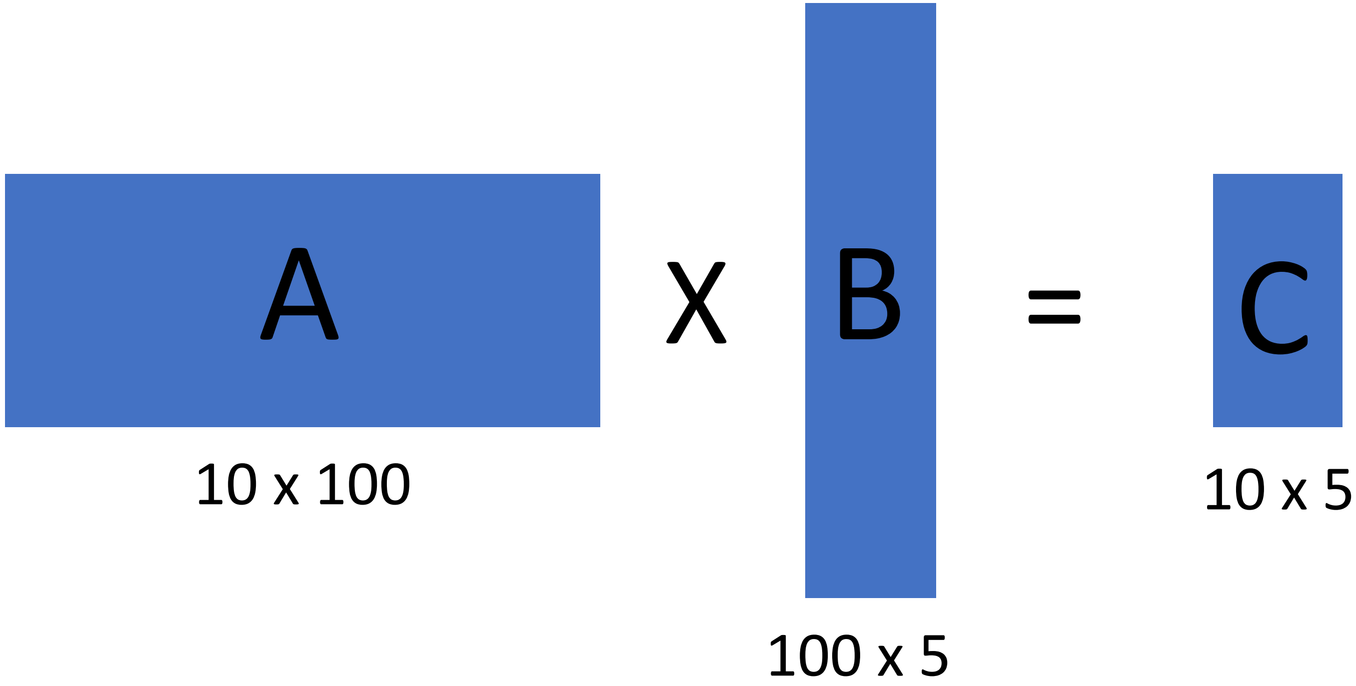
Dynamic Programming
CSCI 432

Matrix-Chain Multiplication



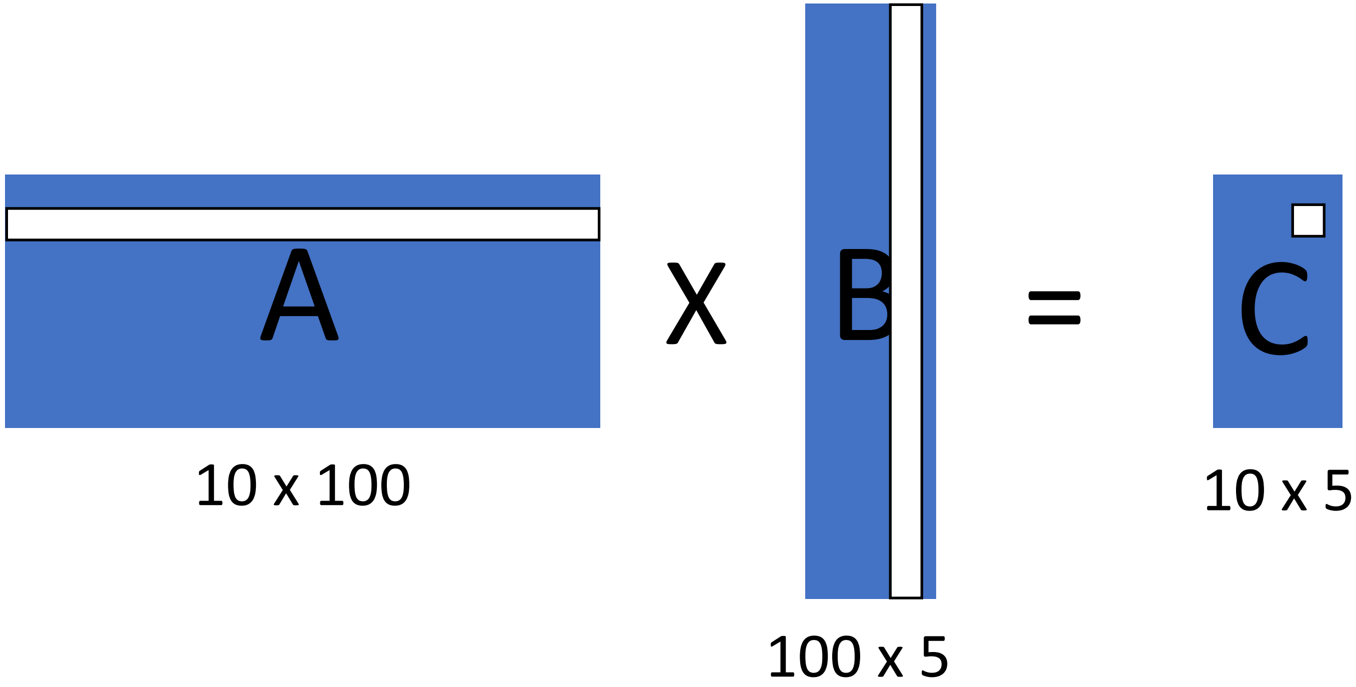
A X B = C

10 x 100

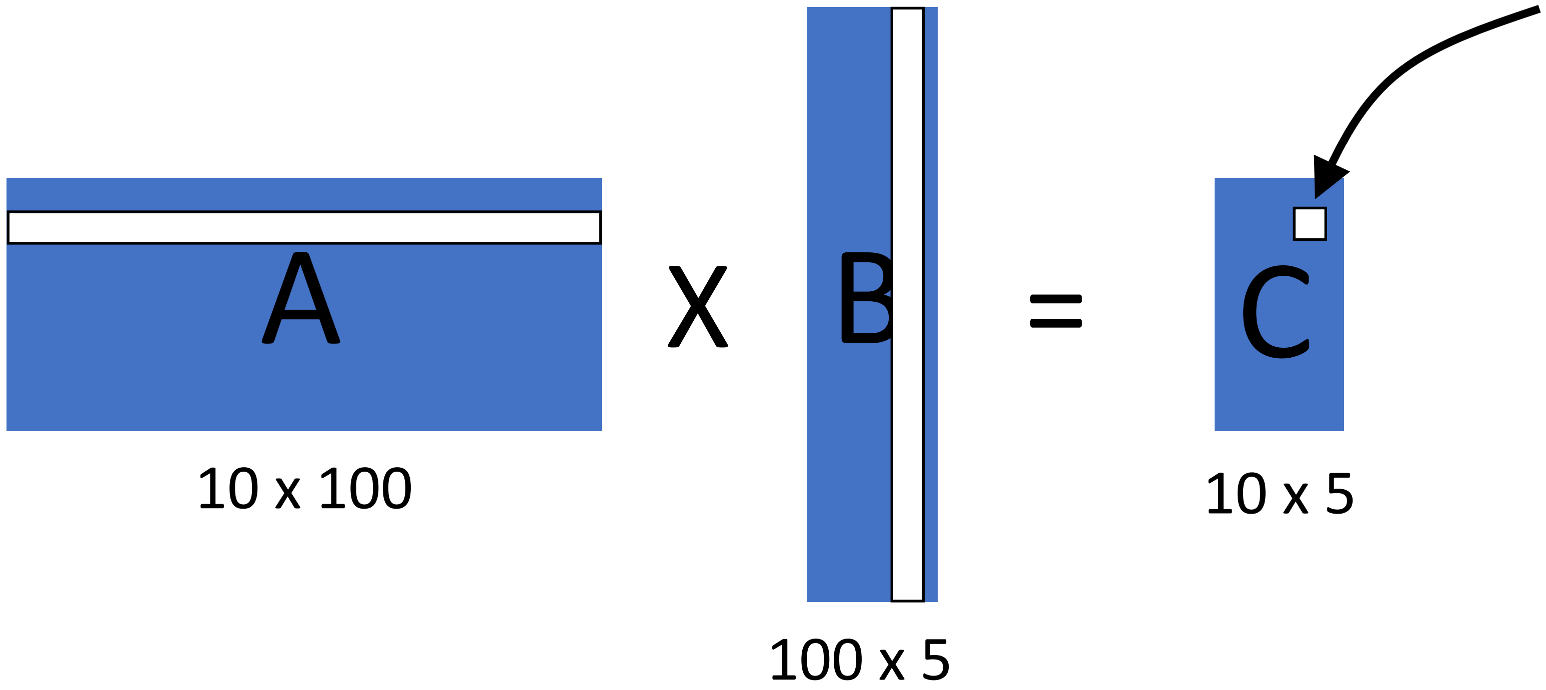
100 x 5

10 x 5

Matrix-Chain Multiplication

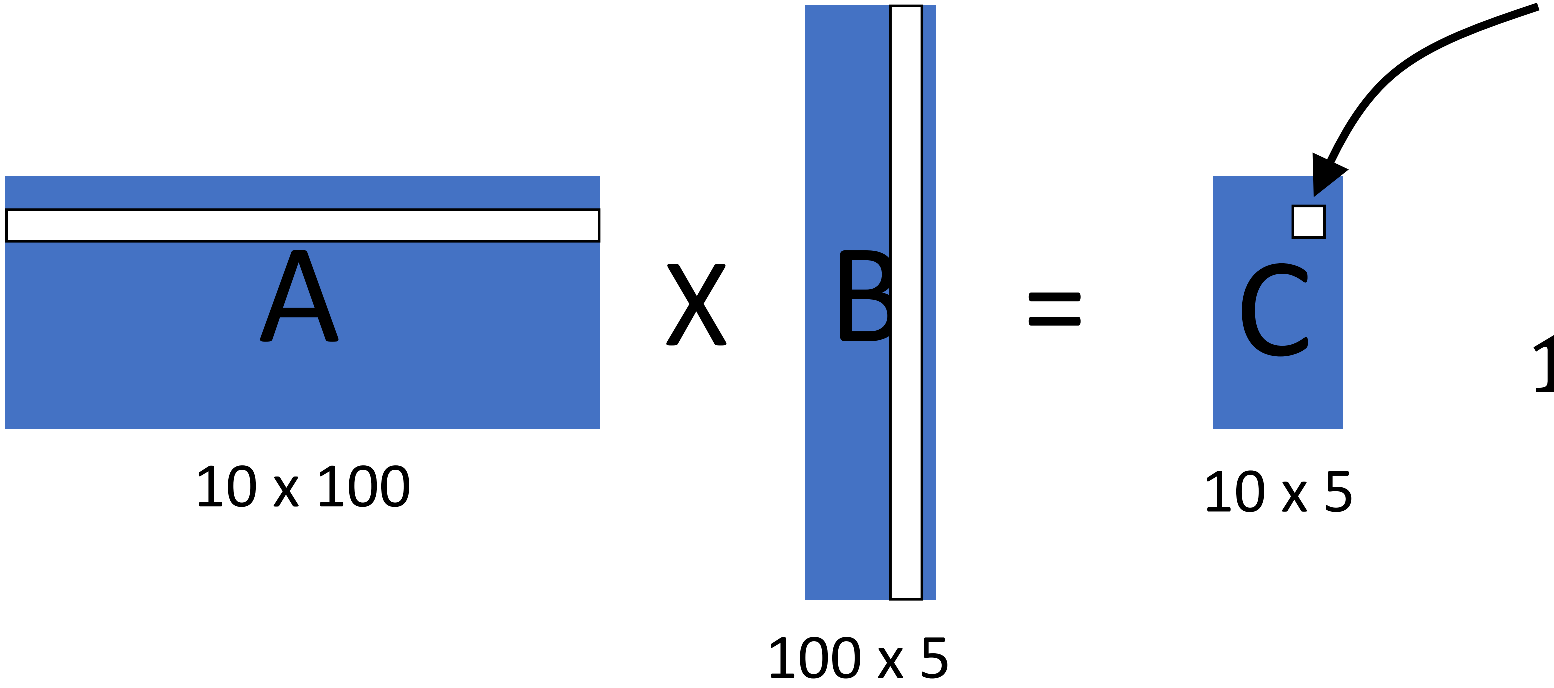


Matrix-Chain Multiplication



Number of scalar multiplications for single spot?

