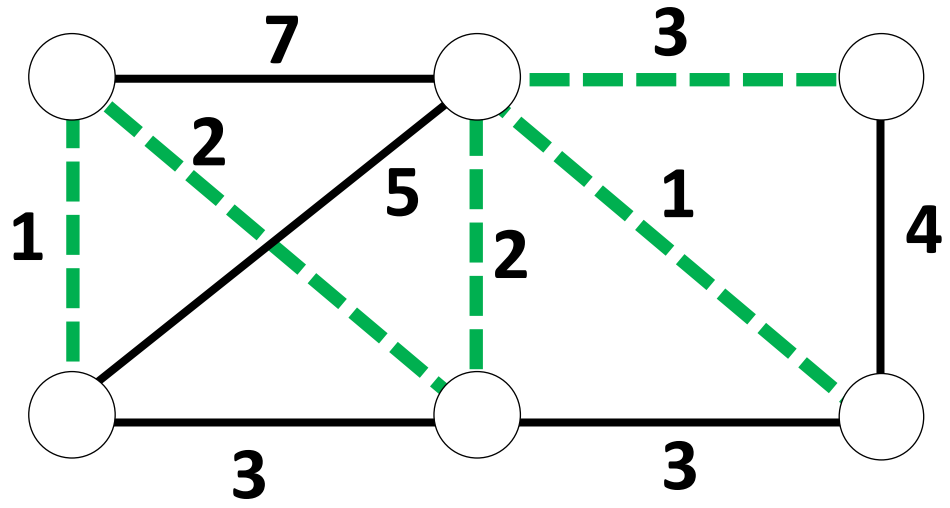
Greedy Algorithms
CSCI 432

Kruskal's MST Algorithm

Algorithm: Add the edge with smallest weight, that does not create a cycle.



Basic algorithm analysis questions:

1. ~~Is the solution valid? (Does it actually find a spanning tree?)~~
2. Is the solution optimal? (Does it find the best spanning tree?)
3. What is the running time?

MST Cut Property

Assume unique edge costs.

Lemma: Suppose that S is a subset of nodes from $G = (V, E)$. Then, the cheapest edge e between S and $V \backslash S$ is part of the MST.

Proof:

We need a plan:

Suppose we show that every edge selected by the algorithm is part of the optimal MST. Can we conclude that the selected edges form the MST?

How could a set of edges not be the MST?

It could have edges not in the MST. ✗

It could be missing edges from the MST. ✗

Thus, the spanning tree constructed *is* the MST.

Spanning Trees all have $|V| - 1$ edges.

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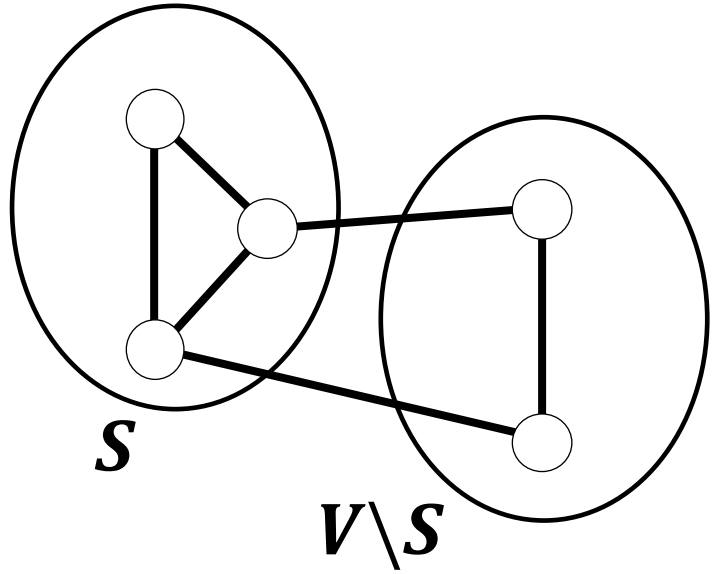

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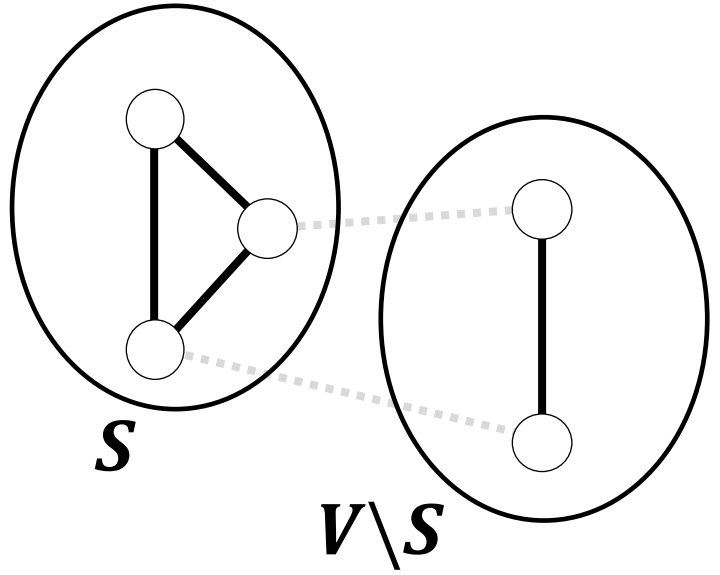

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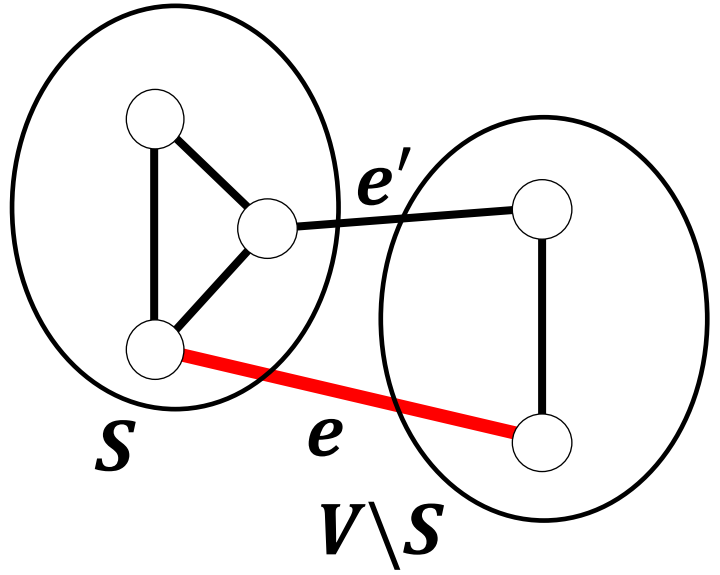

