Does the rod cutting problem have optimal substructure?

Rod Cutting

Assume there is a profit ≥ 0
for each possible length.

length	1	2	3	4	…	n
profit	\$1	\$5	\$8	\$9	…	\$?



Does the rod cutting problem have optimal substructure?

I.e. Does an optimal solution to an instance
imply an optimal solution to a smaller instance?

Rod Cutting

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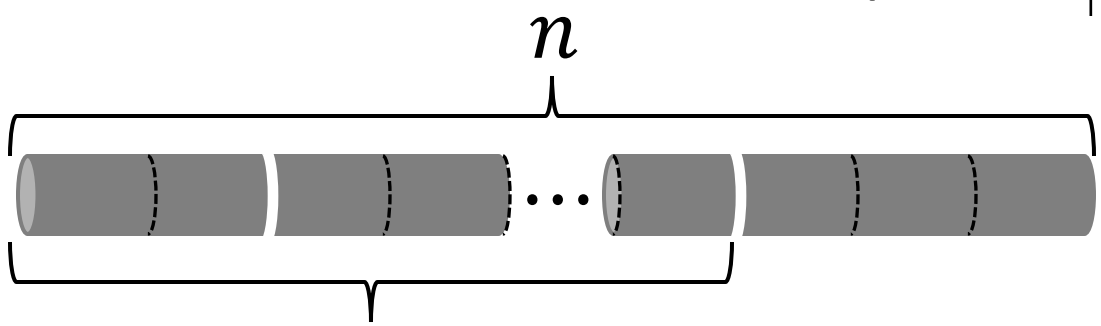

Does the rod cutting problem have optimal substructure?

Yes! Given an optimal partition for n, every subset of that partition must also be optimal for its size!

Rod Cutting

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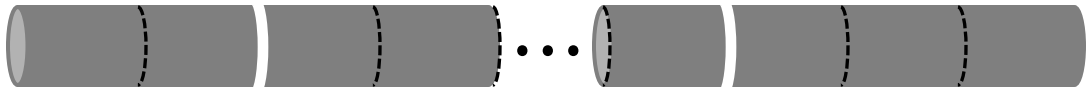
profit	\$1	\$5	\$8	\$9	…	\$?



O_n = optimal profit from partitioning rod of length n.
p_i = profit for rod of length i.

Rod Cutting

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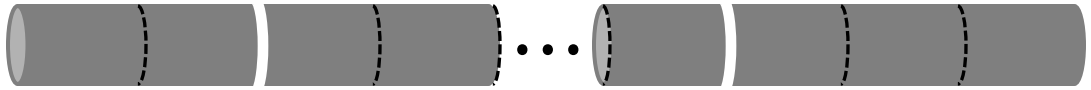

O_n = optimal profit from partitioning rod of length n.
p_i = profit for rod of length i.

$$O_n = ?$$

How can we recursively calculate O_n from smaller solutions, leveraging the optimal substructure?

Rod Cutting

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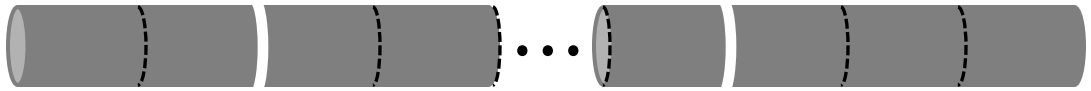

O_n = optimal profit from partitioning rod of length n.
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$$O_n = \max_{1 \le i \le n} (p_i + O_{n-i})$$

Rod Cutting

Algorithm Plan:

?

Rod Cutting

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Rod Cutting

```
partition(n, p)
   r[0,...,n] = [0,...,0]
   for m = 1 to n
      max = 0
      for length l ≤ m
         if r[m – l] + p_l > max
            max = r[m – l] + p_l
      r[m] = max
   return r[n]
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Running Time?

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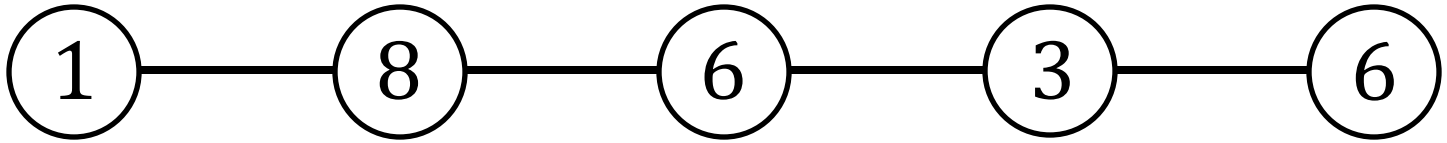
Running Time?

$O(n^2)$

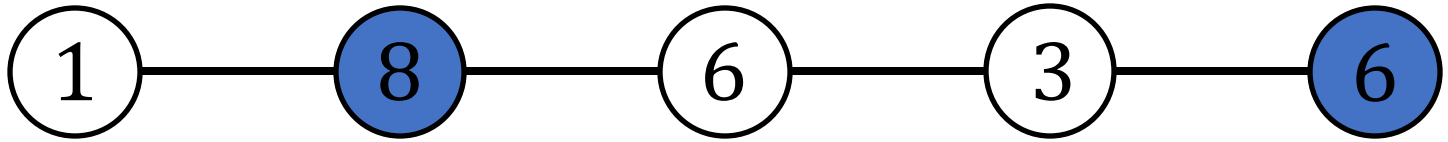
Problem Statement

Suppose we have a graph of this form:



Goal: Find a subset of vertices such that they do not share any edges with each other, and the sum of their weights (w_i) is maximized.

Example:



Find counter example for this algorithm:

```
maxWeight1(G = (V, E))
    S = Ø
    while V ≠ Ø
      Let v_i be node of max weight from V
      Add v_i to S
      Delete v_i and neighbors from G
    return S
```

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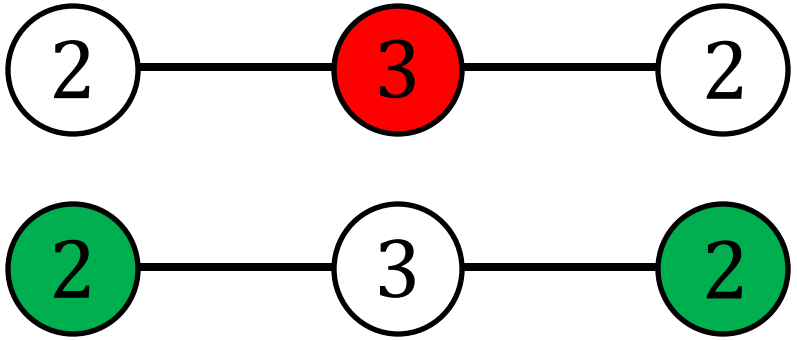

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Find counter example for this algorithm:

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maxWeight2(G = (V, E))
    Let S1 = vi where i is odd
    Let S2 = vi where i is even
    return maxWeight(S1, S2)
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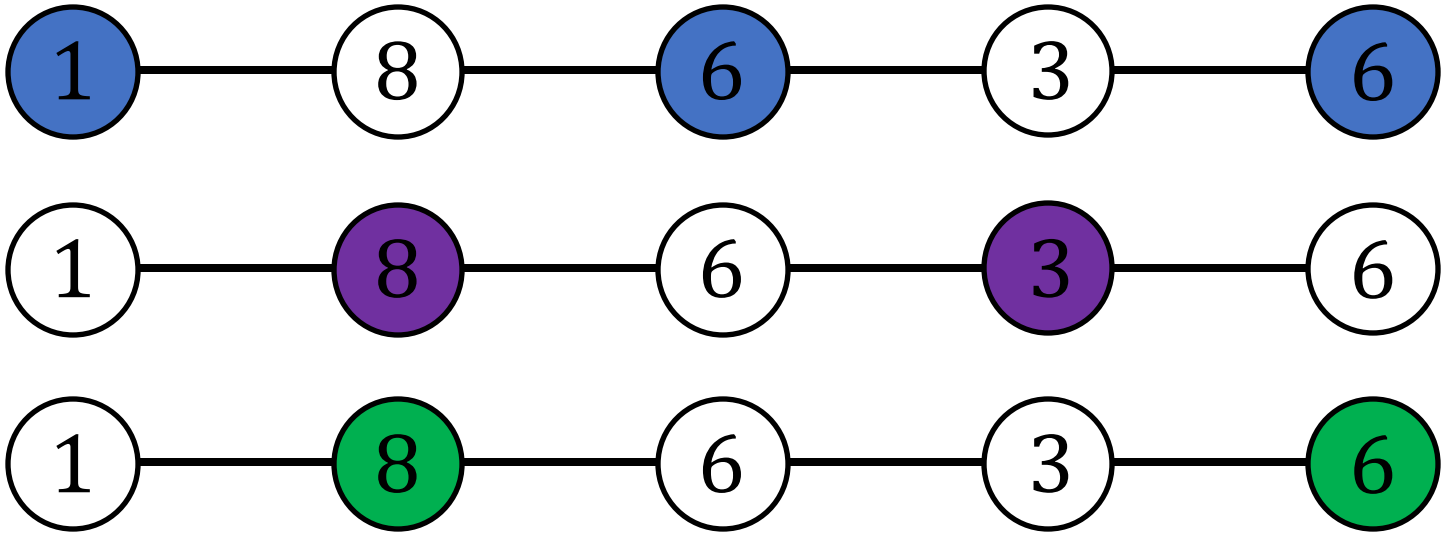

Make an algorithm:

Input: n node graph, $G = (V, E)$, with vertex weights.

Output: Subset of vertices such that they do not share any edges with each other, and the sum of their weights (w_i) is maximized.

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Is there an optimal substructure? Does an optimal solution to $V_{i-1} = \{v_1, \ldots, v_{i-1}\}$ translate to an optimal solution to $V_i = \{v_1, \ldots, v_i\}$?

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Yes(ish). Optimal solution to V_i is either the optimal for V_{i-1} (with vertex i discarded), or the optimal for V_{i-2} (with vertex i included).

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Let $A[i]$ be the weight of the optimal solution for V_i.

$A[i] = ?$

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$$A[i] = \max \begin{pmatrix} A[i-1] \\ A[i-2] + w_i \end{pmatrix}$$

Make an algorithm:

Input: n node graph, $G = (V, E)$, with vertex weights.

Output: Weight of the heaviest vertex subset that do not share edges.

```
maxWeight(G = (V, E))
```

?

$A[i]$ – weight of the optimal solution for V_i.

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Running time ∈ ?

Make an algorithm:

Input: n node graph, $G = (V, E)$, with vertex weights.

Output: Weight of the heaviest vertex subset that do not share edges.

```
maxWeight(G = (V, E))
    A[0,...,n] = [0,w_1,0,...,0]
    for i = 2,...,n
      A[i] = max(A[i - 1], A[i - 2] + w_i)
    return A[n]
```

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Running time $\in O(n)$