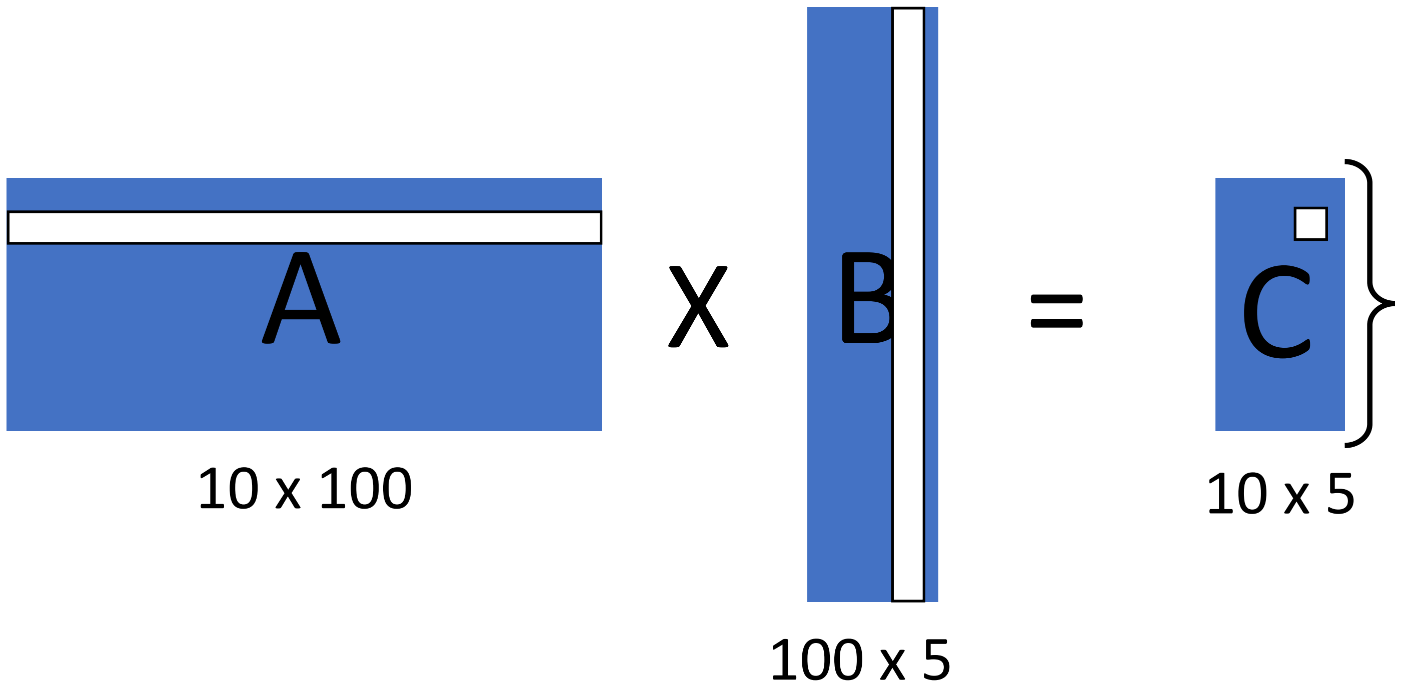
Matrix-Chain Multiplication



Number of scalar multiplications for single spot?

100

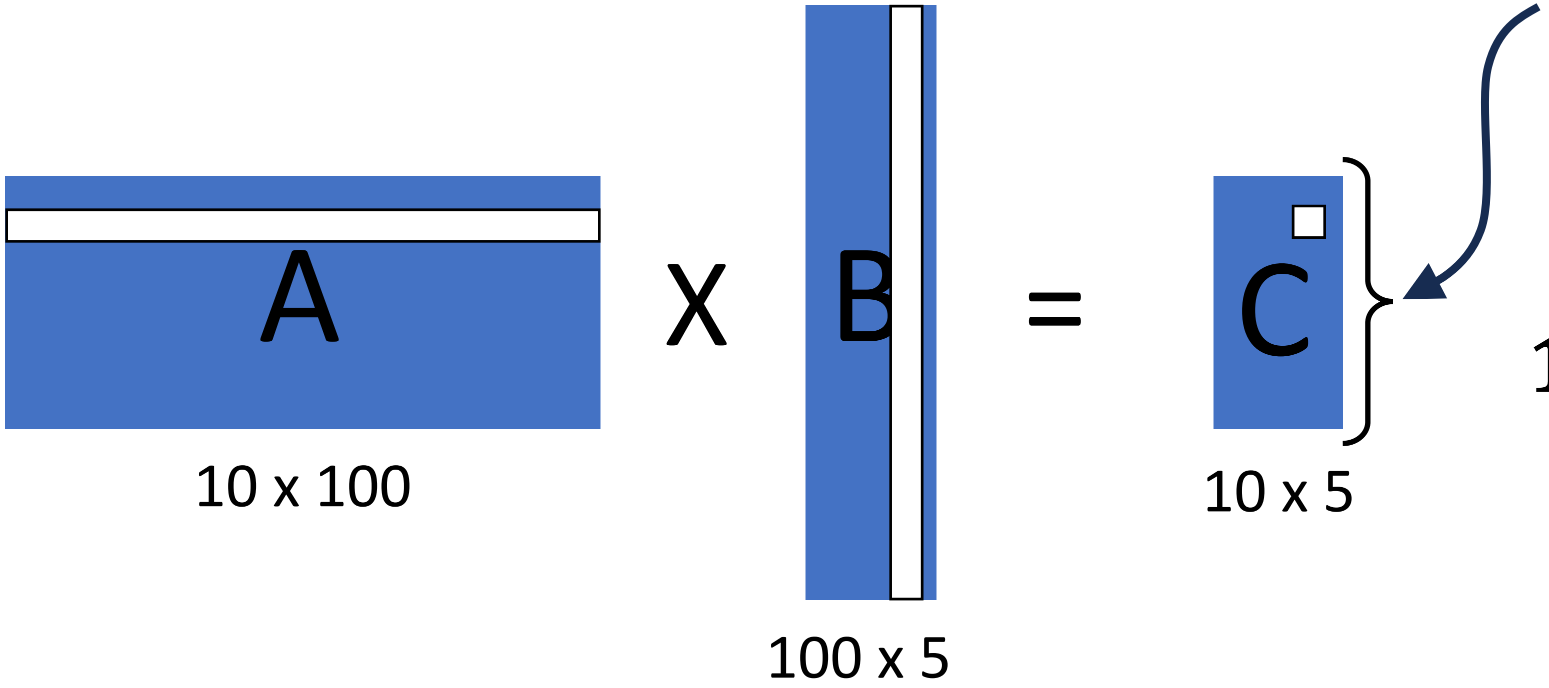
Matrix-Chain Multiplication



Number of scalar multiplications for full array?

100

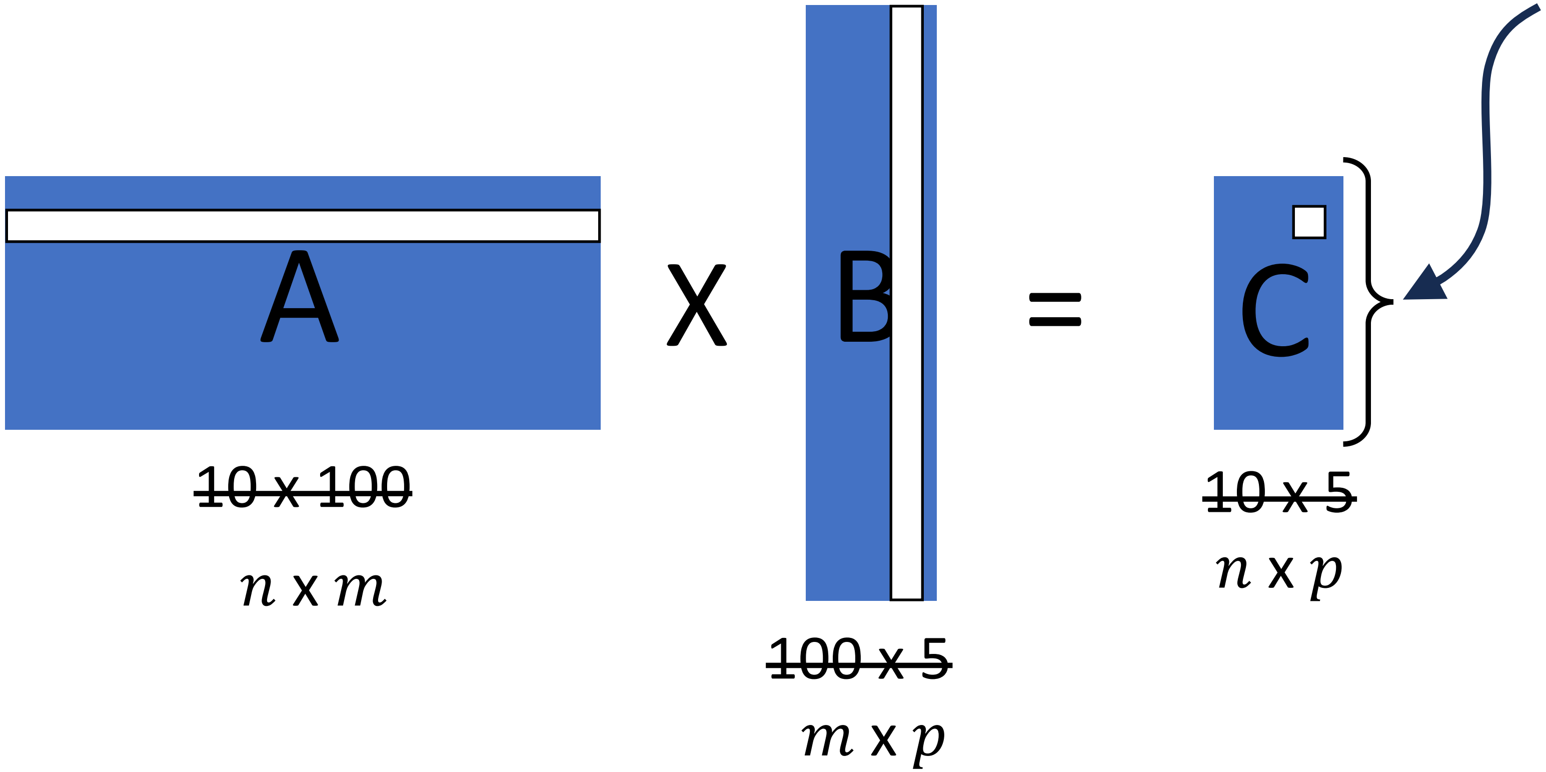
Matrix-Chain Multiplication



Number of scalar multiplications for full array?

$$\mathbf{100 \times 10 \times 5 = 5000}$$

Matrix-Chain Multiplication



A X B = C

~~10 x 100~~

n x m

~~100 x 5~~

m x p

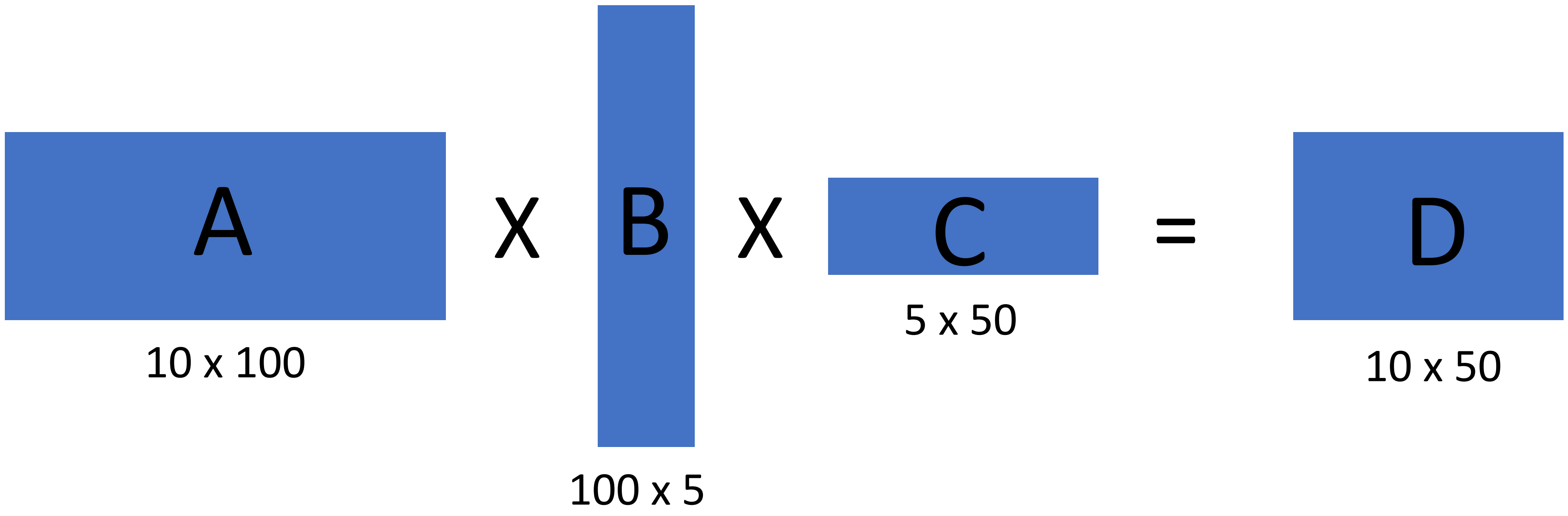
~~10 x 5~~

n x p

Number of scalar multiplications for full array?