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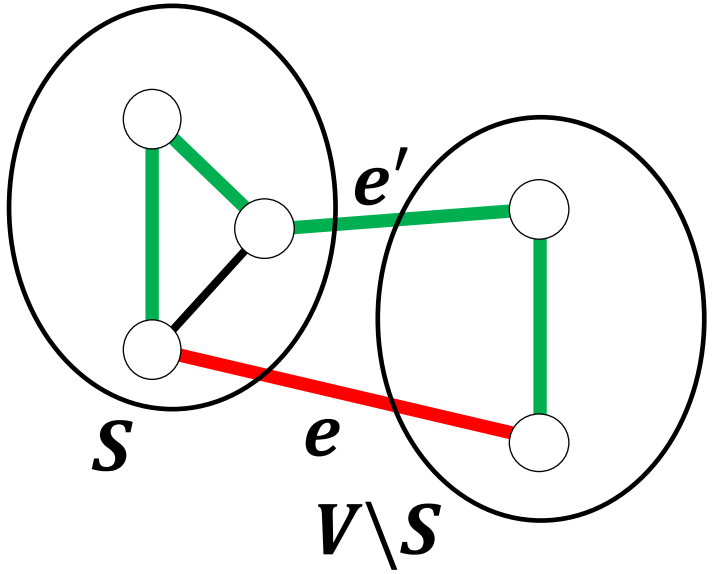

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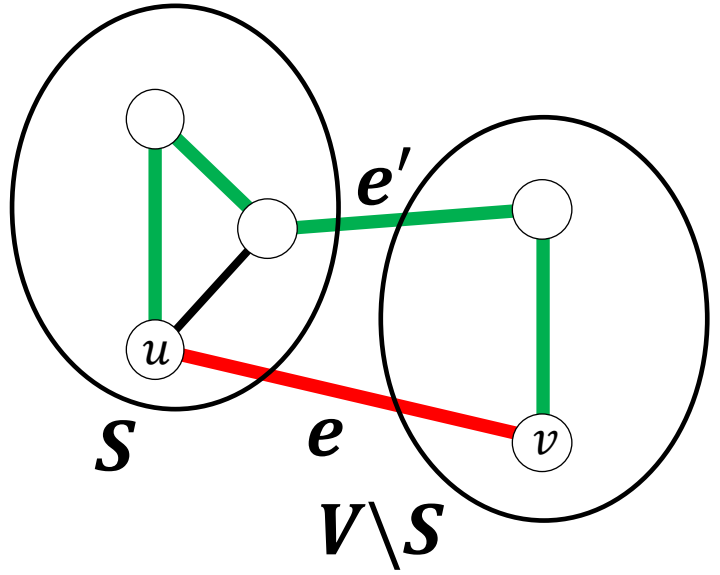
1. $\boldsymbol{T} \cup \{\boldsymbol{e}\}$ must have a cycle. Because?



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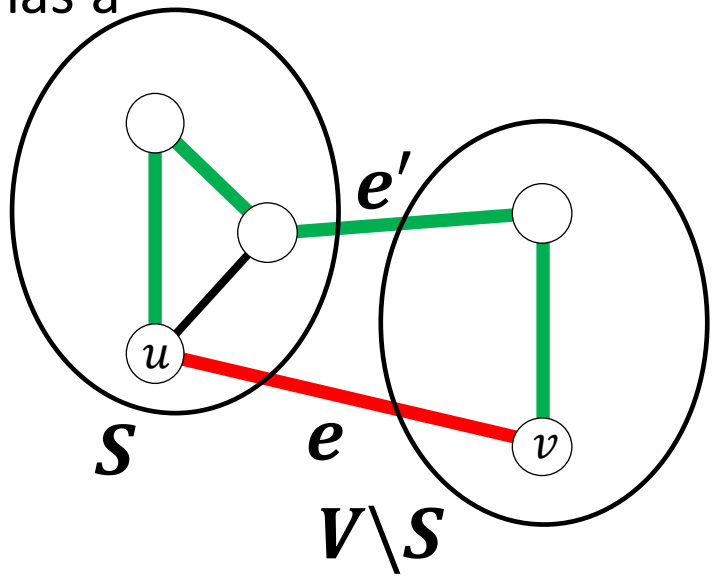

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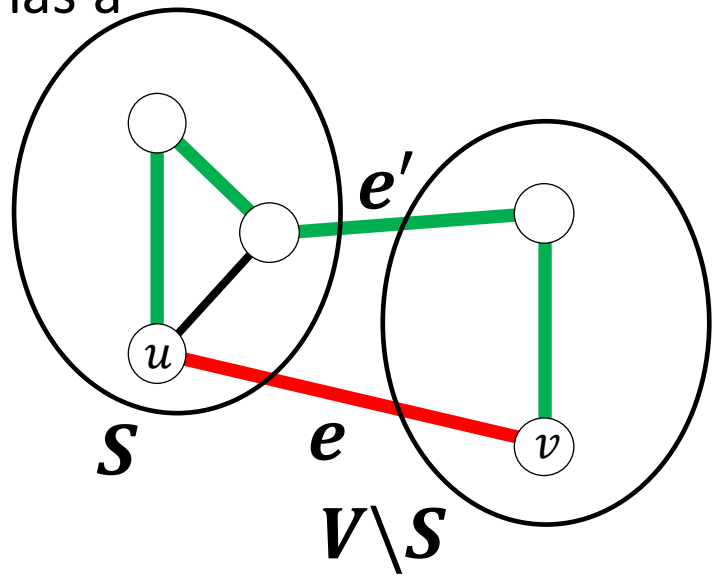

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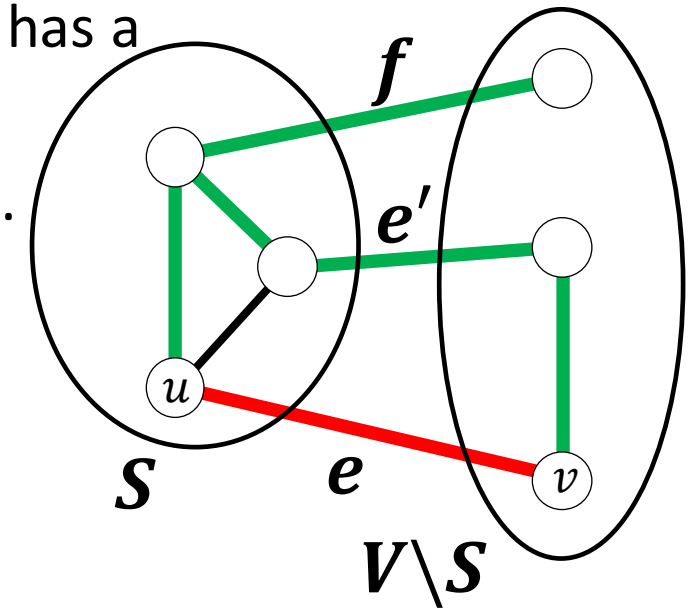

