

# Dynamic Programming

## CSCI 432

# Matrix-Chain Multiplication

$$\begin{matrix} A & \times & B & = & C \\ 10 \times 100 & & 100 \times 5 & & 10 \times 5 \end{matrix}$$

# Matrix-Chain Multiplication

$$\begin{matrix} \text{A} \\ 10 \times 100 \end{matrix} \times \begin{matrix} \text{B} \\ 100 \times 5 \end{matrix} = \begin{matrix} \text{C} \\ 10 \times 5 \end{matrix}$$

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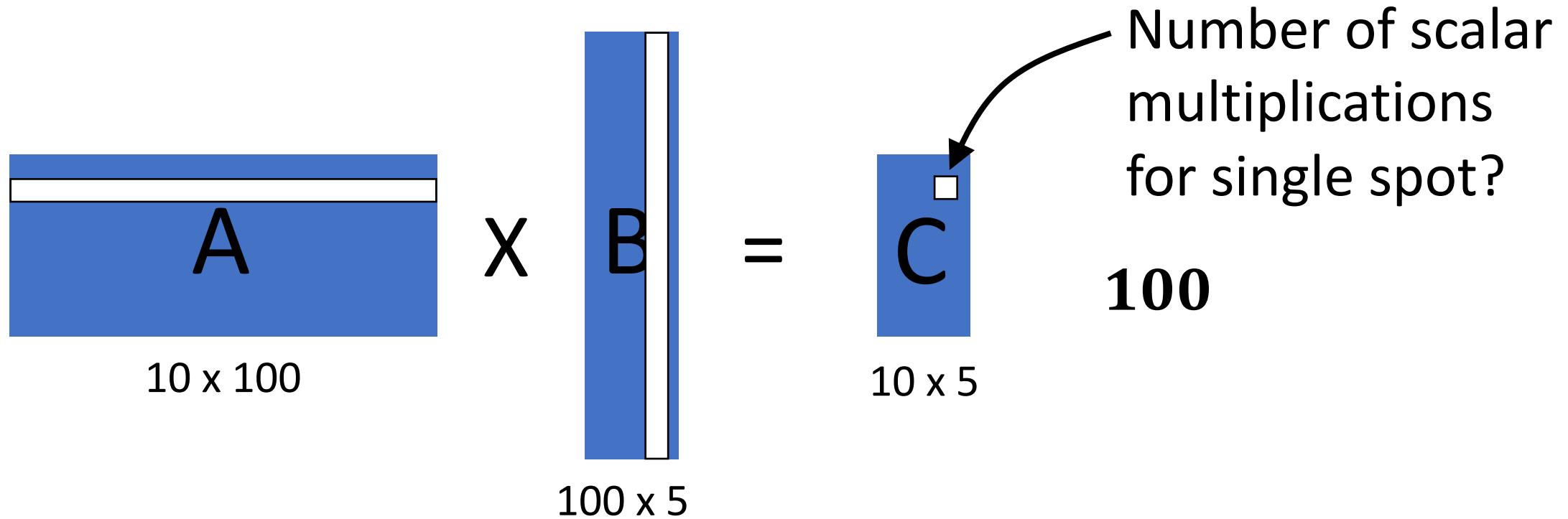
Number of scalar multiplications for single spot?

A diagram illustrating matrix-chain multiplication. It shows three matrices: A (10x100), B (100x5), and C (10x5). Matrix A is represented by a blue rectangle with 'A' in the center. Matrix B is represented by a blue rectangle with 'B' in the center. Matrix C is represented by a blue rectangle with 'C' in the center. The dimensions 10x100, 100x5, and 10x5 are written below their respective matrices. An arrow points from the text 'Number of scalar multiplications for single spot?' to a small white square in the top-left corner of matrix C.

# Matrix-Chain Multiplication

$$\begin{matrix} \text{A} \\ 10 \times 100 \end{matrix} \times \begin{matrix} \text{B} \\ 100 \times 5 \end{matrix} = \begin{matrix} \text{C} \\ 10 \times 5 \end{matrix}$$

Number of scalar multiplications for single spot?  
100



# Matrix-Chain Multiplication

$$\begin{matrix} \text{A} \\ 10 \times 100 \end{matrix} \times \begin{matrix} \text{B} \\ 100 \times 5 \end{matrix} = \begin{matrix} \text{C} \\ 10 \times 5 \end{matrix}$$

Number of scalar multiplications for full array?  
100

The diagram illustrates the multiplication of two matrices, A and B, resulting in matrix C. Matrix A is a blue rectangle labeled 'A' with dimensions '10 x 100'. Matrix B is a blue rectangle labeled 'B' with dimensions '100 x 5'. The multiplication is shown as A × B = C, where C is a blue rectangle labeled 'C' with dimensions '10 x 5'. A curly brace groups the dimensions of C, and an arrow points from the text 'Number of scalar multiplications for full array?' to the number '100'.

# Matrix-Chain Multiplication

$$\begin{matrix} \text{A} \\ 10 \times 100 \end{matrix} \times \begin{matrix} \text{B} \\ 100 \times 5 \end{matrix} = \begin{matrix} \text{C} \\ 10 \times 5 \end{matrix}$$

Number of scalar multiplications for full array?  
 $100 \times 10 \times 5 = 5000$

# Matrix-Chain Multiplication

$$\begin{array}{c} \text{A} \\ \text{---} \\ 10 \times 100 \\ n \times m \end{array} \times \begin{array}{c} \text{B} \\ \text{---} \\ 100 \times 5 \\ m \times p \end{array} = \begin{array}{c} \text{C} \\ \text{---} \\ 10 \times 5 \\ n \times p \end{array}$$

Number of scalar multiplications for full array?

$m \times n \times p$

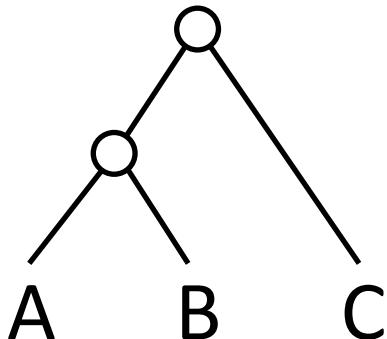
# Matrix-Chain Multiplication

$$\begin{matrix} A & \times & B & \times & C & = & D \\ 10 \times 100 & & 100 \times 5 & & 5 \times 50 & & 10 \times 50 \end{matrix}$$

# Matrix-Chain Multiplication

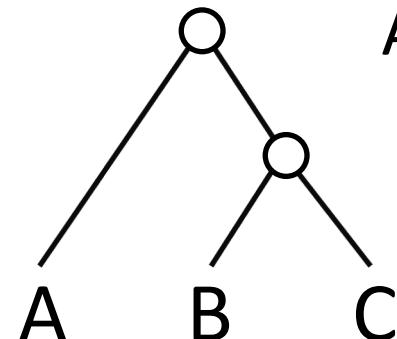
$$\begin{matrix} A & \times & B & \times & C & = & D \end{matrix}$$

A:  $10 \times 100$       B:  $100 \times 5$       C:  $5 \times 50$       D:  $10 \times 50$



$(A \times B) \times C :$

Both give same answer.  
Any reason to pick one?