$$\boldsymbol{m \times n \times p}$$

Matrix-Chain Multiplication



A x B x C = D

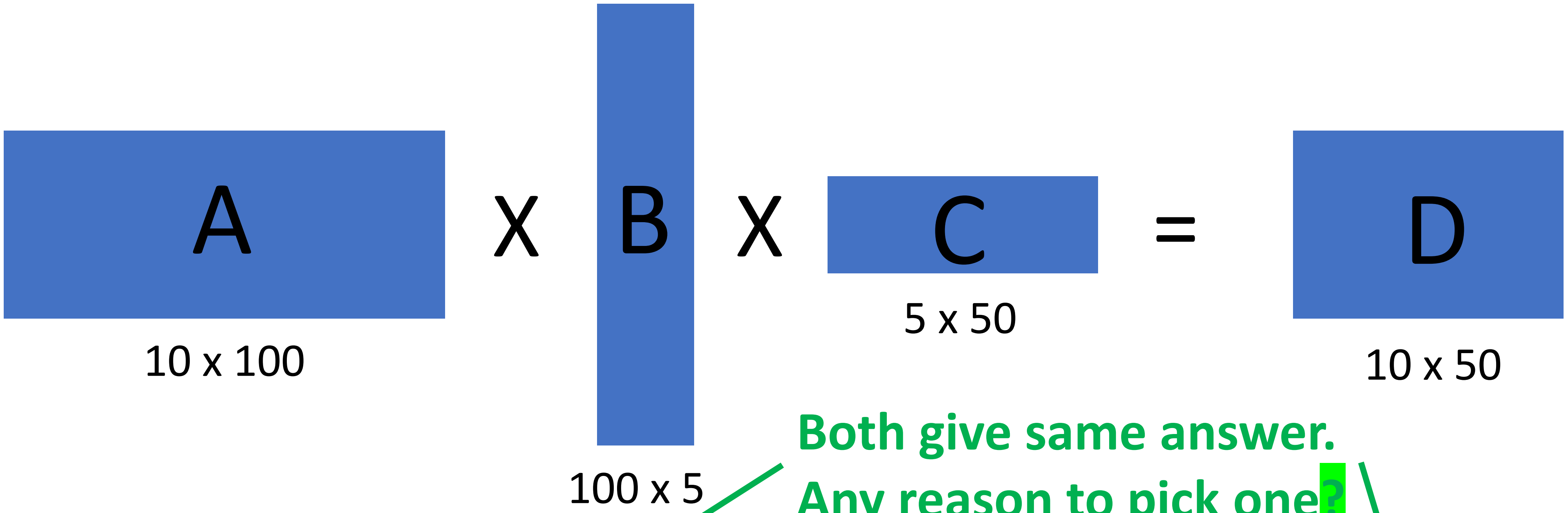
A: 10 x 100

B: 100 x 5

C: 5 x 50

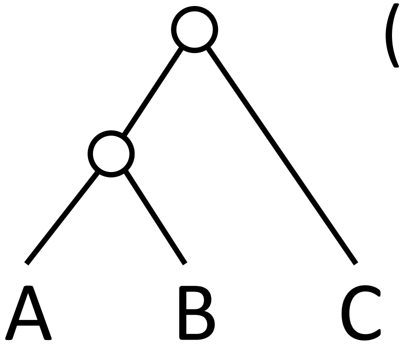
D: 10 x 50

Matrix-Chain Multiplication

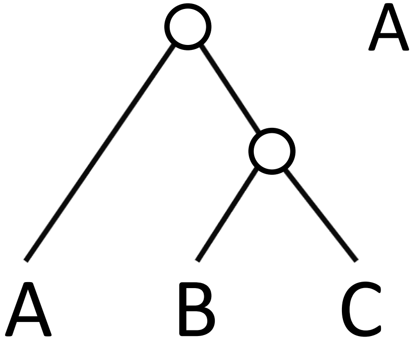


Both give same answer.

Any reason to pick one?

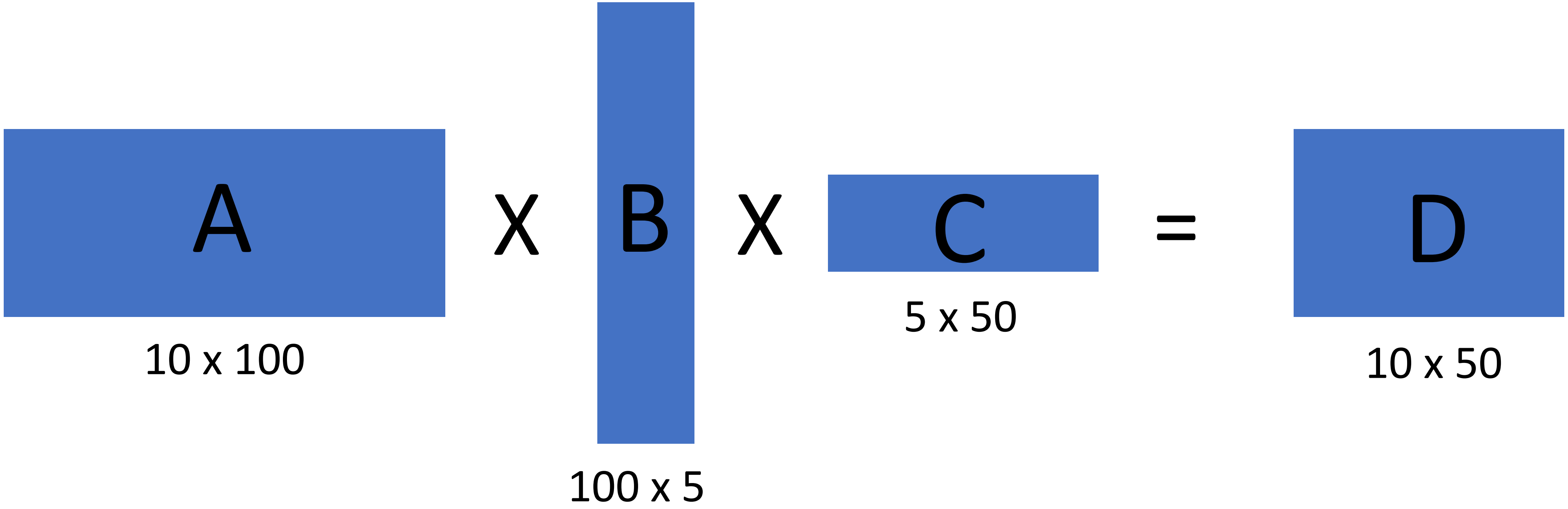


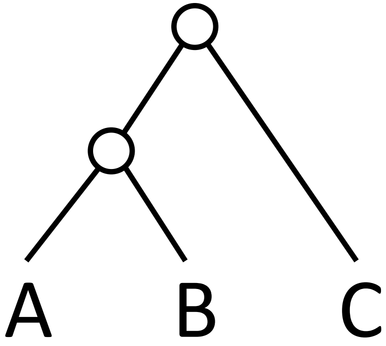
(A x B) x C :



A x (B x C) :

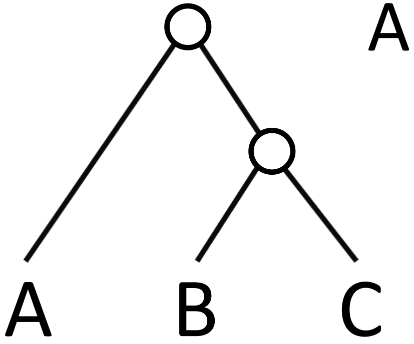
Matrix-Chain Multiplication





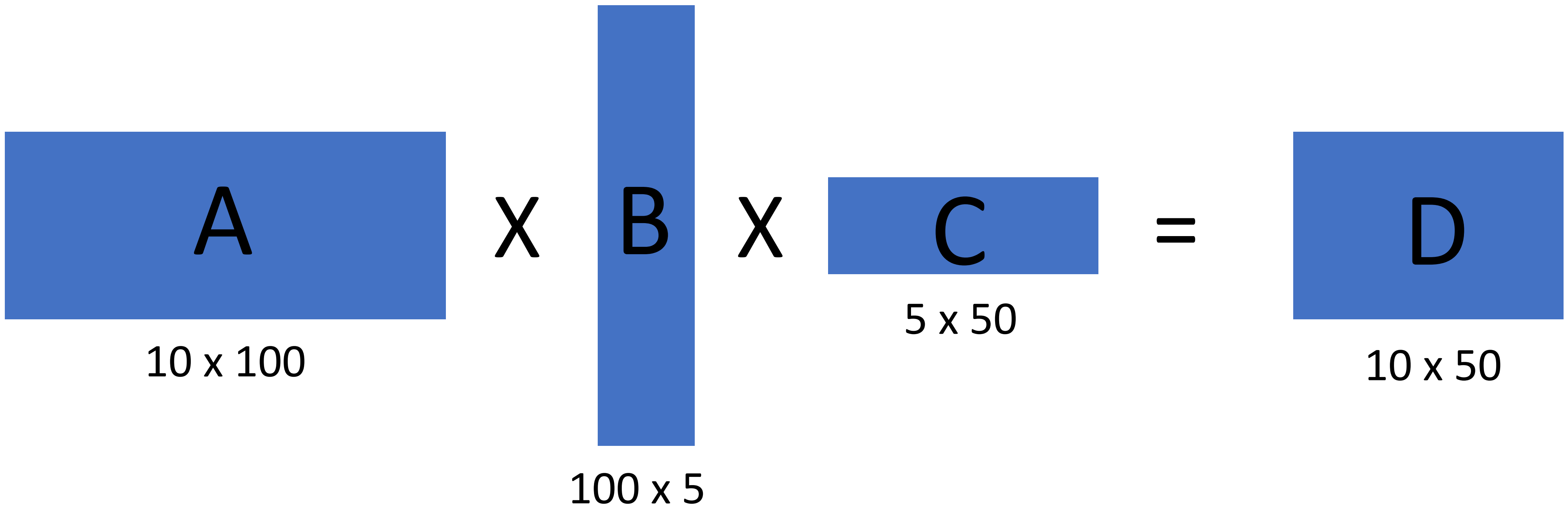
(A x B) x C :

$100 \times 10 \times 5$



A x (B x C) :

Matrix-Chain Multiplication



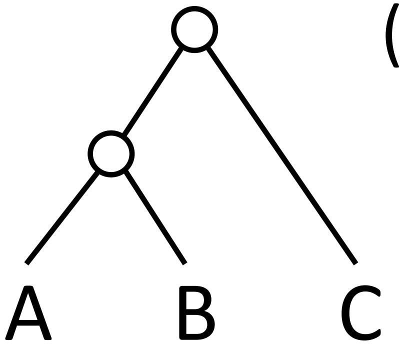
A x B x C = D

10 x 100

100 x 5

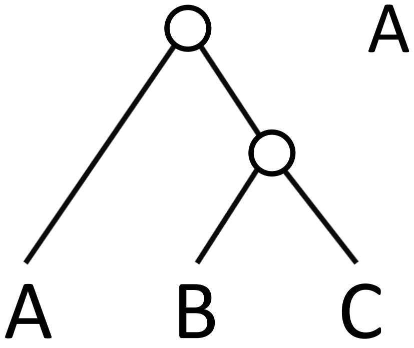
5 x 50

10 x 50



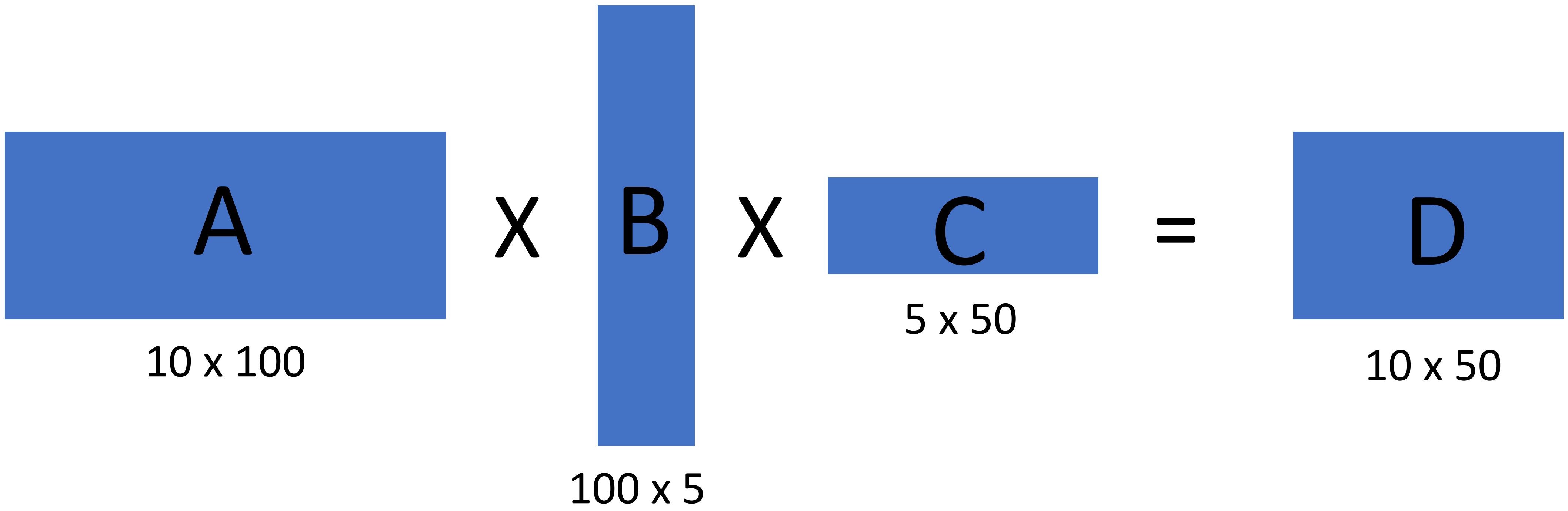
(A x B) x C :

$$100\times10\times5$$
$$+5\times10\times50$$



A x (B x C) :