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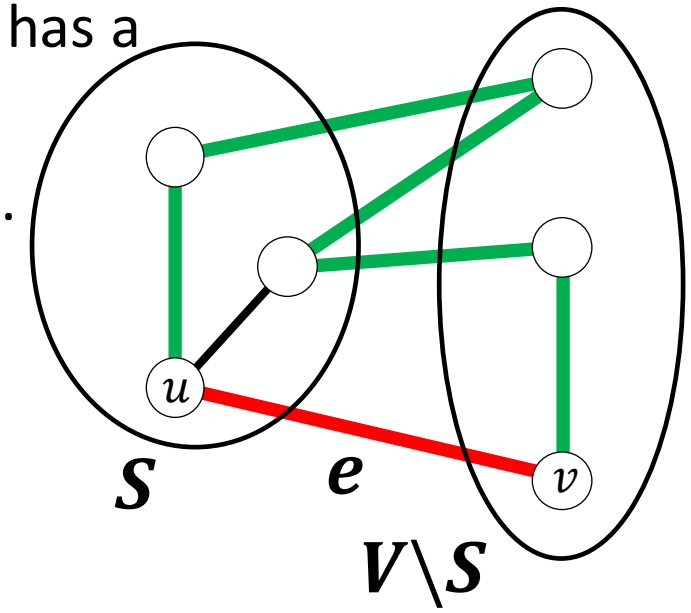

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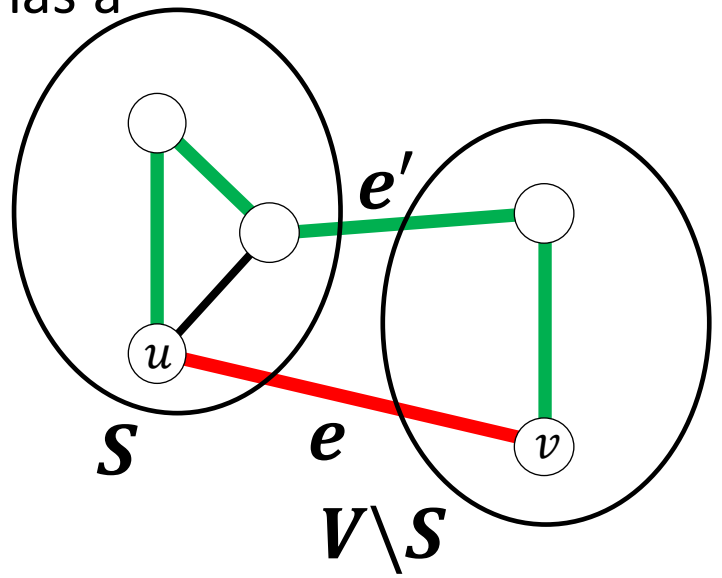

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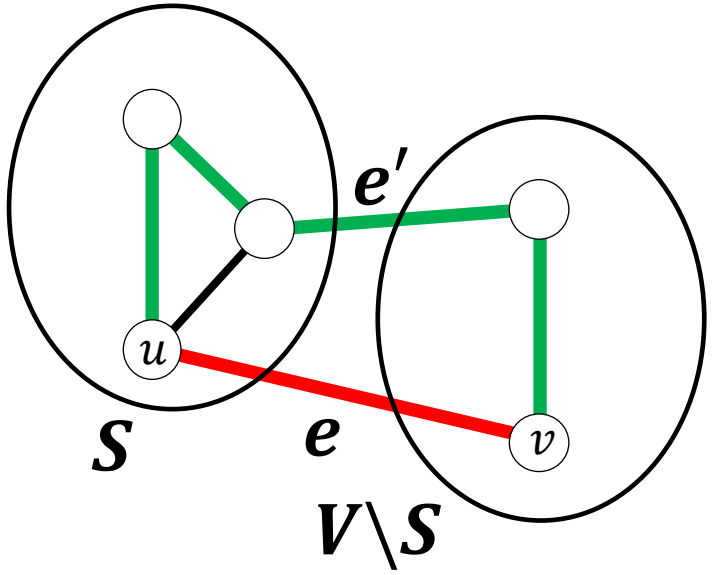

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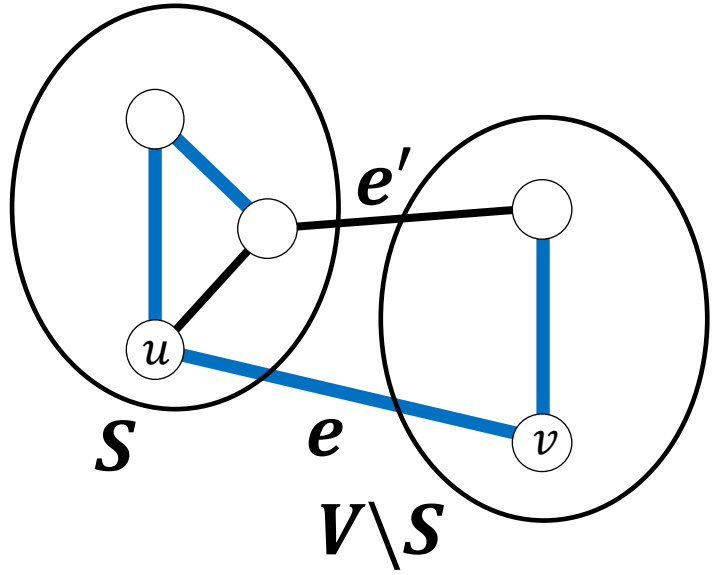

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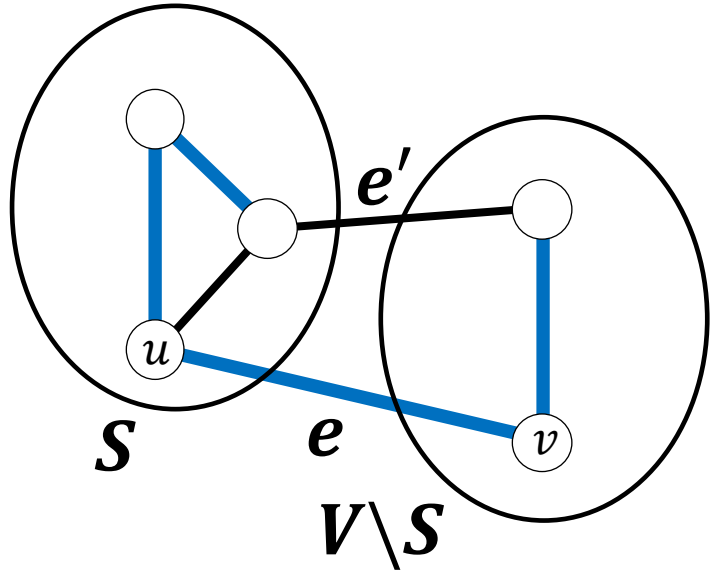