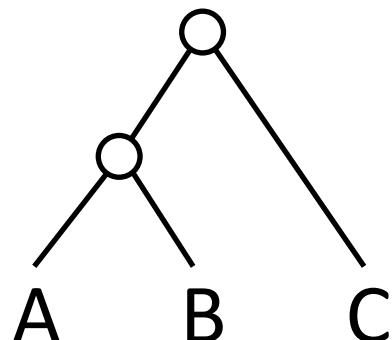


$A \times (B \times C) :$

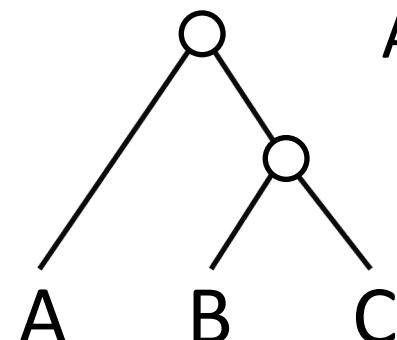
# Matrix-Chain Multiplication

$$\begin{matrix} A & \times & B & \times & C & = & D \end{matrix}$$

$A: 10 \times 100$        $B: 100 \times 5$        $C: 5 \times 50$        $D: 10 \times 50$



$(A \times B) \times C :$   
 $100 \times 10 \times 5$

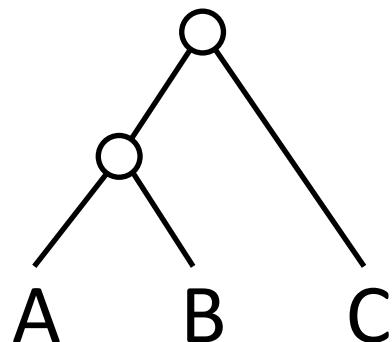


$A \times (B \times C) :$

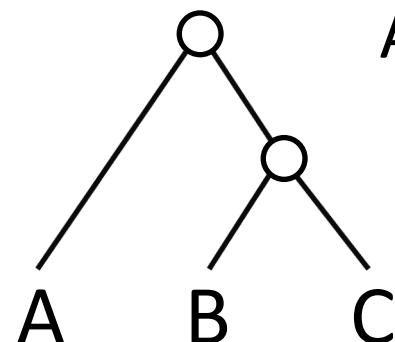
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$A: 10 \times 100$        $B: 100 \times 5$        $C: 5 \times 50$        $D: 10 \times 50$



$(A \times B) \times C :$   
 $100 \times 10 \times 5$   
 $+ 5 \times 10 \times 50$

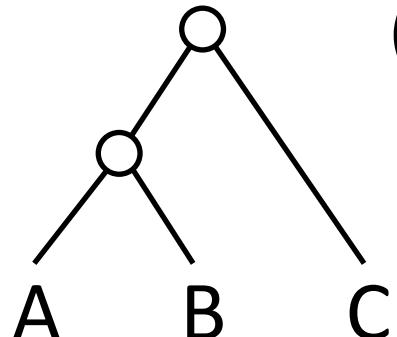


$A \times (B \times C) :$

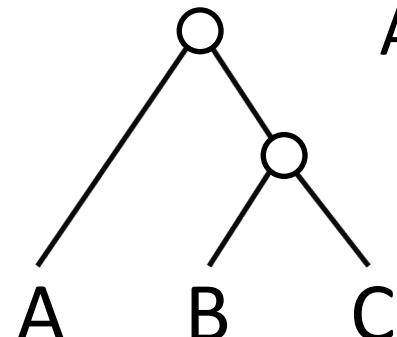
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$$\begin{matrix} A & \times & B & \times & C & = & D \end{matrix}$$

$A: 10 \times 100$        $B: 100 \times 5$        $C: 5 \times 50$        $D: 10 \times 50$



$$\begin{aligned} (A \times B) \times C : \\ 100 \times 10 \times 5 \\ + 5 \times 10 \times 50 \\ = 7500 \end{aligned}$$

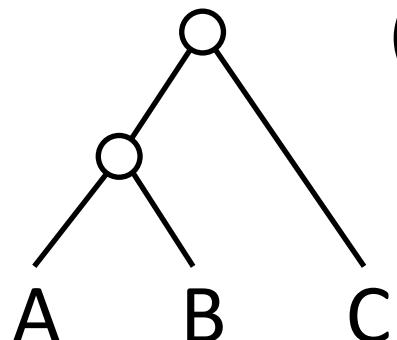


$$A \times (B \times C) :$$

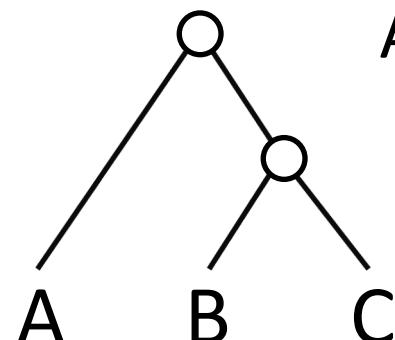
# Matrix-Chain Multiplication

$$\begin{matrix} A & \times & B & \times & C & = & D \end{matrix}$$

$A: 10 \times 100$        $B: 100 \times 5$        $C: 5 \times 50$        $D: 10 \times 50$



$$\begin{aligned} (A \times B) \times C : \\ 100 \times 10 \times 5 \\ + 5 \times 10 \times 50 \\ = 7500 \end{aligned}$$

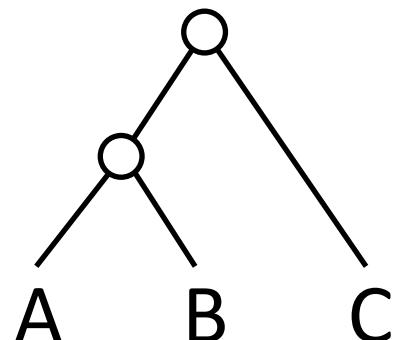


$$\begin{aligned} A \times (B \times C) : \\ 5 \times 100 \times 50 \\ + 100 \times 10 \times 50 \end{aligned}$$

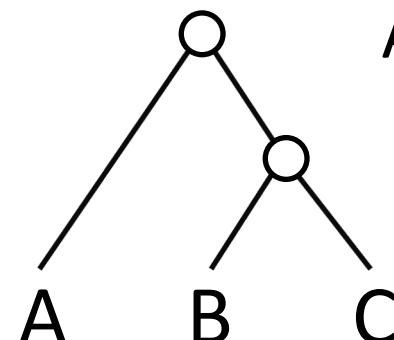
# Matrix-Chain Multiplication

$$\begin{matrix} A & \times & B & \times & C & = & D \end{matrix}$$

$A: 10 \times 100$        $B: 100 \times 5$        $C: 5 \times 50$        $D: 10 \times 50$