Matrix-Chain Multiplication



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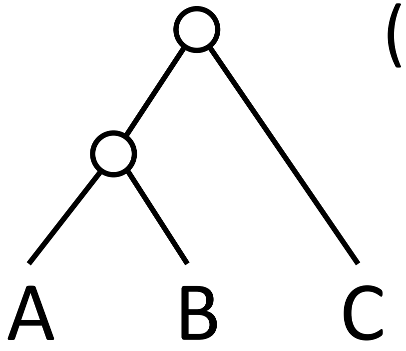
10 x 100

100 x 5

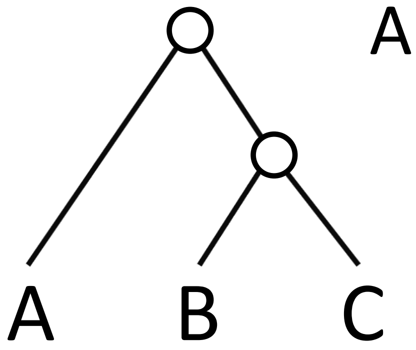
5 x 50

10 x 50



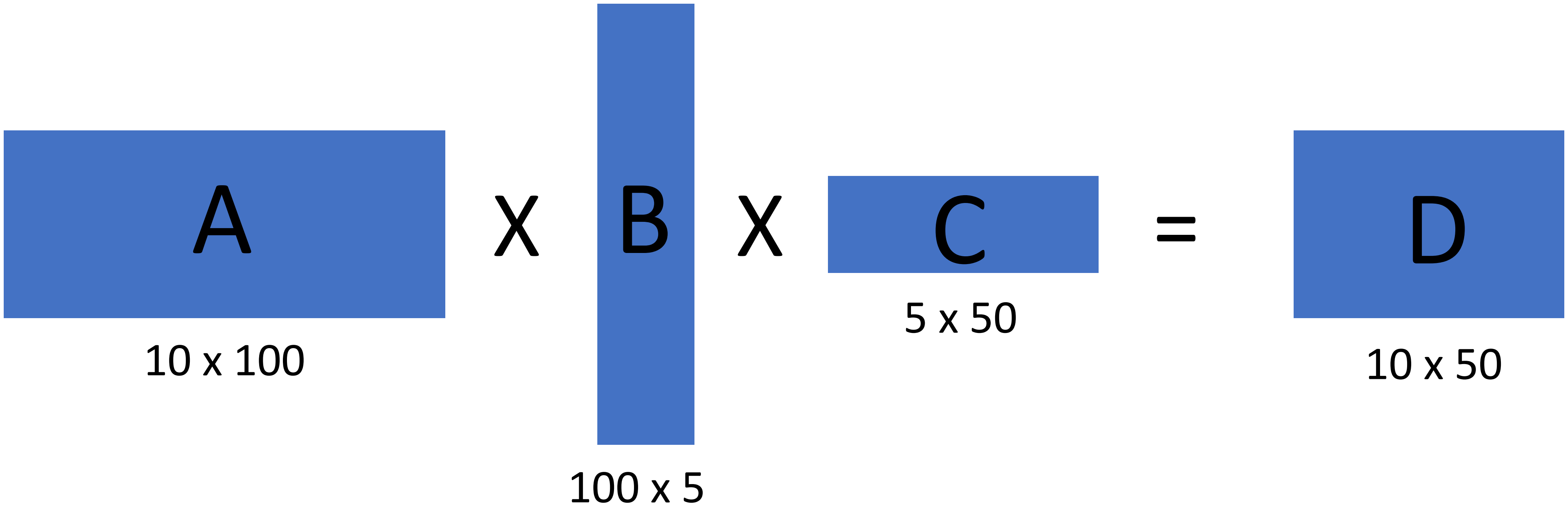
(A x B) x C :

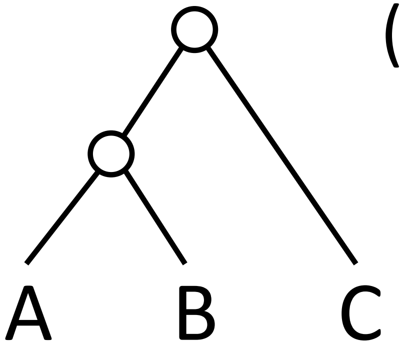
$$\begin{aligned} &100\times10\times5 \\ &+5\times10\times50 \\ &=\ 7500 \end{aligned}$$



A x (B x C) :

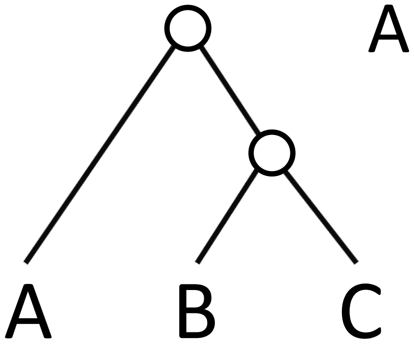
Matrix-Chain Multiplication





(A x B) x C :

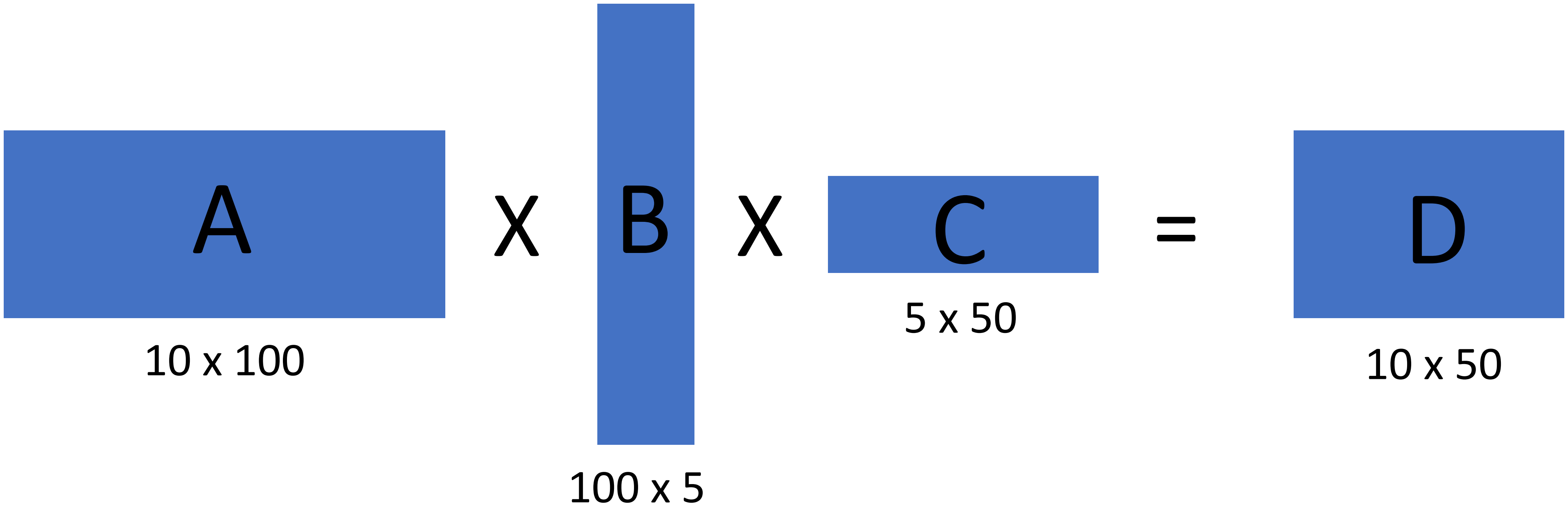
$$\begin{aligned} &100\times10\times5 \\ +&5\times10\times50 \\ =&\ 7500 \end{aligned}$$

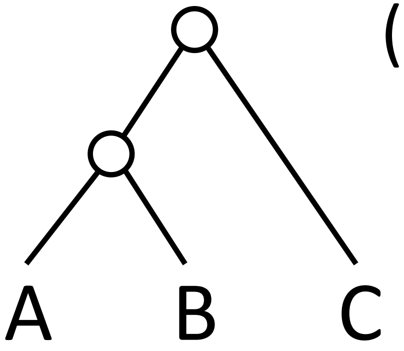


A x (B x C) :

$$\begin{aligned} &5\times100\times50 \\ +&\ 100\times10\times50 \end{aligned}$$

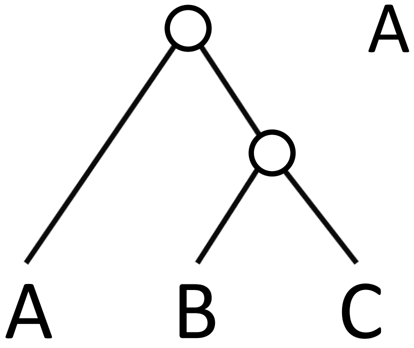
Matrix-Chain Multiplication





(A x B) x C :

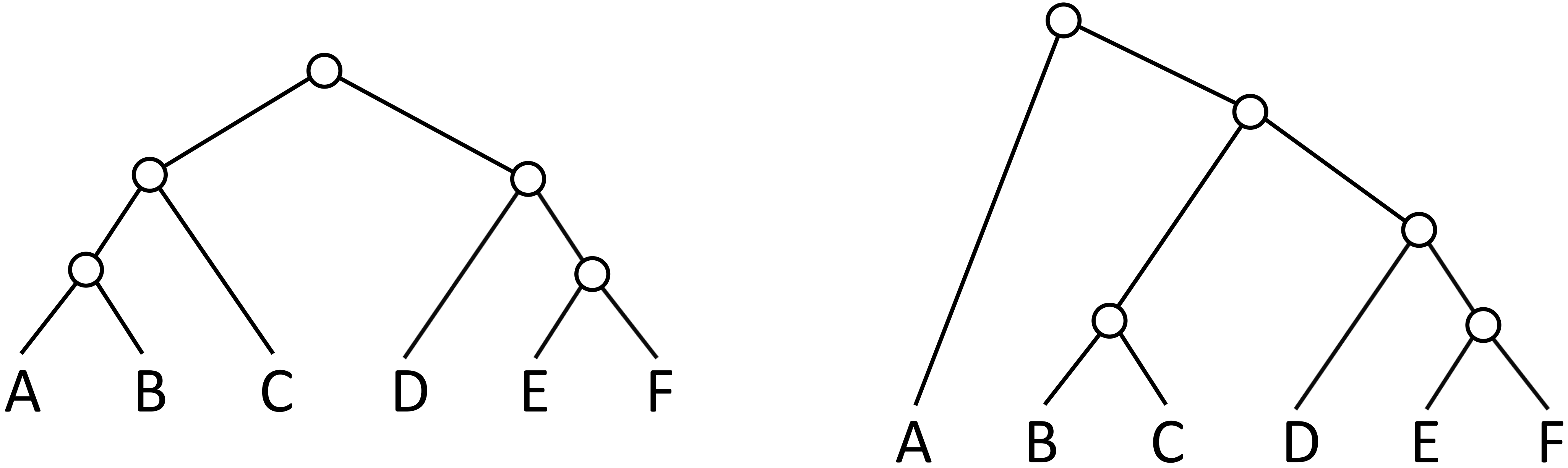
$$100 \times 10 \times 5$$
$$+5 \times 10 \times 50$$
$$= 7500$$



A x (B x C) :

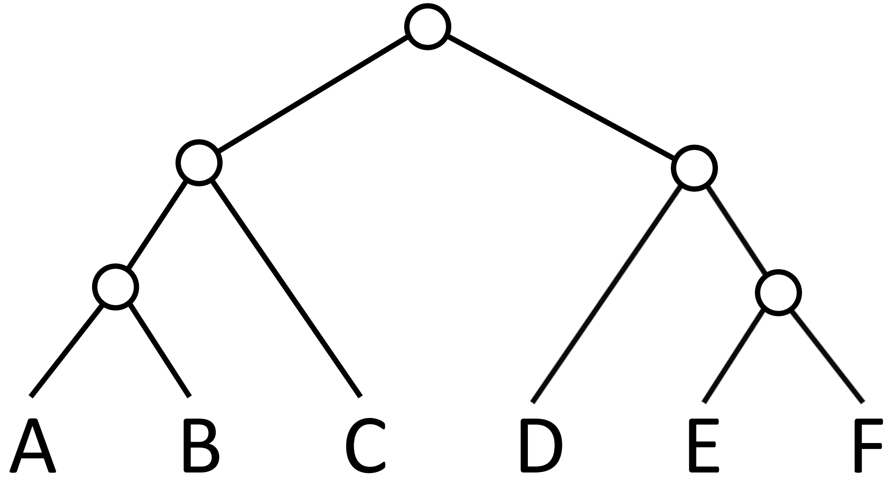
$$5 \times 100 \times 50$$
$$+ 100 \times 10 \times 50$$
$$= 75000$$

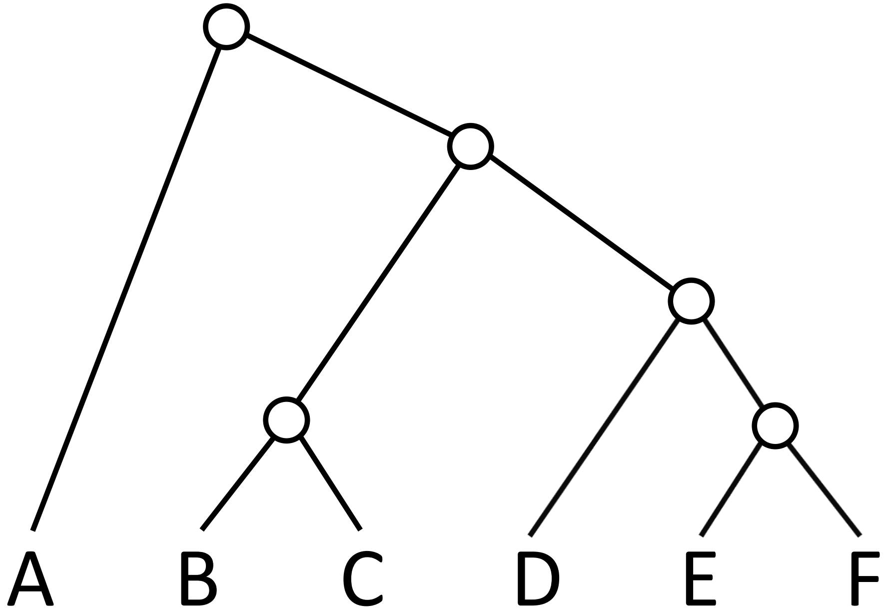
Matrix-Chain Multiplication



Can we use dynamic programming?