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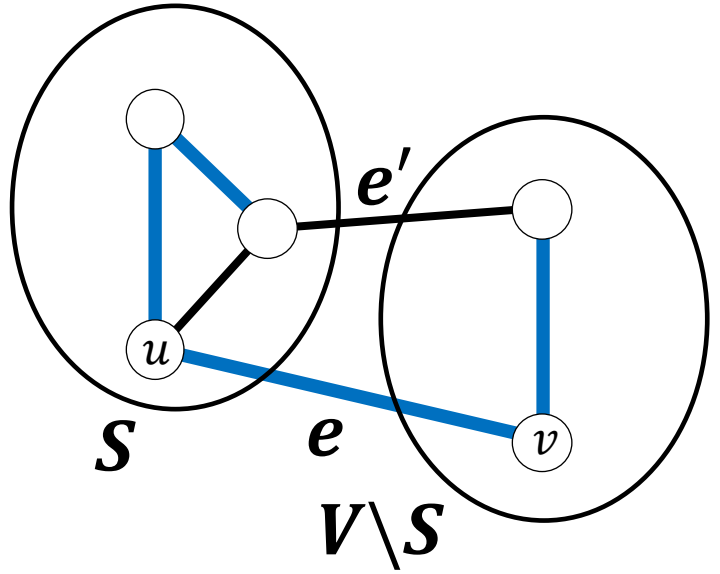

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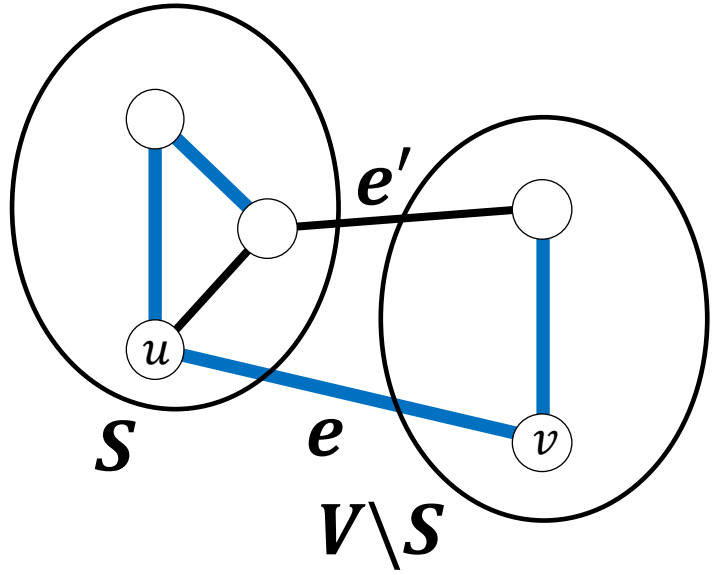

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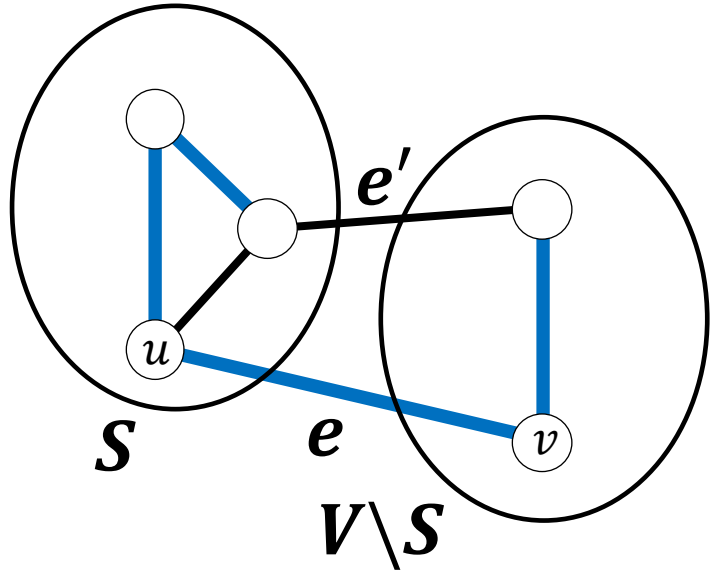

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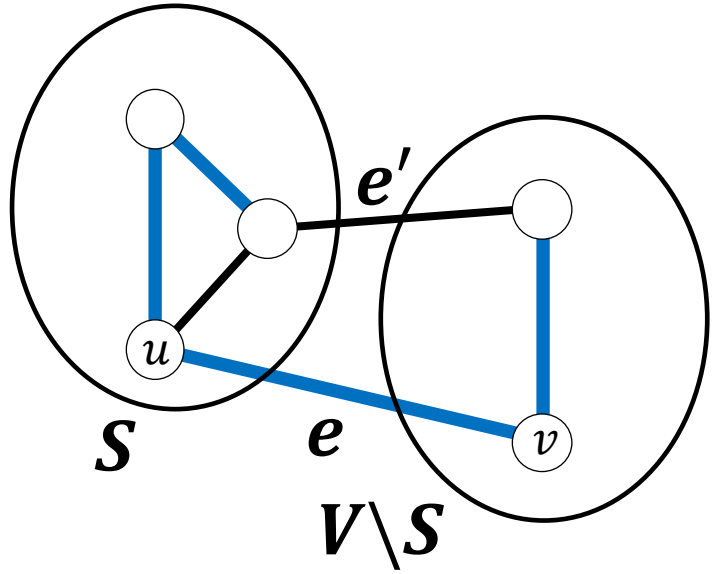

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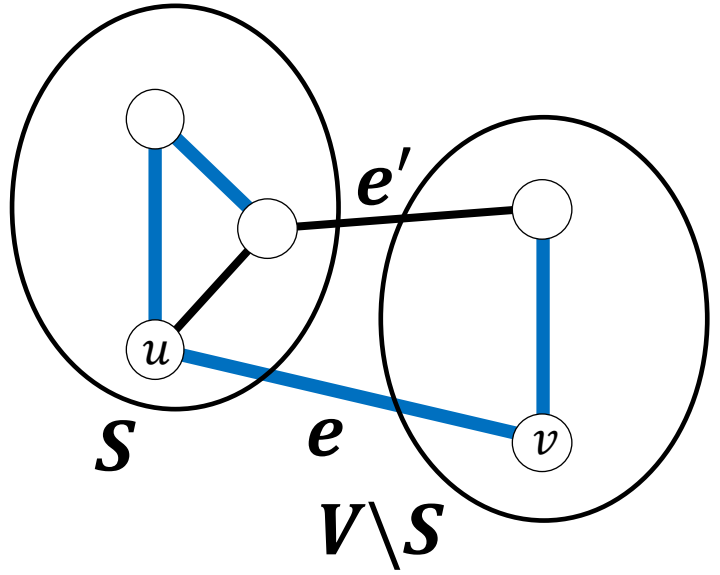
⇒ The MST must include e.