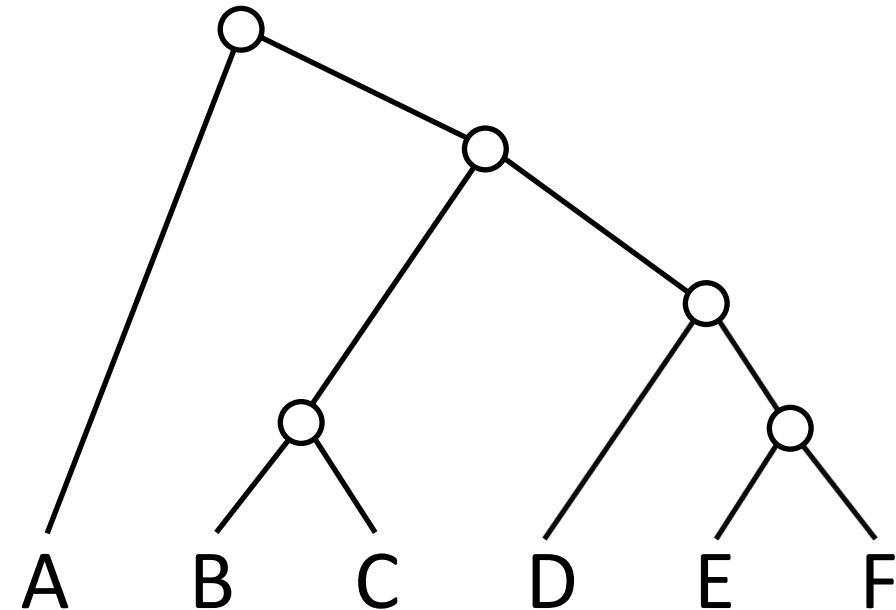
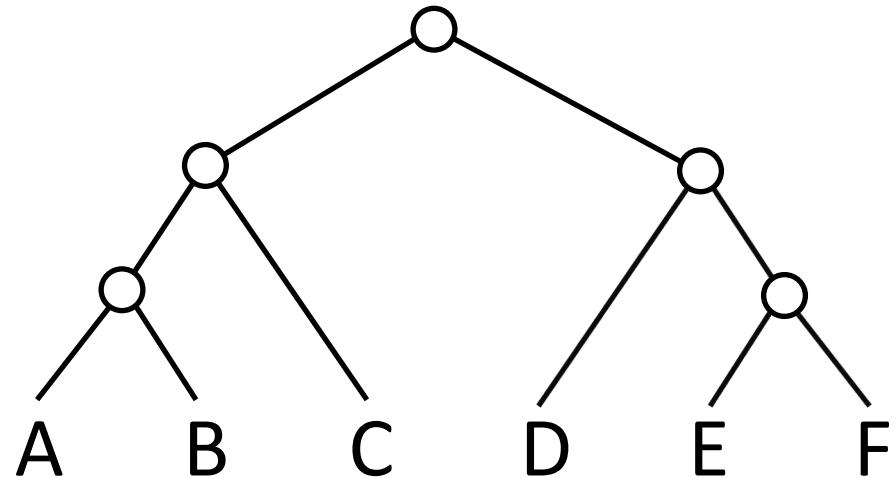


$$\begin{aligned} (A \times B) \times C : \\ 100 \times 10 \times 5 \\ + 5 \times 10 \times 50 \\ = 7500 \end{aligned}$$



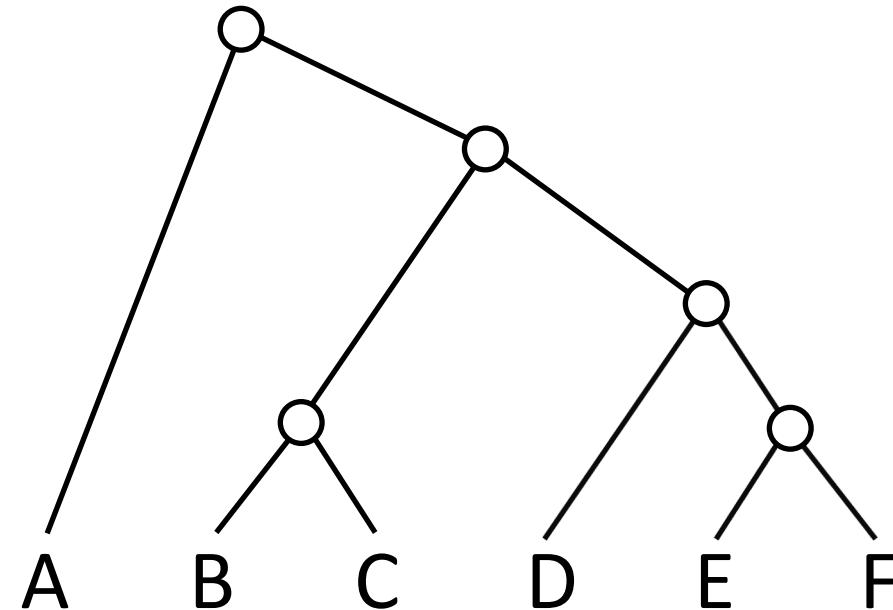
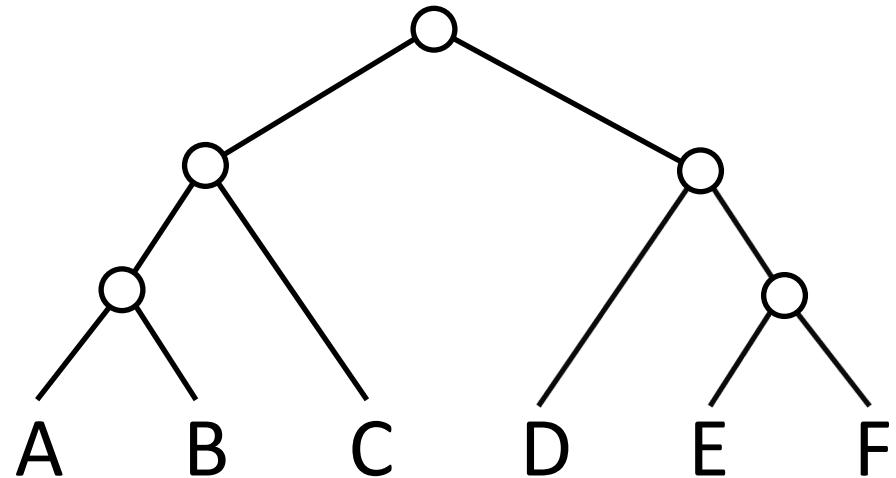
$$\begin{aligned} A \times (B \times C) : \\ 5 \times 100 \times 50 \\ + 100 \times 10 \times 50 \\ = 75000 \end{aligned}$$