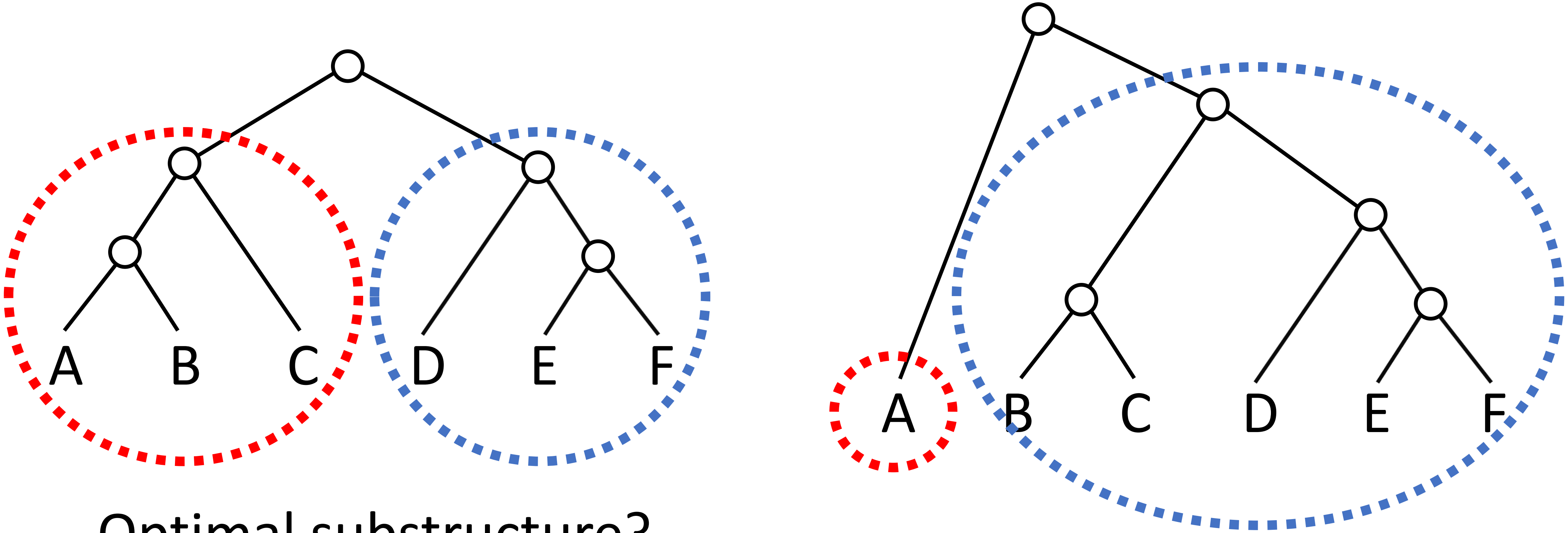
Matrix-Chain Multiplication





Optimal substructure?

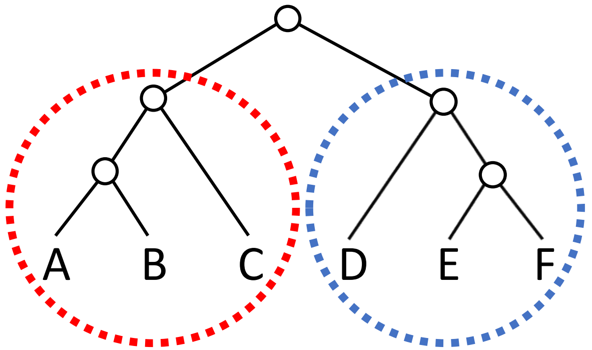
Matrix-Chain Multiplication



Optimal substructure?

An optimal parenthesization must end with multiplying two matrices. The parenthesizations that led to those two matrices must also be optimal.

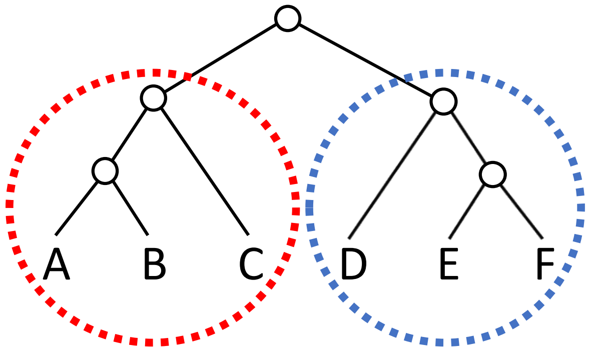
Matrix-Chain Multiplication



Input: n matrices $M_1, M_2, \ldots, M_n$ with dimensions of:

$$m_0 \times m_1,\ m_1 \times m_2,\ \ldots,\ m_{n-1} \times m_n$$

Matrix-Chain Multiplication



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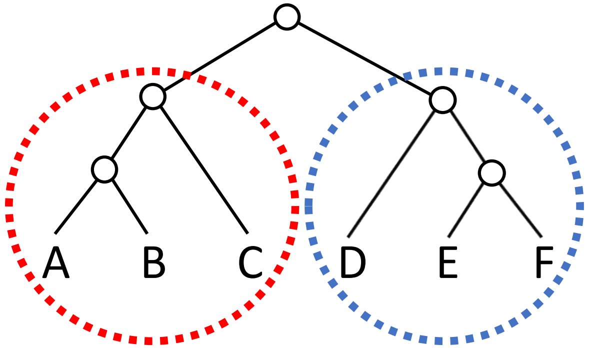
Output: Minimum number of scalar multiplications needed to calculate $M_1 \times M_2 \times \cdots \times M_n$.

Matrix-Chain Multiplication



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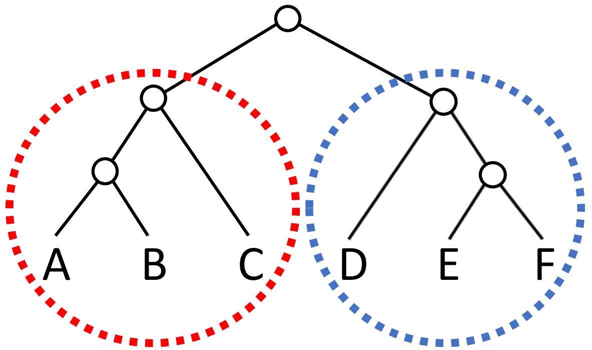
Let $C[1,n] =$ Minimum number of scalar multiplications needed for multiplying $M_1 \times M_2 \times \cdots \times M_n$

Matrix-Chain Multiplication



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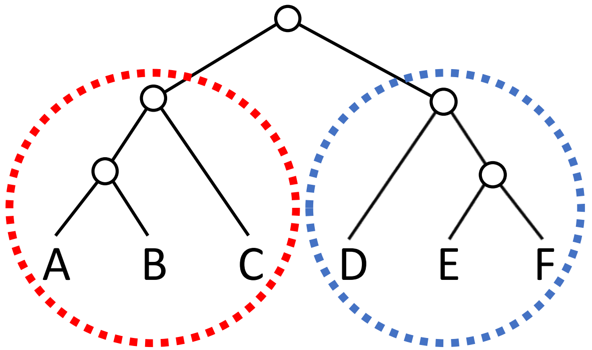
Let $C[1, n] =$ Minimum number of scalar multiplications needed for multiplying $M_1 \times M_2 \times \cdots \times M_n$

Suppose the final multiplication in the optimal solution occurs at k:

$$(M_1 \times M_2 \times \cdots \times M_k)(M_{k+1} \times M_{k+2} \times \cdots \times M_n)$$

Matrix-Chain Multiplication



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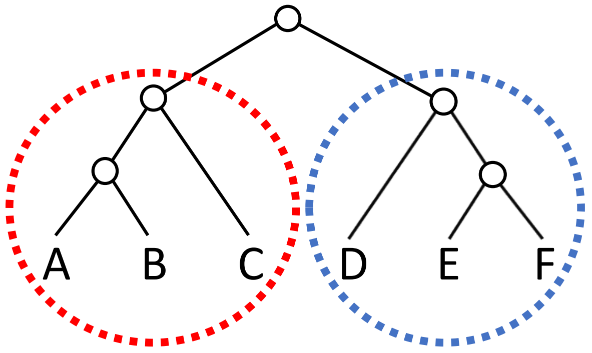
$$\boldsymbol{C[1, n]} = \textbf{???}$$

Matrix-Chain Multiplication



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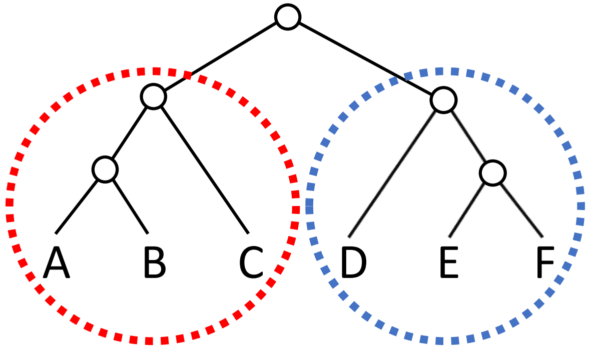
$$\boldsymbol{C[1,n]} = \overbrace{\boldsymbol{C[1,k]}}^{\text{Optimal \# mults for left side}}$$

Matrix-Chain Multiplication



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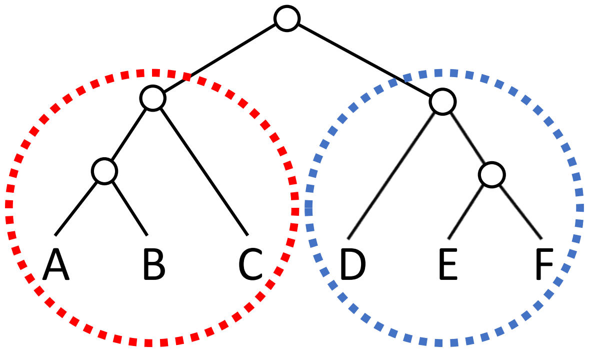
Let $C[1, n]$ = Minimum number of scalar multiplications needed for multiplying $M_1 \times M_2 \times \cdots \times M_n$

Suppose the final multiplication in the optimal solution occurs at k:

$$(M_1 \times M_2 \times \cdots \times M_k)(M_{k+1} \times M_{k+2} \times \cdots \times M_n)$$

$$\boldsymbol{C[1,n] = \overbrace{C[1,k]}^{\text{Optimal \# mults for left side}} + \overbrace{C[k+1,n]}^{\text{Optimal \# mults for right side}}}$$

Matrix-Chain Multiplication



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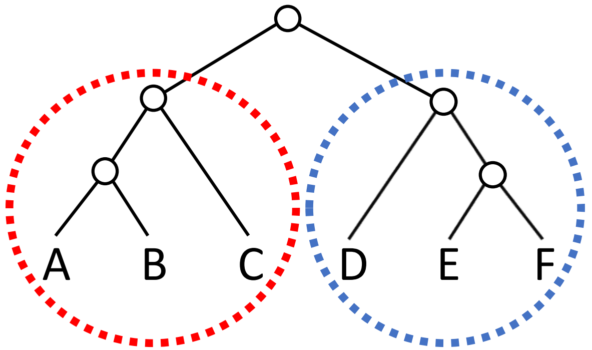
Suppose the final multiplication in the optimal solution occurs at k:
$(M_1 \times M_2 \times \cdots \times M_k)(M_{k+1} \times M_{k+2} \times \cdots \times M_n)$

$$\boldsymbol{C[1,n] = \overbrace{C[1,k]}^{\text{Optimal \# mults for left side}} + \overbrace{C[k+1,n]}^{\text{Optimal \# mults for right side}} + \overbrace{m_k \times m_0 \times m_n}^{\text{\# mults to combine}}}$$

Matrix-Chain Multiplication



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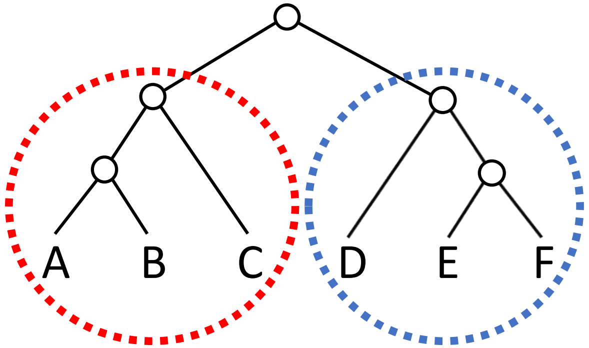
Let $C[i,j] =$ Minimum number of scalar multiplications needed for multiplying $M_i \times M_{i+1} \times \cdots \times M_j$

Suppose the final multiplication in the optimal solution occurs at k:

$$(M_i \times \cdots \times M_k)(M_{k+1} \times \cdots \times M_j)$$

$$\boldsymbol{C[i,j] = \overbrace{C[i,k]}^{\text{Optimal \# mults for left side}} + \overbrace{C[k+1,j]}^{\text{Optimal \# mults for right side}} + \overbrace{m_k \times m_{i-1} \times m_j}^{\text{\# mults to combine}}}$$

Matrix-Chain Multiplication



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Let $C[i,j] =$ Minimum number of scalar multiplications needed for multiplying $M_i \times M_{i+1} \times \cdots \times M_j$

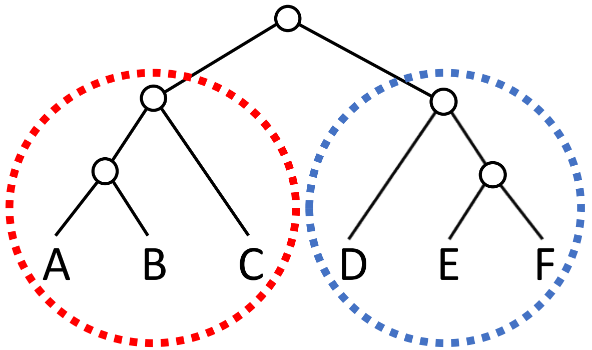
Suppose the final multiplication in the optimal solution occurs at k:

$$(M_i \times \cdots \times M_k)\big(M_{k+1} \times \cdots \times M_j\big)$$

?