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Algorithm: Add the edge with smallest weight, that does not create a cycle.

Proof of optimality: Let $G = (V, E)$, and $T \subseteq E$ be the set of edges resulting from Kruskal's algorithm.

How do we use the Cut Property to show that Kruskal's is optimal?

Lemma: The cheapest edge between $S \subseteq V$ and $V \backslash S$ is part of the MST.

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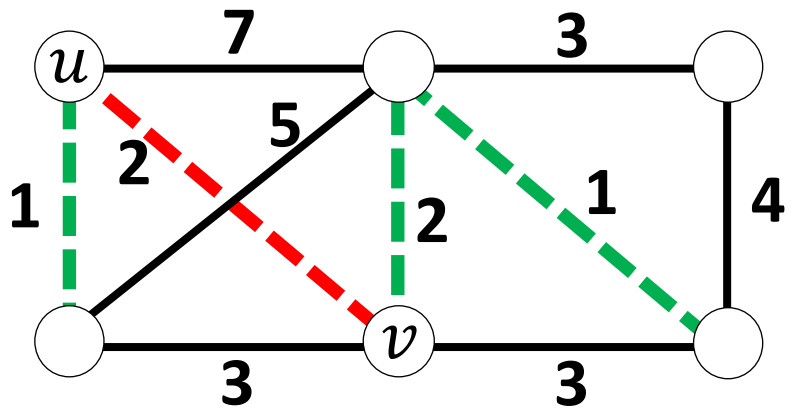

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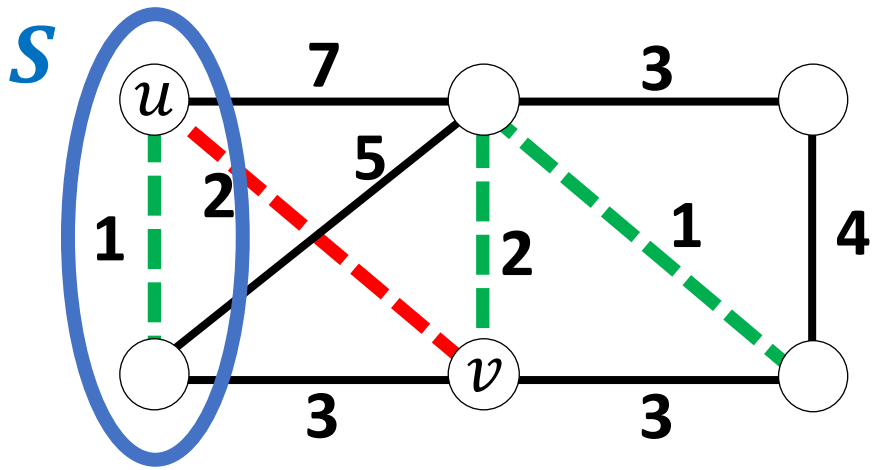

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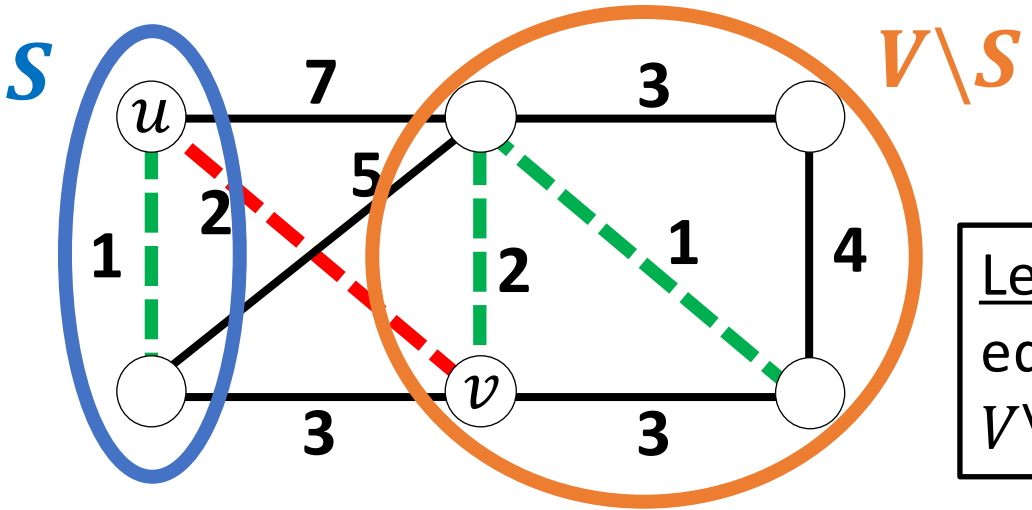

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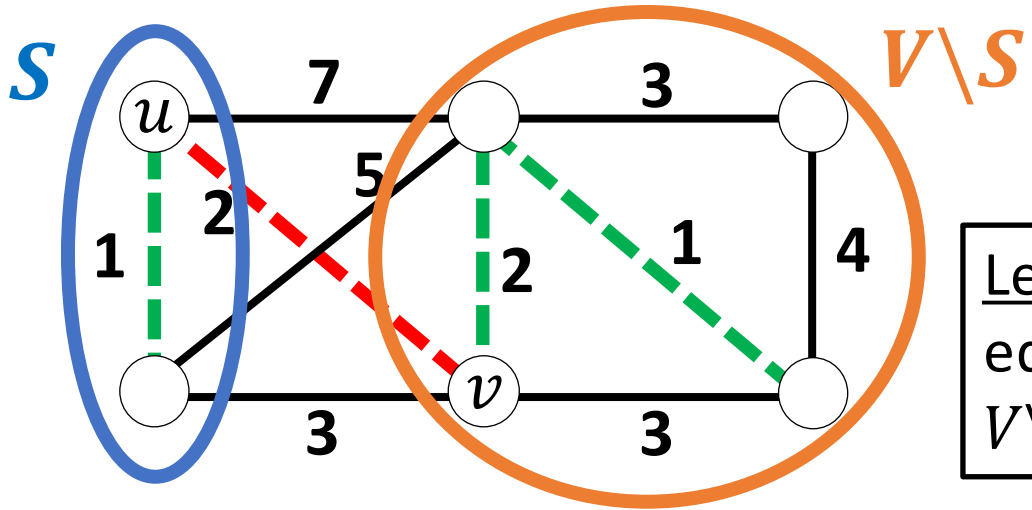