# Matrix-Chain Multiplication



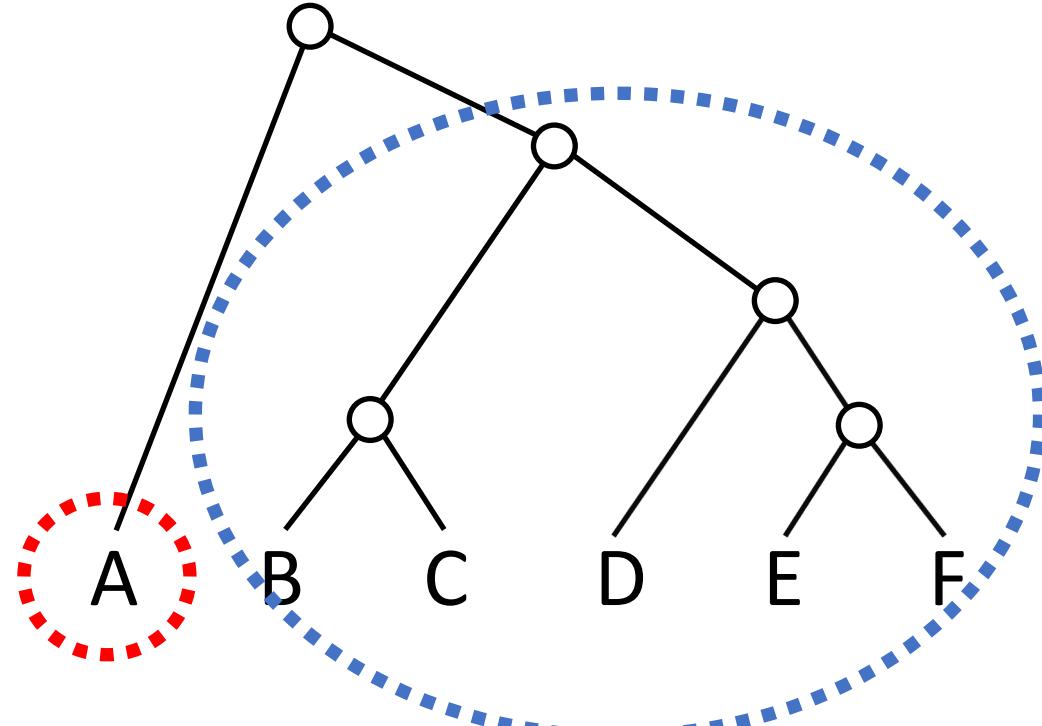
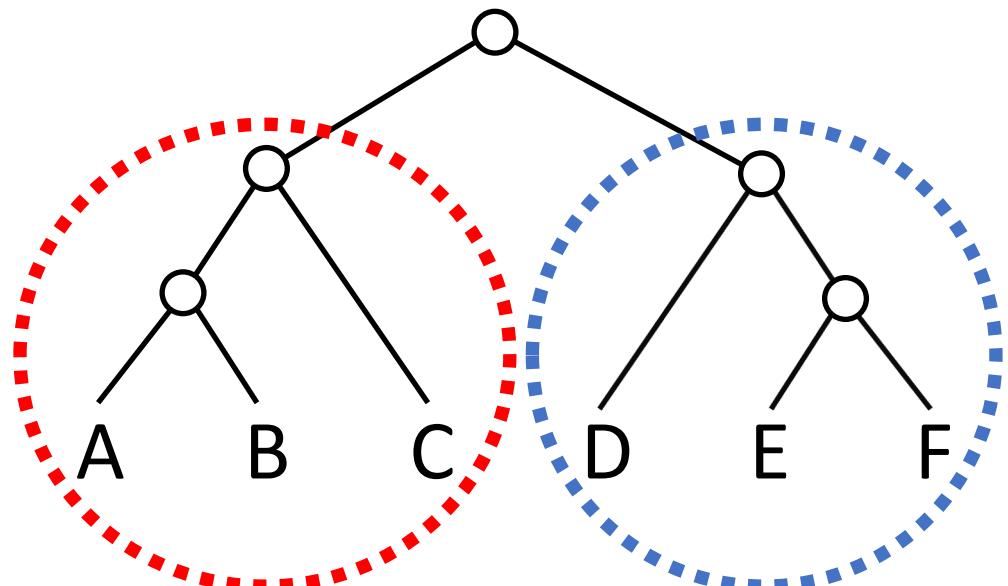
Can we use dynamic programming?

# Matrix-Chain Multiplication



Optimal substructure?

# Matrix-Chain Multiplication

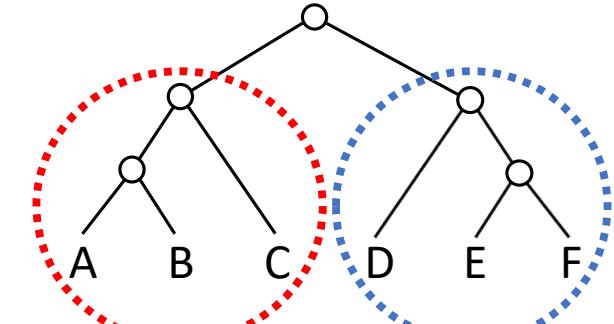


Optimal substructure?

An optimal parenthesization must end with multiplying two matrices. The parenthesizations that led to those two matrices must also be optimal.

# Matrix-Chain Multiplication

Input:  $n$  matrices  $M_1, M_2, \dots, M_n$  with dimensions of:  
 $m_0 \times m_1, m_1 \times m_2, \dots, m_{n-1} \times m_n$

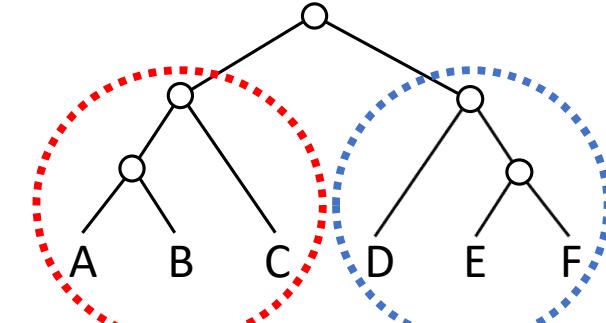


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Output: Minimum number of scalar multiplications needed to calculate  $M_1 \times M_2 \times \dots \times M_n$ .



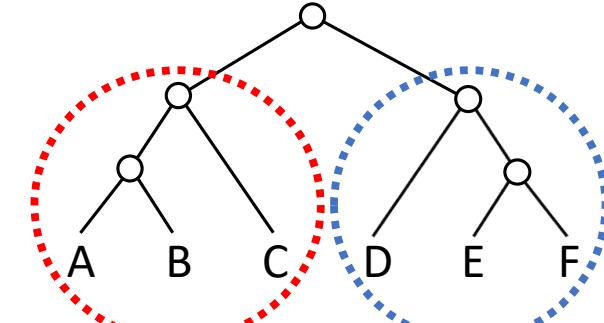
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Let  $C[1, n]$  = Minimum number of scalar multiplications needed for multiplying  $M_1 \times M_2 \times \dots \times M_n$



# Matrix-Chain Multiplication

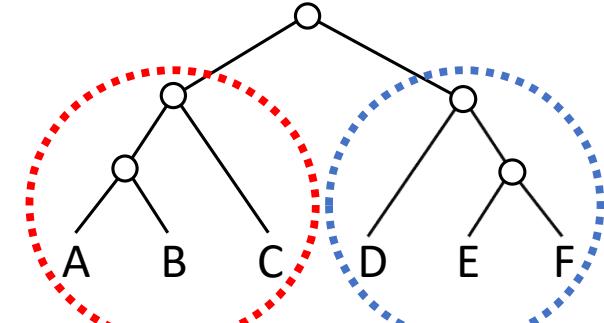
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Suppose the final multiplication in the optimal solution occurs at  $k$ :  
 $(M_1 \times M_2 \times \dots \times M_k)(M_{k+1} \times M_{k+2} \times \dots \times M_n)$



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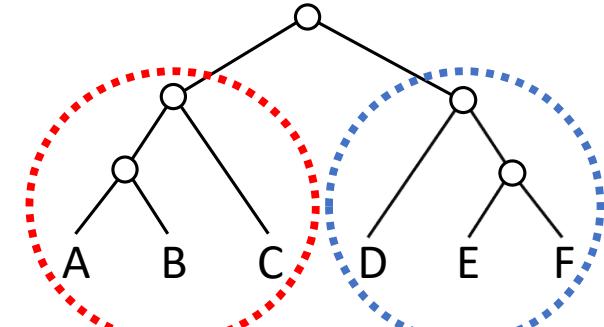
$$m_0 \times m_1, m_1 \times m_2, \dots, m_{n-1} \times m_n$$