$\boldsymbol{C[i,j]} =$ **???**

Matrix-Chain Multiplication



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Let $C[i,j]$ = Minimum number of scalar multiplications needed for multiplying $M_i \times M_{i+1} \times \cdots \times M_j$

Suppose the final multiplication in the optimal solution occurs at k:

$$(M_i \times \cdots \times M_k)(M_{k+1} \times \cdots \times M_j)$$

$$\boldsymbol{C[i,j] = \min_{i \le k < j} \left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j \right)}$$

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$

Matrix-Chain Multiplication

$$\boldsymbol{C[i,j] = \min_{i\le k<j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)}$$

i

		1	2	3	4	5	6
	1						
	2						
j	3						
	4						
	5						
	6						

Matrix-Chain Multiplication

$$\boldsymbol{C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)}$$

		1	2	3	4	5	6
				i			
	1						
	2						
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	4						
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Matrix-Chain Multiplication

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i

j	1	2	3	4	5	6
1		-	-	-	-	-
2			-	-	-	-
3				-	-	-
4					-	-
5						-
6						

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$

i

j

	1	2	3	4	5	6
1		-	-	-	-	-
2			-	-	-	-
3				-	-	-
4					-	-
5						-
6						

What can we say about $C[i,i]$?

Let $C[i,j]$ = Minimum number of scalar multiplications needed for multiplying $M_i \times M_{i+1} \times \cdots \times M_j$

Matrix-Chain Multiplication

$$\boldsymbol{C[i,j] = \min_{i\le k<j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)}$$

i

j

	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

What can we say about $C[i,i]$?

$$C[i,i] = 0$$

Let $C[i,j]$ = Minimum number of scalar multiplications needed for multiplying $M_i \times M_{i+1} \times \cdots \times M_j$

Matrix-Chain Multiplication

$$\boldsymbol{C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)}$$

i

j

	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

Which element of $C[]$ do we want in the end?

Matrix-Chain Multiplication

$$\boldsymbol{C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)}$$

		i					
		1	2	3	4	5	6
	1	0	-	-	-	-	-
	2		0	-	-	-	-
j	3			0	-	-	-
	4				0	-	-
	5					0	-
	6						0

Which element of $C[]$ do we want in the end?

$$C[1,n]$$

Matrix-Chain Multiplication

$$\boldsymbol{C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)}$$

i

j

	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

What elements of $C[]$ need to be filled out to compute $C[1,6]$?

Matrix-Chain Multiplication

$$\boldsymbol{C[i,j] = \min_{i\le k<j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)}$$

i

j

	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

$$C[i,j]$$
$$= C[1,6]$$

Matrix-Chain Multiplication

$$C[i,j] = \min_{i\le k<j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$

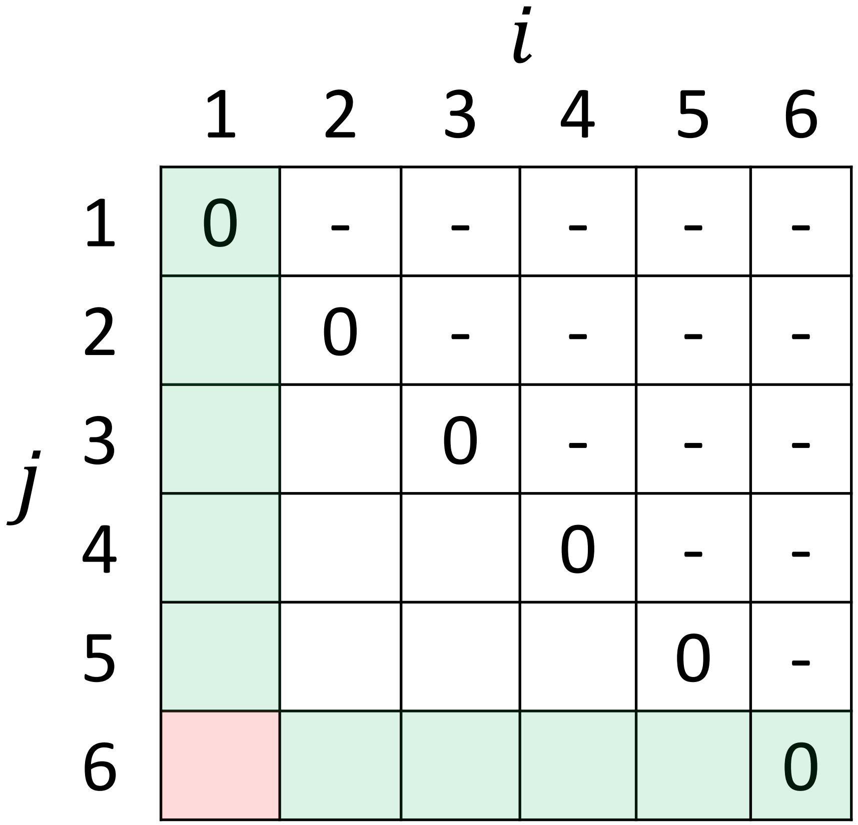
i

		1	2	3	4	5	6
	1	0	-	-	-	-	-
	2		0	-	-	-	-
j	3			0	-	-	-
	4				0	-	-
	5					0	-
	6						0

$$\begin{aligned} &C[i,j] \\ &\quad = C[1,6] \\ &\quad = \min_{1\le k<6}\left(C[1,k] + C[k+1,6] + ??\right) \end{aligned}$$

Matrix-Chain Multiplication

$$\boldsymbol{C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)}$$

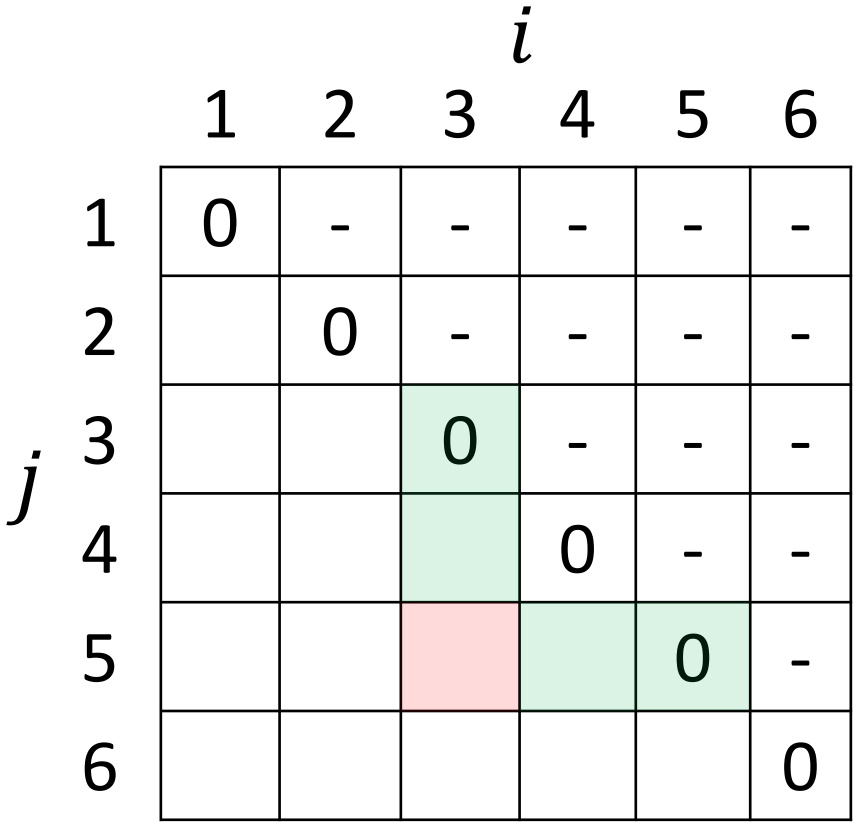


		i					
		1	2	3	4	5	6
	1	0	-	-	-	-	-
	2		0	-	-	-	-
j	3			0	-	-	-
	4				0	-	-
	5					0	-
	6						0

$$\begin{aligned} &C[i,j] \\ &\quad = C[1,6] \\ &\quad = \min_{1 \le k < 6}\left(C[1,k] + C[k+1,6] + ??\right) \\ &\quad = \min \begin{cases} C[1,1] + C[2,6] + ?? \\ C[1,2] + C[3,6] + ?? \\ C[1,3] + C[4,6] + ?? \\ C[1,4] + C[5,6] + ?? \\ C[1,5] + C[6,6] + ?? \end{cases} \end{aligned}$$

Matrix-Chain Multiplication

$$C[i,j] = \min_{i\le k<j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$



i

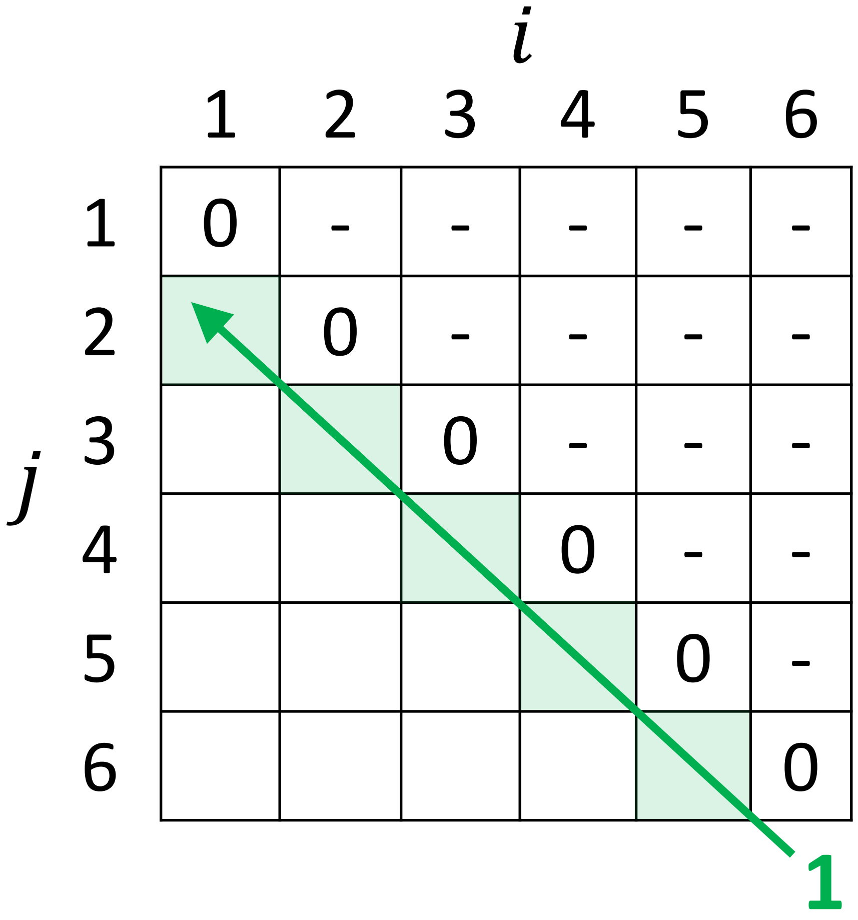
j

	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4			🟩	0	-	-
5			🟥	🟩	0	-
6						0

$$\begin{aligned} C[i,j] &= C[3,5] \\ &= \min_{3\le k<5}\left(C[3,k] + C[k+1,5] + ??\right) \\ &= \min\begin{cases} C[3,3] + C[4,5] + ?? \\ C[3,4] + C[5,5] + ?? \end{cases} \end{aligned}$$