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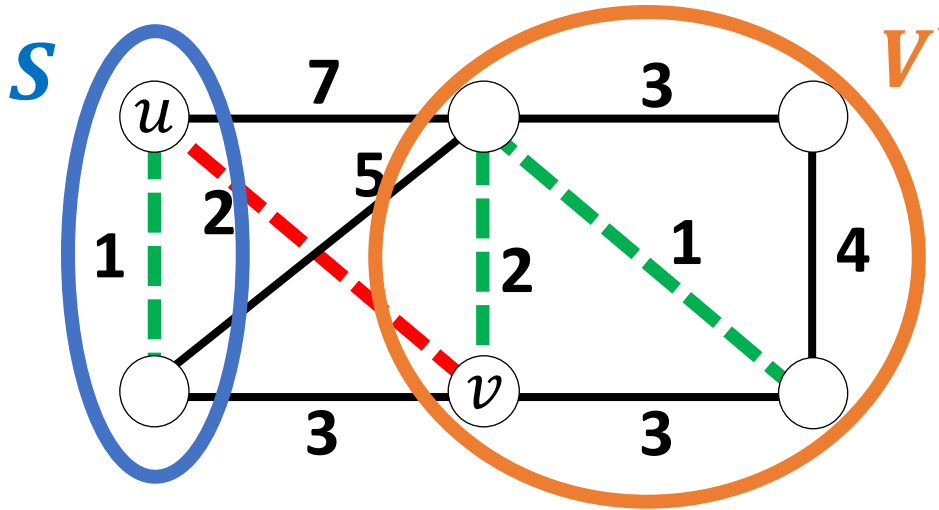

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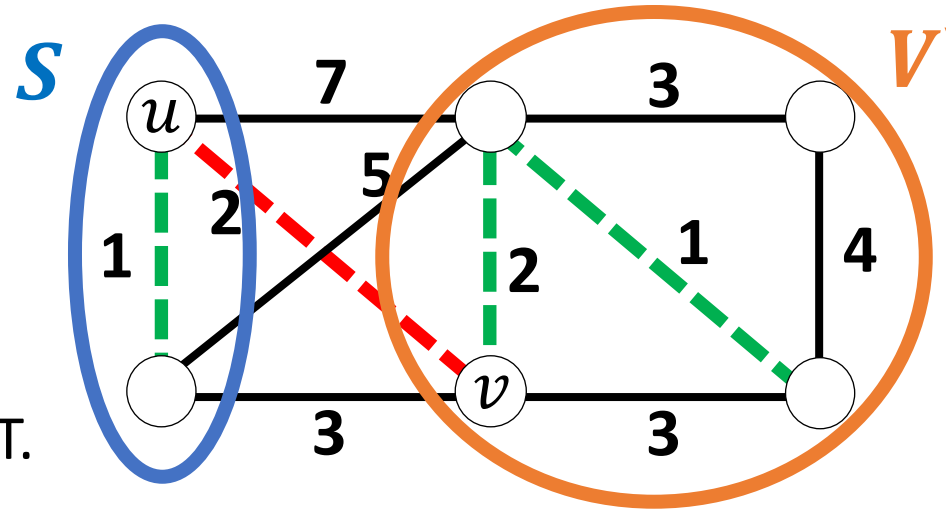
Thus, every edge found by Kruskal's algorithm is part of the MST, and since the edges found form a spanning tree, it is the MST.



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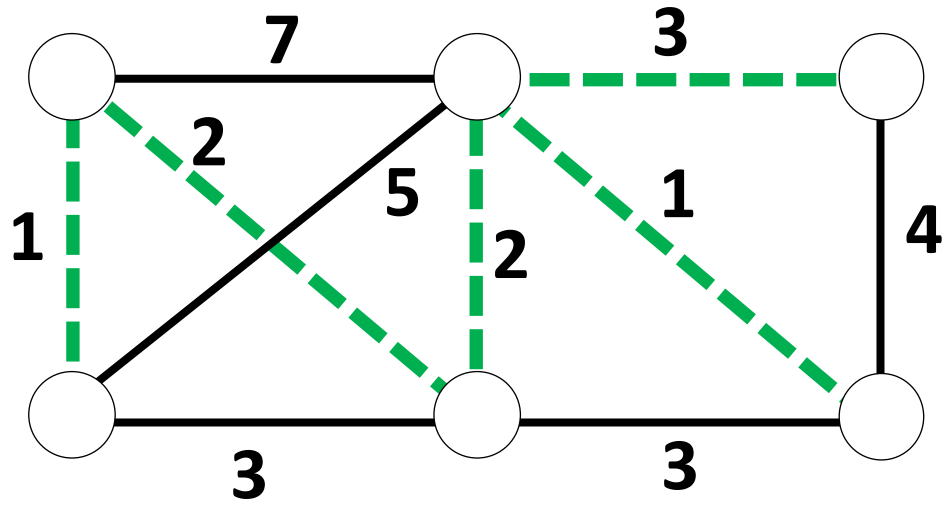

Basic algorithm analysis questions:

1. ~~Is the solution valid? (Does it actually find a spanning tree?)~~
2. ~~Is the solution optimal? (Does it find the best spanning tree?)~~
3. What is the running time?

Kruskal's MST Algorithm

Algorithm: Add the edge with smallest weight, that does not create a cycle.

Running Time:

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Running Time:

```
findMST(G=(V,E)) {
  T = ∅
  sort(E) //smallest to largest weight
  for (e in E) {
    if (T U {e} is acyclic) {
      T = T U {e}
    }
  }
  return T
}
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$\leftarrow O(|E| \log(|E|))$

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  for (e in E) { ← O(|E|)
    if (T U {e} is acyclic) { ← O(|V| + |E|) using BFS
      T = T U {e}
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$\leftarrow O(|E| \log(|E|))$

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$\leftarrow O(|V| + |E|)$ **using BFS**

Running time

$$\in O\big(|E| \log(|E|) + |E|(|V| + |E|)\big)$$

$$\in O\big(|E|^2 + |E||V|\big)$$

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```

Can be improved to $O(1)$, thus $O(|E| \log(|E|))$ overall

$O(|E| \log(|E|))$ ← sort(E) //smallest to largest weight

$O(|E|)$ ← for (e in E) {

$O(|V| + |E|)$ **using BFS** ← if (T U {e} is acyclic) {

Running time

$$\in O\big(|E| \log(|E|) + |E|(|V| + |E|)\big)$$

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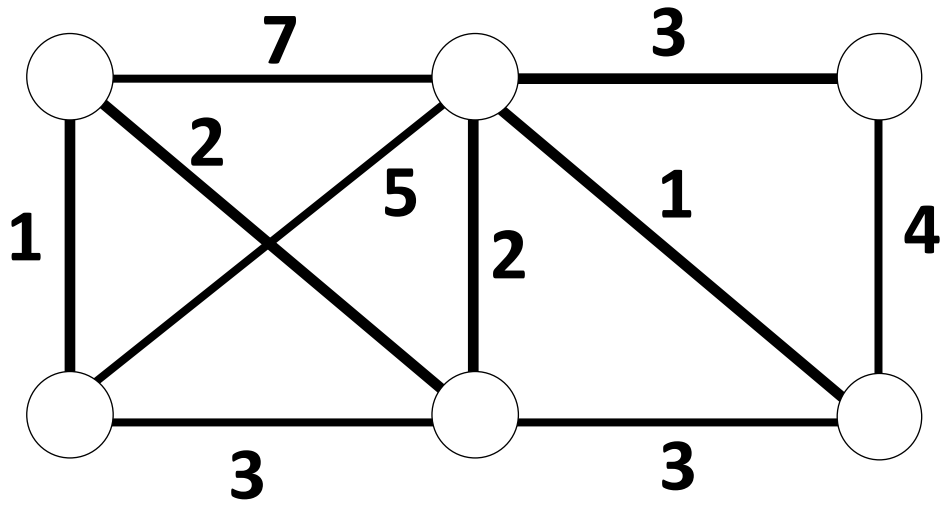
Prim's MST Algorithm