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$$C[1, n] = ???$$



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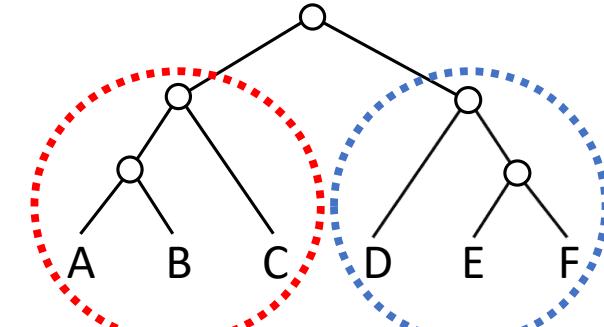
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Optimal # mults

for left side

$$C[1, n] = \overbrace{C[1, k]}$$



# Matrix-Chain Multiplication

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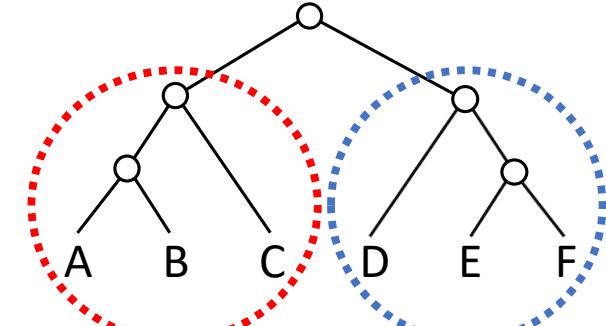
$$m_0 \times m_1, m_1 \times m_2, \dots, m_{n-1} \times m_n$$

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$$C[1, n] = \overbrace{C[1, k]}^{\text{Optimal # mults for left side}} + \overbrace{C[k + 1, n]}^{\text{Optimal # mults for right side}}$$



# Matrix-Chain Multiplication

Input:  $n$  matrices  $M_1, M_2, \dots, M_n$  with dimensions of:

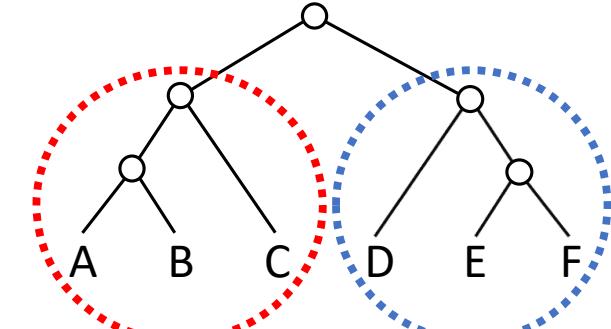
$$m_0 \times m_1, m_1 \times m_2, \dots, m_{n-1} \times m_n$$

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Suppose the final multiplication in the optimal solution occurs at  $k$ :  
 $(M_1 \times M_2 \times \dots \times M_k)(M_{k+1} \times M_{k+2} \times \dots \times M_n)$

$$C[1, n] = \underbrace{C[1, k]}_{\text{Optimal # mults for left side}} + \underbrace{C[k + 1, n]}_{\text{Optimal # mults for right side}} + \underbrace{m_k \times m_0 \times m_n}_{\text{\# mults to combine}}$$



# Matrix-Chain Multiplication

Input:  $n$  matrices  $M_1, M_2, \dots, M_n$  with dimensions of:

$$m_0 \times m_1, m_1 \times m_2, \dots, m_{n-1} \times m_n$$

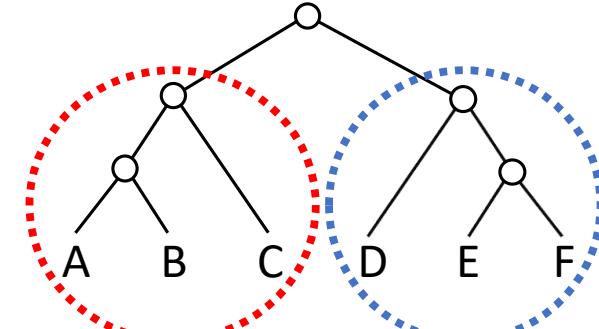
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Let  $C[i, j] =$  Minimum number of scalar multiplications needed for multiplying  $M_i \times M_{i+1} \times \dots \times M_j$

Suppose the final multiplication in the optimal solution occurs at  $k$ :

$$(M_i \times \dots \times M_k)(M_{k+1} \times \dots \times M_j)$$

$$C[i, j] = \underbrace{C[i, k]}_{\text{Optimal # mults for left side}} + \underbrace{C[k+1, j]}_{\text{Optimal # mults for right side}} + \underbrace{m_k \times m_{k-1} \times \dots \times m_j}_{\text{\# mults to combine}}$$



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Input:  $n$  matrices  $M_1, M_2, \dots, M_n$  with dimensions of:

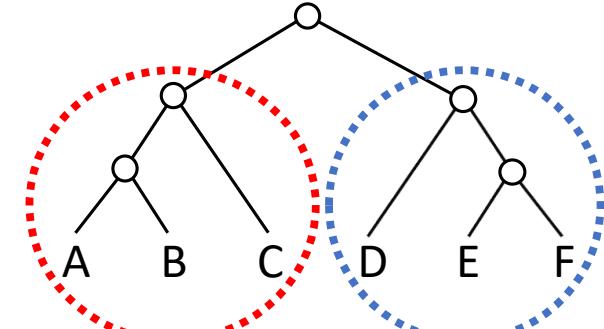
$$m_0 \times m_1, m_1 \times m_2, \dots, m_{n-1} \times m_n$$

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Suppose the final multiplication in the optimal solution occurs at  $k$ :  
 $(M_i \times \dots \times M_k)(M_{k+1} \times \dots \times M_j)$  ?!

$$C[i, j] = ???$$



# Matrix-Chain Multiplication

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$$m_0 \times m_1, m_1 \times m_2, \dots, m_{n-1} \times m_n$$

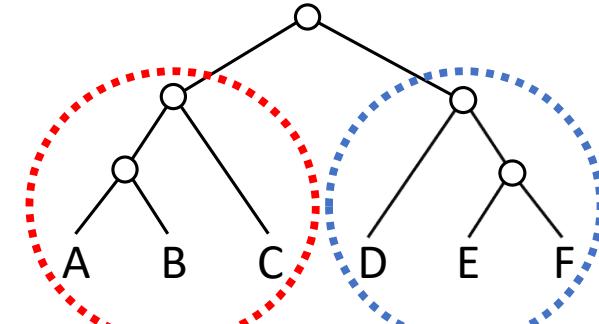
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Let  $C[i, j]$  = Minimum number of scalar multiplications needed for multiplying  $M_i \times M_{i+1} \times \dots \times M_j$

Suppose the final multiplication in the optimal solution occurs at  $k$ :

$$(M_i \times \dots \times M_k)(M_{k+1} \times \dots \times M_j)$$

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{i-1} \times m_j)$$



# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

						<i>i</i>
						1 2 3 4 5 6
<i>j</i>	1					
	2					
	3					
	4					
	5					
	6					

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

						i
						1 2 3 4 5 6
1						
2						
3						
4						
5						
6						

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>	1	2	3	4	5	6
		<i>j</i>	1	-	-	-	-	-
		2	-	-	-	-	-	-
		3	-	-	-	-	-	-
		4	-	-	-	-	-	-
		5	-	-	-	-	-	-
		6	-	-	-	-	-	-

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>					
		1	2	3	4	5	6
1		-	-	-	-	-	-
2			-	-	-	-	-
3				-	-	-	-
4					-	-	-
5						-	
6							

What can we say about  $C[i, i]$ ?

Let  $C[i, j] =$  Minimum number  
of scalar multiplications needed  
for multiplying  $M_i \times M_{i+1} \times \dots \times M_j$

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		$i$					
		1	2	3	4	5	6
1		0	-	-	-	-	-
2		0	-	-	-	-	-
3		0	-	-	-	-	-
4		0	-	-	-	-	-
5		0	-	-	-	-	-
6		0	-	-	-	-	-

What can we say about  $C[i, i]$ ?

$$C[i, i] = 0$$

Let  $C[i, j]$  = Minimum number  
of scalar multiplications needed  
for multiplying  $M_i \times M_{i+1} \times \dots \times M_j$

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>					
		1	2	3	4	5	6
1		0	-	-	-	-	-
2		0	-	-	-	-	-
3		0	-	-	-	-	-
4		0	-	-	-	-	-
5		0	-	-	-	-	-
6		0	-	-	-	-	-

Which element of  $C[]$  do we want in the end?