Matrix-Chain Multiplication

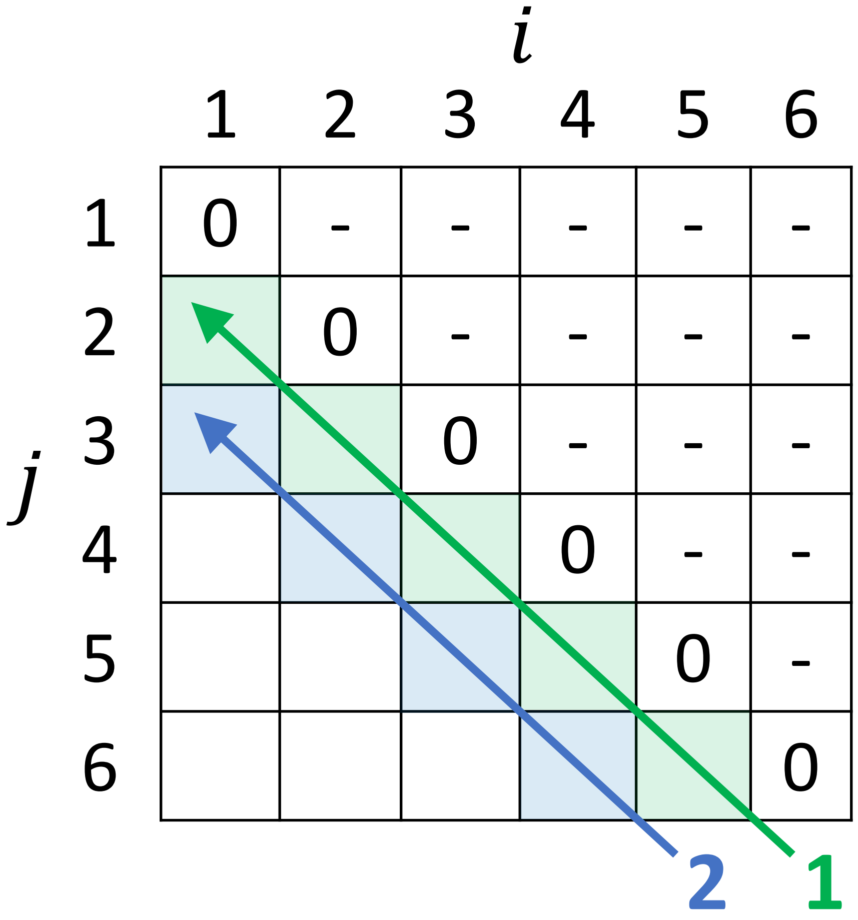
$$\boldsymbol{C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)}$$



j \ i	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

Matrix-Chain Multiplication

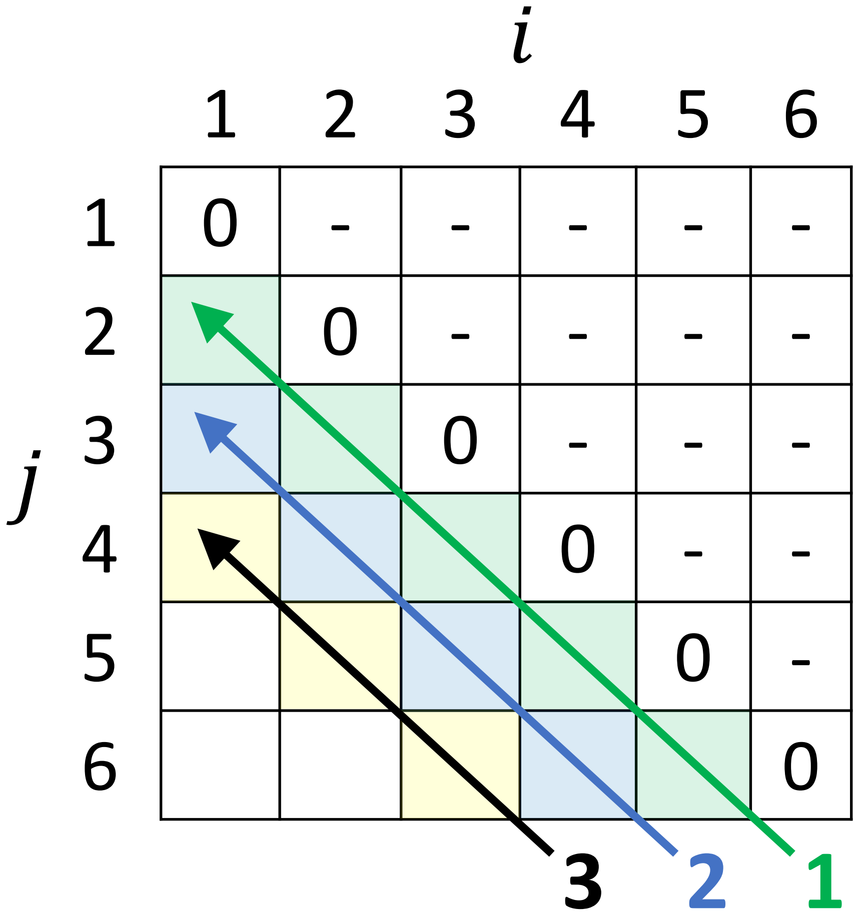
$$C[i,j] = \min_{i\le k<j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$



	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

Matrix-Chain Multiplication

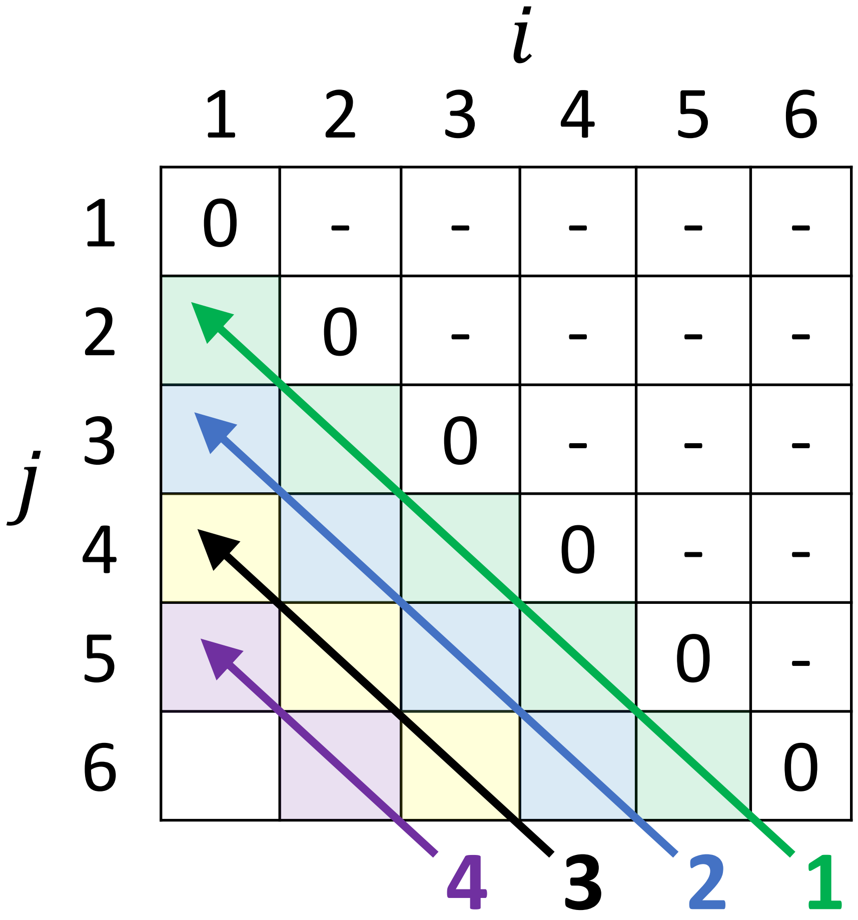
$$\boldsymbol{C[i,j] = \min_{i\le k<j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)}$$



	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

Matrix-Chain Multiplication

$$C[i,j] = \min_{i\le k<j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$



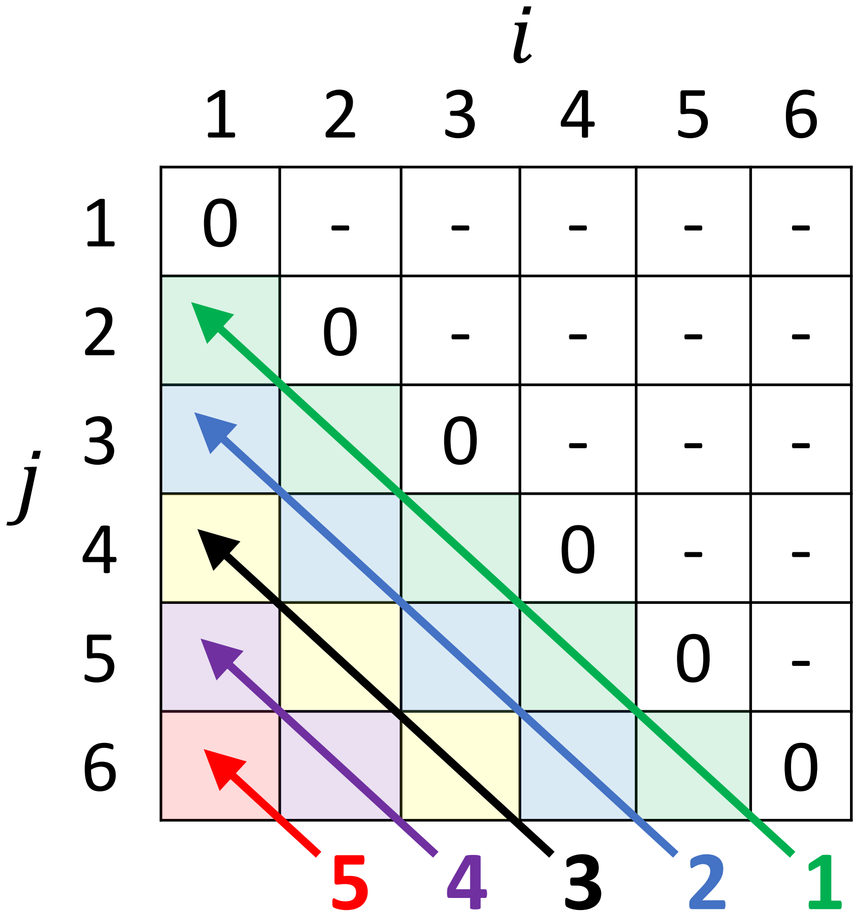
i

j

	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$



	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

Matrix-Chain Multiplication

$$C[i,j] = \min_{i\le k<j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$

i

j	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

`1`

`2`

`i =`

`j =`

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$

i

j	1	2	3	4	5	6
1	0	-	-	-	-	-
2	🟩	0	-	-	-	-
3		🟩	0	-	-	-
4				0	-	-
5					0	-
6						0

`1,2`

`i =`

`2,3`

`j =`

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$

i

j

	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

`1,2,3`

`2,3,4`

`i =`

`j =`

Matrix-Chain Multiplication

$$\boldsymbol{C[i,j] = \min_{i\leq k<j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)}$$

i

j	1	2	3	4	5	6
1	0	-	-	-	-	-
2	🟩	0	-	-	-	-
3		🟩	0	-	-	-
4			🟩	0	-	-
5				🟩	0	-
6						0

`1,2,3,4`

`2,3,4,5`

`i =`

`j =`

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$

i

j	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

`1,2,3,4,5`

`2,3,4,5,6`

`i =`

`j =`

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$

i

j

	1	2	3	4	5	6
1	0	-	-	-	-	-
2	🟩	0	-	-	-	-
3	🟦	🟩	0	-	-	-
4			🟩	0	-	-
5				🟩	0	-
6					🟩	0