Algorithm: Beginning at any node, add the node that can be connected as cheaply as possible to the tree we are building.



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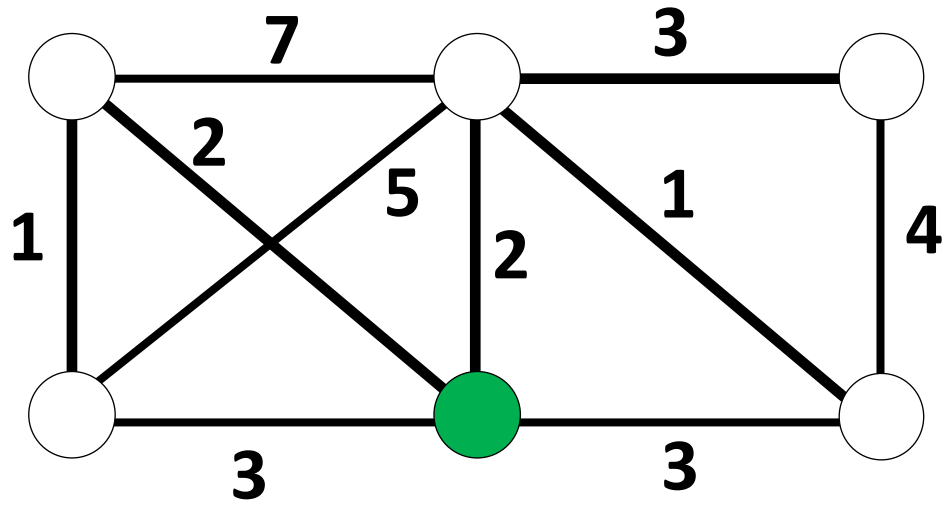

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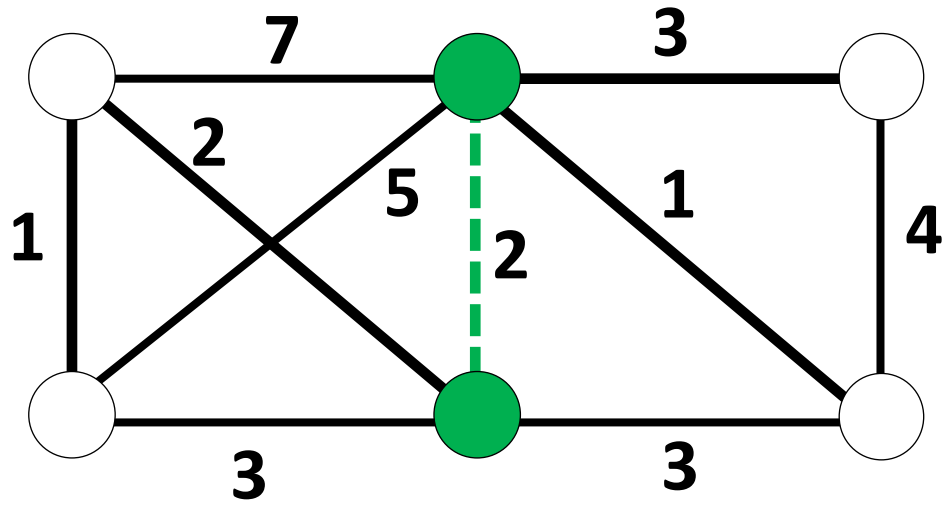

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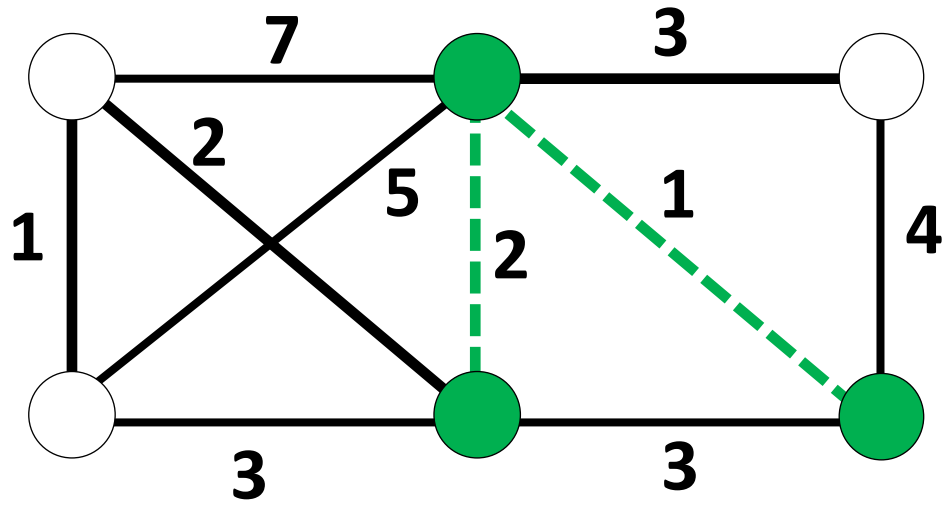

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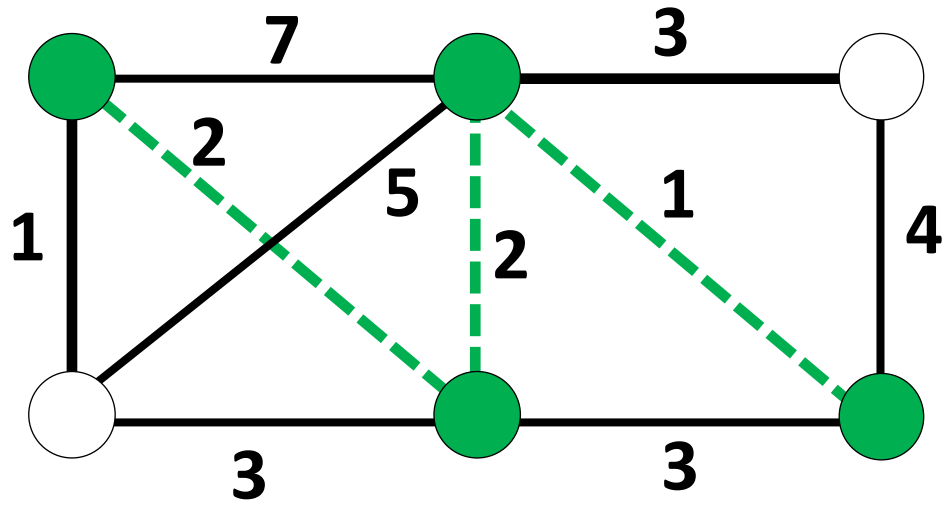

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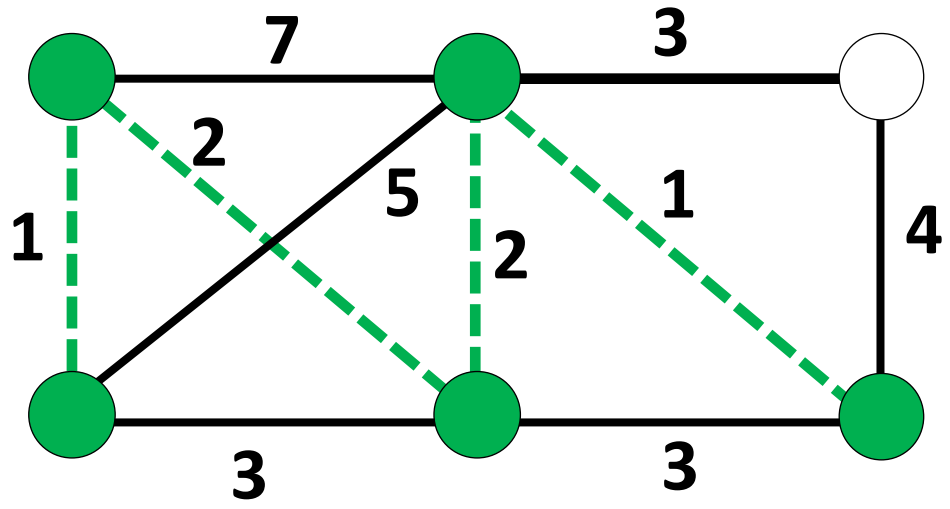

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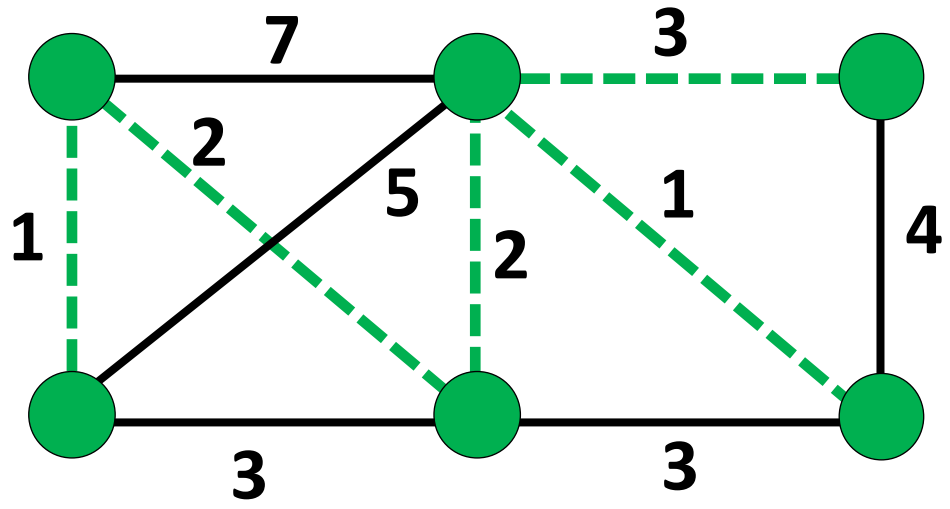

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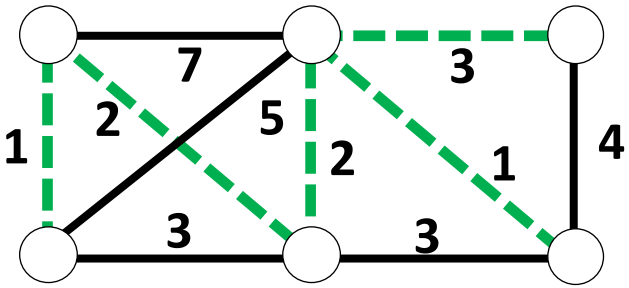
Proof of validity & optimality: Let $G = (V, E)$, and $T \subseteq E$ be the set of edges resulting from Prim's algorithm.

T is a tree because ??



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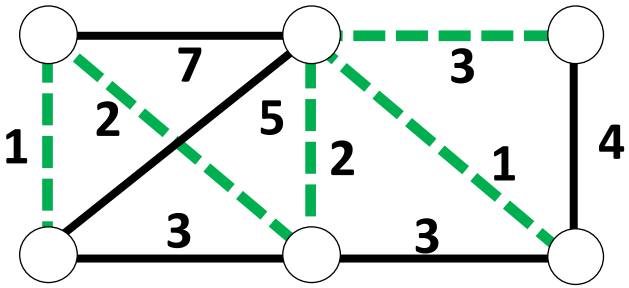

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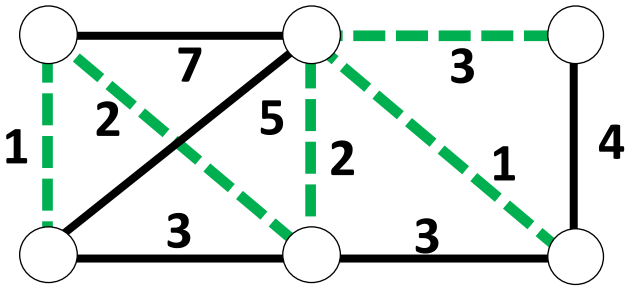

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T is an MST because at each step we add the cheapest edge between the current tree and the remaining nodes (cut property).