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>					
		1	2	3	4	5	6
1		0	-	-	-	-	-
2		0	-	-	-	-	-
3		0	-	-	-	-	-
4		0	-	-	-	-	-
5		0	-	-	-	-	-
6							0

Which element of  $C[]$  do we want in the end?

$$C[1, n]$$

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>	1	2	3	4	5	6
		1	0	-	-	-	-	-
		2	0	-	-	-	-	-
		3	0	-	-	-	-	-
		4	0	-	-	-	-	-
		5	0	-				
		6						0

What elements of  $C[]$  need to be filled out to compute  $C[1,6]$ ?

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		$i$					
		1	2	3	4	5	6
1		0	-	-	-	-	-
2		0	-	-	-	-	-
3		0	-	-	-	-	-
4		0	-	-	-	-	-
5		0	-	-	-	-	-
6		0	-	-	-	-	-

$$C[i, j] = C[1, 6]$$

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>					
		1	2	3	4	5	6
1		0	-	-	-	-	-
2		0	-	-	-	-	-
3		0	-	-	-	-	-
4		0	-	-	-	-	-
5		0	-	-	-	-	-
6		0	-	-	-	-	-

$$\begin{aligned}C[i, j] &= C[1, 6] \\&= \min_{1 \leq k < 6} (C[1, k] + C[k + 1, 6] + ??)\end{aligned}$$

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		$i$					
		1	2	3	4	5	6
1		0	-	-	-	-	-
2		0	-	-	-	-	-
3		0	-	-	-	-	-
4		0	-	-	-	-	-
5		0	-	-	-	-	-
6		0	-	-	-	-	-

$$\begin{aligned} C[i, j] &= C[1, 6] \\ &= \min_{1 \leq k < 6} (C[1, k] + C[k + 1, 6] + ??) \\ &= \min \left\{ \begin{array}{l} C[1, 1] + C[2, 6] + ?? \\ C[1, 2] + C[3, 6] + ?? \\ C[1, 3] + C[4, 6] + ?? \\ C[1, 4] + C[5, 6] + ?? \\ C[1, 5] + C[6, 6] + ?? \end{array} \right. \end{aligned}$$

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		$i$					
		1	2	3	4	5	6
1		0	-	-	-	-	-
2		0	-	-	-	-	-
3		0	-	-	-	-	-
4		0	-	-	-	-	-
5				0	-	-	-
6						0	-

$$\begin{aligned} C[3, 5] &= \min_{3 \leq k < 5} (C[3, k] + C[k + 1, 5] + m_3 \times m_4 \times m_5) \\ &= \min \left\{ C[3, 3] + C[4, 5] + m_3 \times m_4 \times m_5, C[3, 4] + C[5, 5] + m_3 \times m_4 \times m_5 \right\} \end{aligned}$$

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

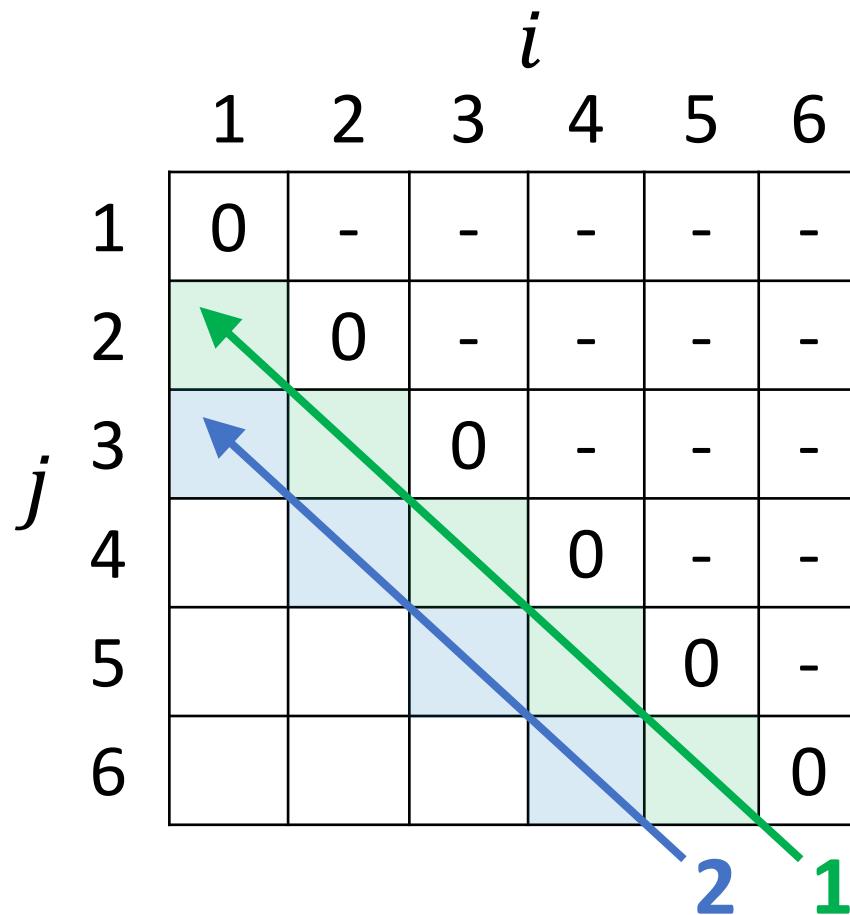
		<i>i</i>	1	2	3	4	5	6
		1	0	-	-	-	-	-
		2	0	-	-	-	-	-
		3	0	-	-	-	-	-
		4	0	-	-	-	-	-
		5	0	-	-	-	-	-
		6	0	-	-	-	-	-

*j*

1

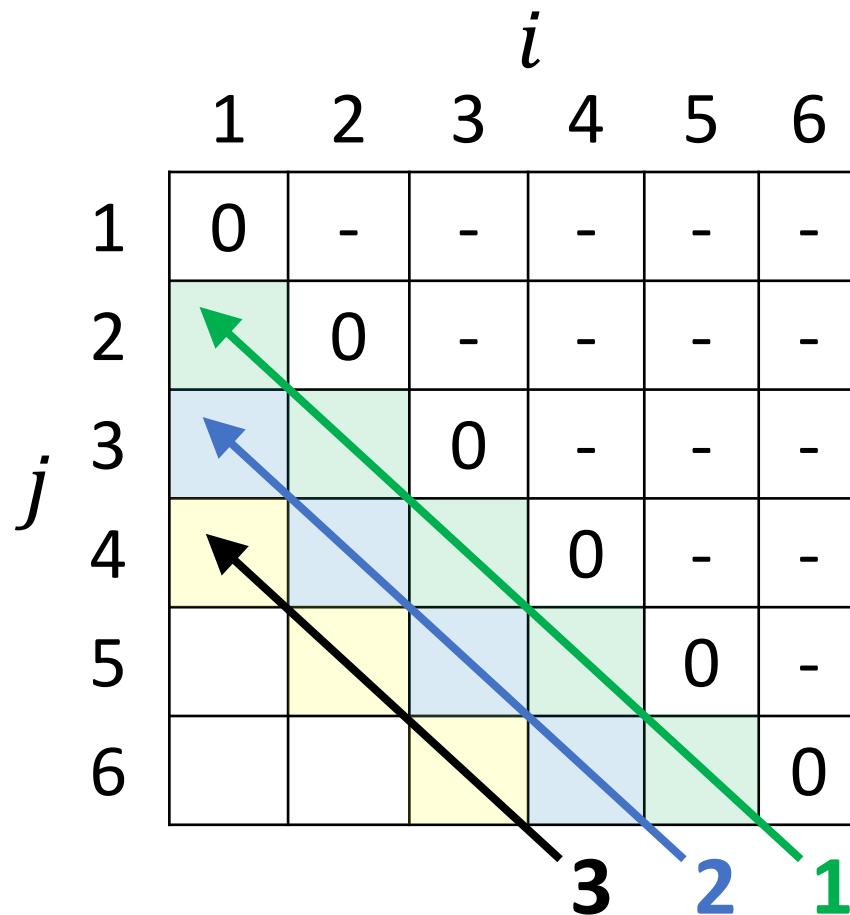
# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$



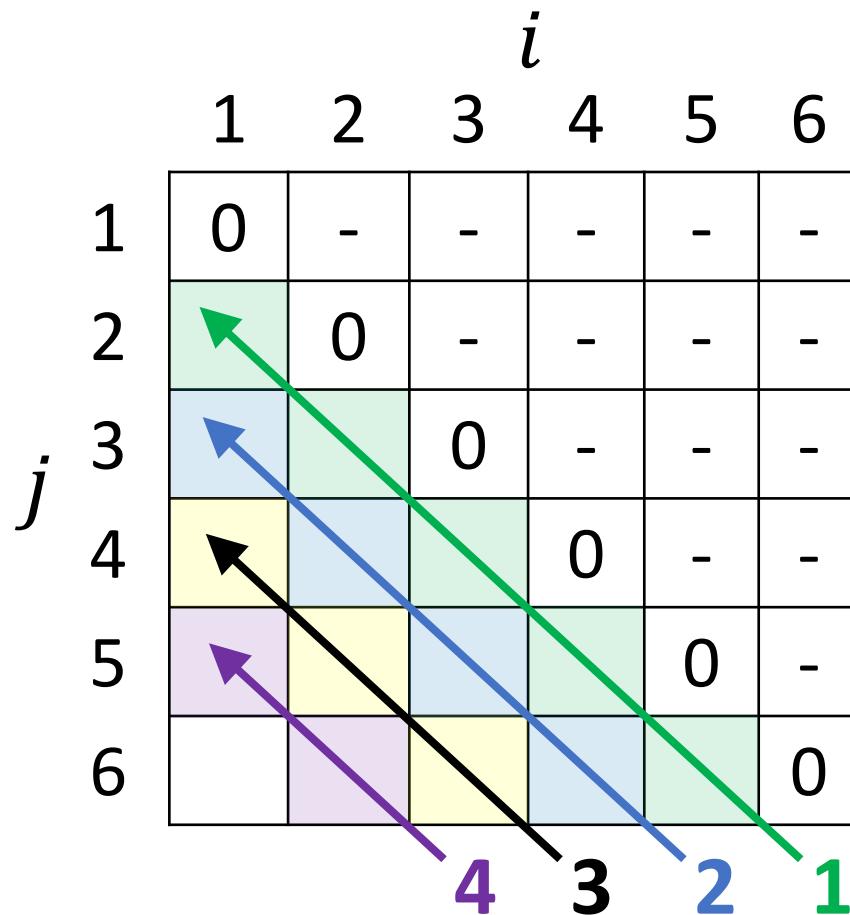
# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$



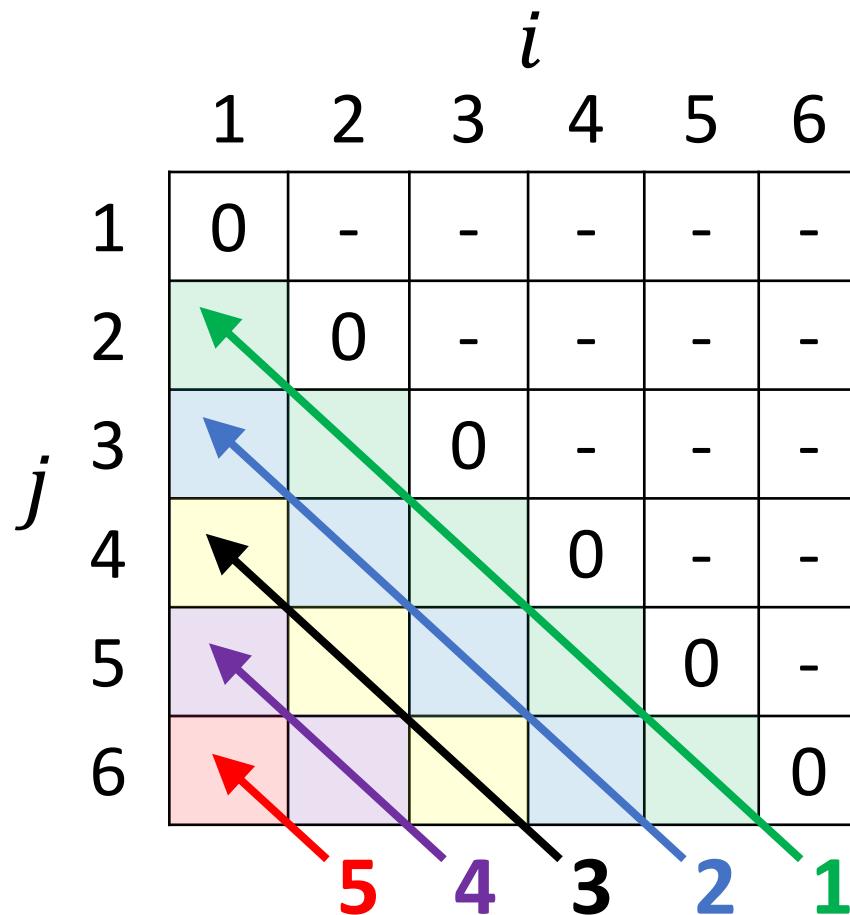
# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$



# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$



# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>					
		1	2	3	4	5	6
1		0	-	-	-	-	-
2		0	-	-	-	-	-
3		0	-	-	-	-	-
4		0	-	-	-	-	-
5		0	-	-	-	-	-
6		0	-	-	-	-	-

*i* = 1      *j* = 2

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>	1	2	3	4	5	6
		1	0	-	-	-	-	-
		2	0	-	-	-	-	-
		3	0	-	-	-	-	-
		4	0	-	-	-	-	-
		5	0	-				
		6	0					

i = 1, 2      j = 2, 3

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>	1	2	3	4	5	6
		1	0	-	-	-	-	-
		2	0	-	-	-	-	-
		3	0	-	-	-	-	-
		4	0	-	-	-	-	-
		5	0	-	-	-	-	-
		6	0	-	-	-	-	-

*i* = 1, 2, **3**      *j* = 2, 3, **4**

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>	1	2	3	4	5	6
		1	0	-	-	-	-	-
		2	0	-	-	-	-	-
		3	0	-	-	-	-	-
		4	0	-	-	-	-	-
		5	0	-	-	-	-	-
		6	0	-	-	-	-	-

*i* = 1, 2, 3, 4      *j* = 2, 3, 4, 5

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>	1	2	3	4	5	6
		1	0	-	-	-	-	-
		2	0	-	-	-	-	-
		3	0	-	-	-	-	-
		4	0	-	-	-	-	-
		5	0	-	-	-	-	-
		6	0	-	-	-	-	-

*i* = 1, 2, 3, 4, 5      *j* = 2, 3, 4, 5, 6

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>					
		1	2	3	4	5	6
1		0	-	-	-	-	-
2		0	-	-	-	-	-
3		0	-	-	-	-	-
4		0	-	-	-	-	-
5		0	-	-	-	-	-
6		0	-	-	-	-	-

*i* =   
1, 2, 3, 4, 5  
1

*j* =   
2, 3, 4, 5, 6  
3

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

	<i>i</i>	1	2	3	4	5	6
1	0	-	-	-	-	-	-
2		0	-	-	-	-	-
3			0	-	-	-	-
4				0	-	-	-
5					0	-	-
6						0	-

$i =$ 1,2,3 1,2 1	$j =$ 4,5,6 5,6 6
----------------------------	----------------------------

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

	<i>i</i>	1	2	3	4	5	6
<i>j</i>	1	0	-	-	-	-	-
2		0	-	-	-	-	-
3			0	-	-	-	-
4				0	-	-	-
5					0	-	-
6						0	

i = 1,2,3      j = 4,5,6

1,2,3,4,5      2,3,4,5,6

1,2,3,4      3,4,5,6

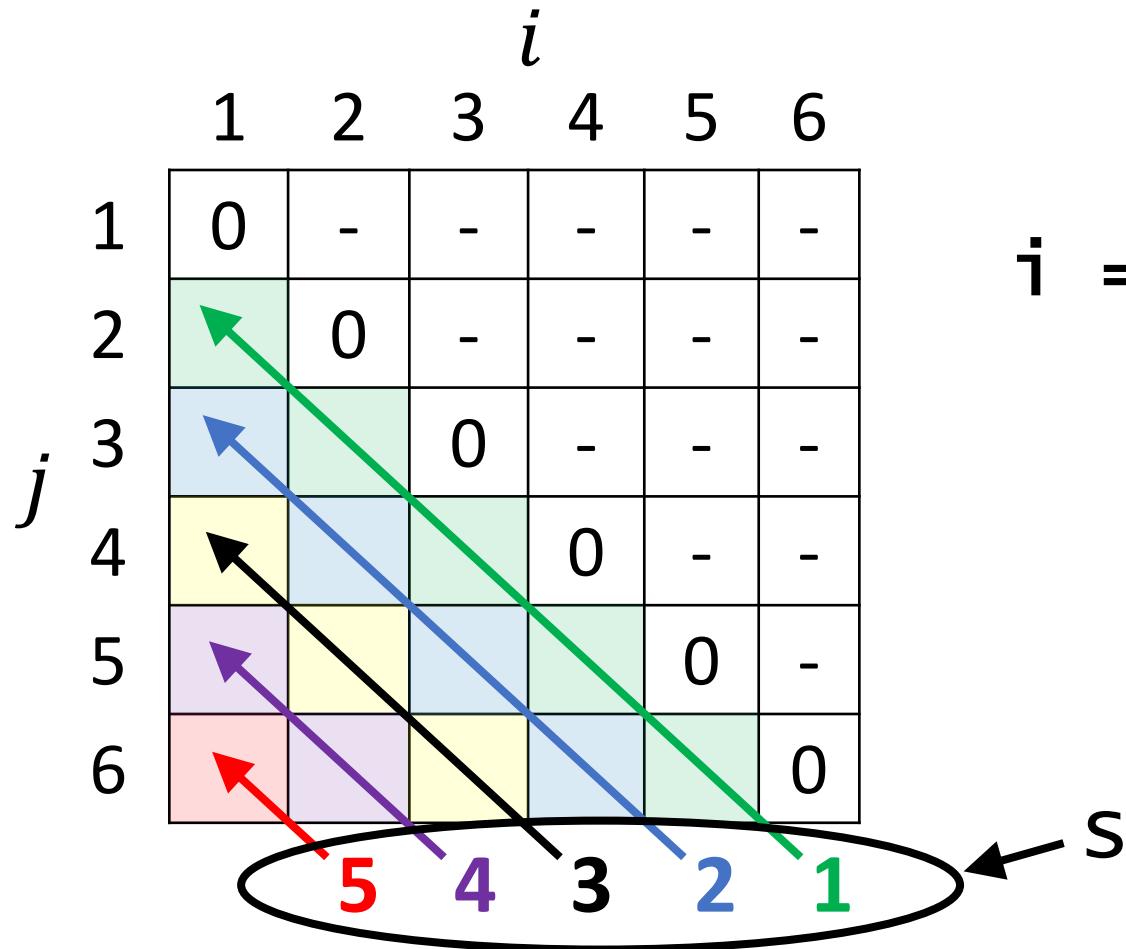
1,2      5,6

1      6

For loop(s) that will give us  
these i/j paired values?

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$



$i = 1, 2, 3, 4, 5, 6$        $j = 2, 3, 4, 5, 6$   
 $i = 1, 2, 3, 4$        $j = 3, 4, 5, 6$   
 $i = 1, 2$        $j = 4, 5, 6$   
 $i = 1$        $j = 5, 6$   
 $j = 6$

**for**  $s = 1, \dots, n - 1$   
**for**  $i = 1, \dots, n - s$   
 $j = i + s$

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>	1	2	3	4	5	6
		1	0	-	-	-	-	-
		2	0	-	-	-	-	-
		3	0	-	-	-	-	-
		4	0	-	-	-	-	-
		5	0	-	-	-	-	-
		6	0	-	-	-	-	-

```
for s = 1, ..., n - 1
  for i = 1, ..., n - s
    j = i + s
```

???

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>					
		1	2	3	4	5	6
1		0	-	-	-	-	-
2		0	-	-	-	-	-
3		0	-	-	-	-	-
4		0	-	-	-	-	-
5		0	-	-	-	-	-
6		0	-	-	-	-	-

```
for s = 1, ..., n - 1
    for i = 1, ..., n - s
        j = i + s
        min = ∞
        for k = i, ..., j - 1
            a = C[i, k] + C[k + 1, j] + m_k * m_{k+1} * m_j
            if a < min
                min = a
        C[i, j] = min
return C[1, n]
```

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>					
		1	2	3	4	5	6
1		0	-	-	-	-	-
2		0	-	-	-	-	-
3		0	-	-	-	-	-
4		0	-	-	-	-	-
5		0	-	-	-	-	-
6		0	-	-	-	-	-

```
for s = 1, ..., n - 1
    for i = 1, ..., n - s
        j = i + s
        min = ∞
        for k = i, ..., j - 1
            a = C[i, k] + C[k + 1, j] + m_k * m_{k+1} * m_j
            if a < min
                min = a
        C[i, j] = min
return C[1, n]
```

Running Time?

# Matrix-Chain Multiplication

$$C[i, j] = \min_{i \leq k < j} (C[i, k] + C[k + 1, j] + m_k \times m_{k+1} \times m_j)$$

		<i>i</i>					
		1	2	3	4	5	6
1		0	-	-	-	-	-
2		0	-	-	-	-	-
3		0	-	-	-	-	-
4		0	-	-	-	-	-
5		0	-	-	-	-	-
6		0	-	-	-	-	-

```
for s = 1, ..., n - 1
    for i = 1, ..., n - s
        j = i + s
        min = ∞
        for k = i, ..., j - 1
            a = C[i, k] + C[k + 1, j] + m_k * m_{k+1} * m_j
            if a < min
                min = a
        C[i, j] = min
return C[1, n]
```

Running Time?  
 $O(n^3)$