`1,2,3,4,5`

`1`

`i =`

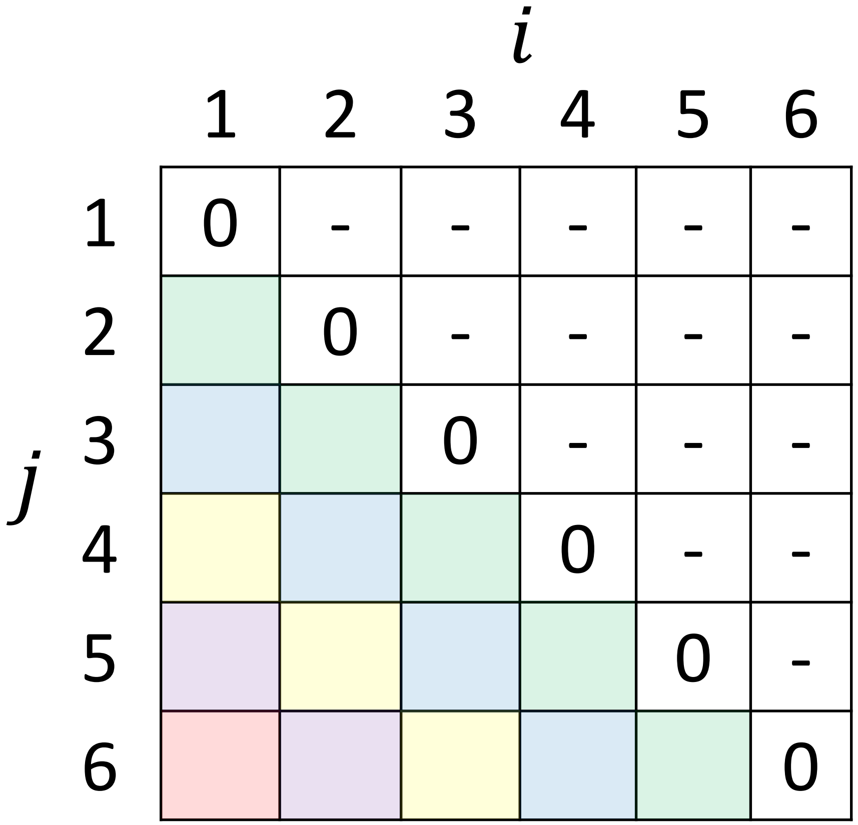
`2,3,4,5,6`

`3`

`j =`

Matrix-Chain Multiplication

$$C[i,j] = \min_{i\le k<j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$

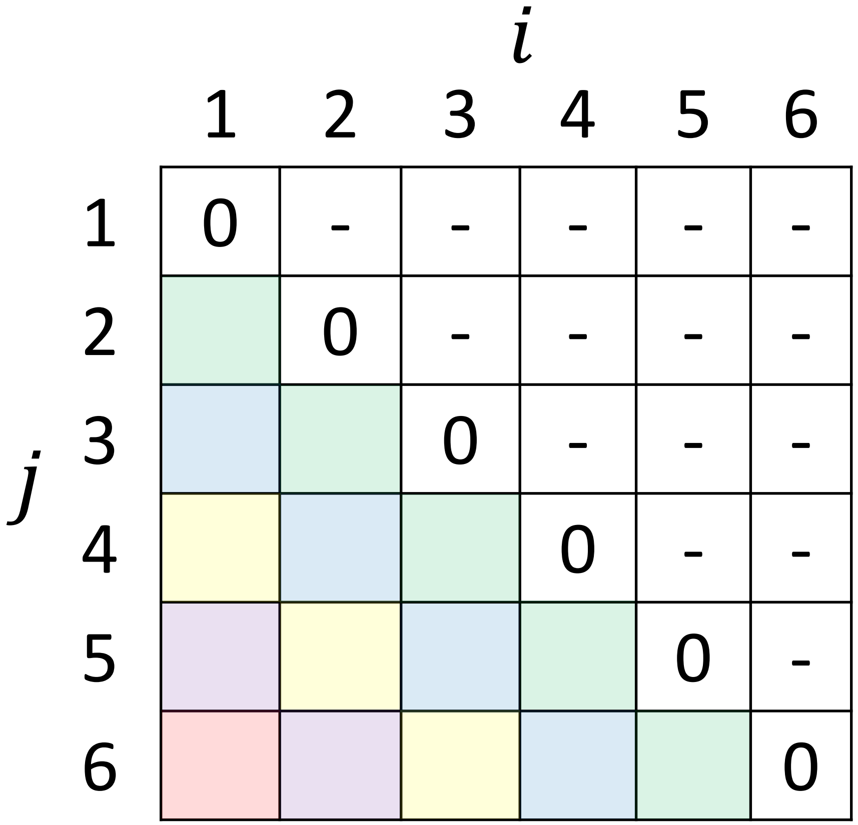


	1	2	3	4	5	6
1	0	-	-	-	-	-
2	🟩	0	-	-	-	-
3	🟦	🟩	0	-	-	-
4	🟨	🟦	🟩	0	-	-
5	🟪	🟨	🟦	🟩	0	-
6	🟥	🟪	🟨	🟦	🟩	0

```
    1,2,3,4,5         2,3,4,5,6
    1,2,3,4           3,4,5,6
i = 1,2,3         j = 4,5,6
    1,2               5,6
    1                 6
```

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$



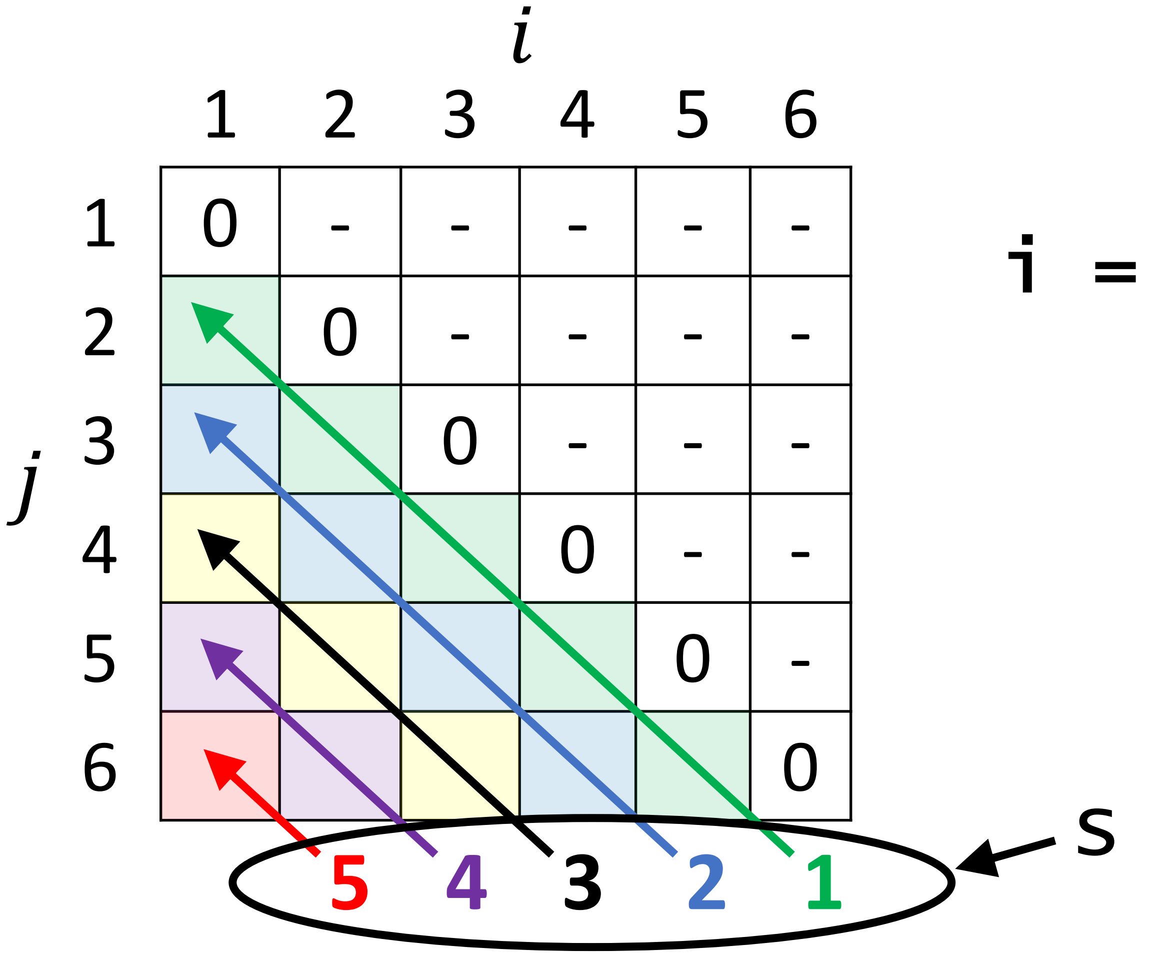
	1	2	3	4	5	6
1	0	-	-	-	-	-
2	🟩	0	-	-	-	-
3	🟦	🟩	0	-	-	-
4	🟨	🟦	🟩	0	-	-
5	🟪	🟨	🟦	🟩	0	-
6	🟥	🟪	🟨	🟦	🟩	0

```
      1,2,3,4,5          2,3,4,5,6
      1,2,3,4            3,4,5,6
i  =  1,2,3        j  =  4,5,6
      1,2                5,6
      1                  6
```

For loop(s) that will give us these `i`/`j` paired values?

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \leq k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$



j \ i	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

```
      1,2,3,4,5          2,3,4,5,6
      1,2,3,4            3,4,5,6
i =   1,2,3        j =   4,5,6
      1,2                5,6
      1                  6
```

```
for s = 1,...,n - 1
  for i = 1,...,n - s
    j = i + s
```

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$

i

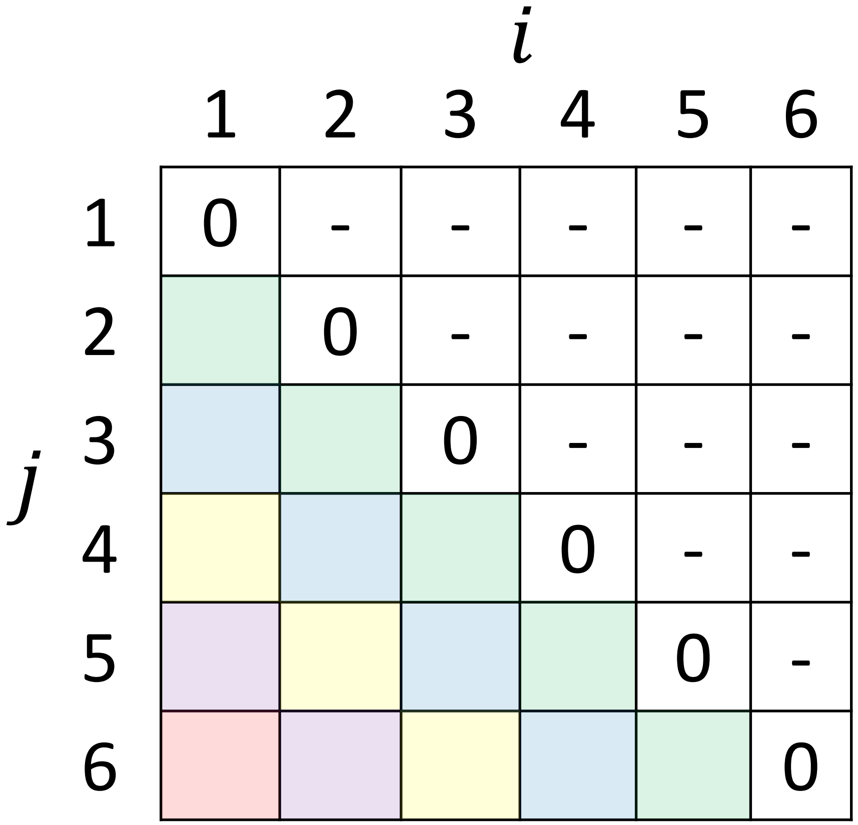
j	1	2	3	4	5	6
1	0	-	-	-	-	-
2	🟩	0	-	-	-	-
3	🟦	🟩	0	-	-	-
4	🟨	🟦	🟩	0	-	-
5	🟪	🟨	🟦	🟩	0	-
6	🟥	🟪	🟨	🟦	🟩	0

```
for s = 1,...,n - 1
    for i = 1,...,n - s
        j = i + s

        ???
```

Matrix-Chain Multiplication

$$C[i,j] = \min_{i\leq k<j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$



i

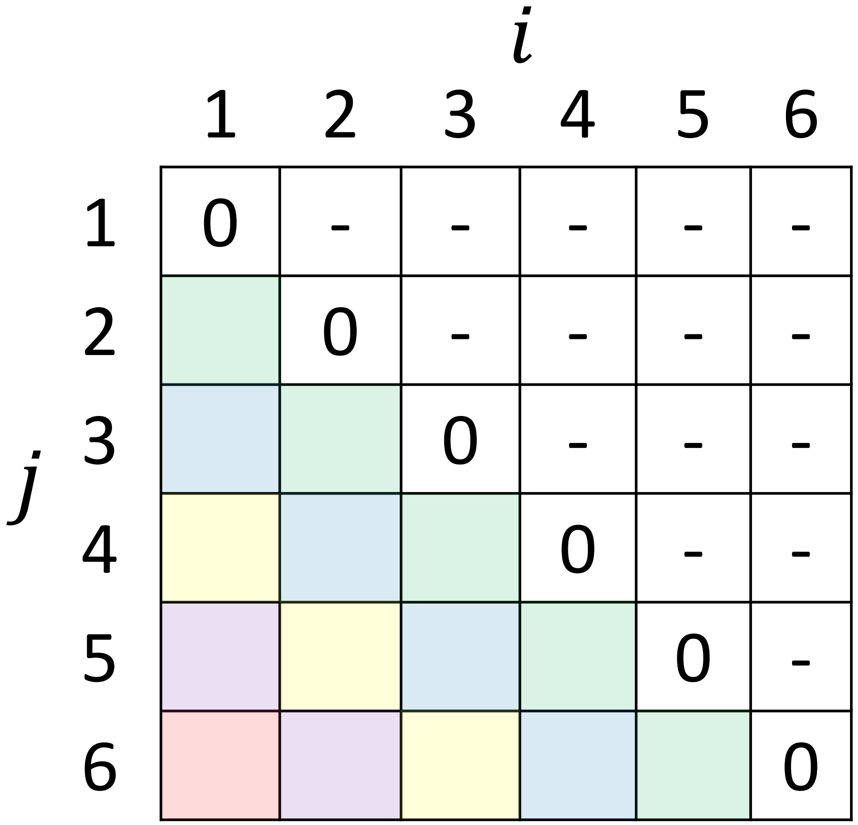
j

	1	2	3	4	5	6
1	0	-	-	-	-	-
2	🟩	0	-	-	-	-
3	🟦	🟩	0	-	-	-
4	🟨	🟦	🟩	0	-	-
5	🟪	🟨	🟦	🟩	0	-
6	🟥	🟪	🟨	🟦	🟩	0

```
for s = 1,...,n - 1
  for i = 1,...,n - s
    j = i + s
    min = ∞
    for k = i,...,j - 1
      a = C[i,k]+C[k+1,j]+$m_k m_{i-1} m_j$
      if a < min
        min = a
    C[i,j] = min
return C[1,n]
```

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \leq k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$



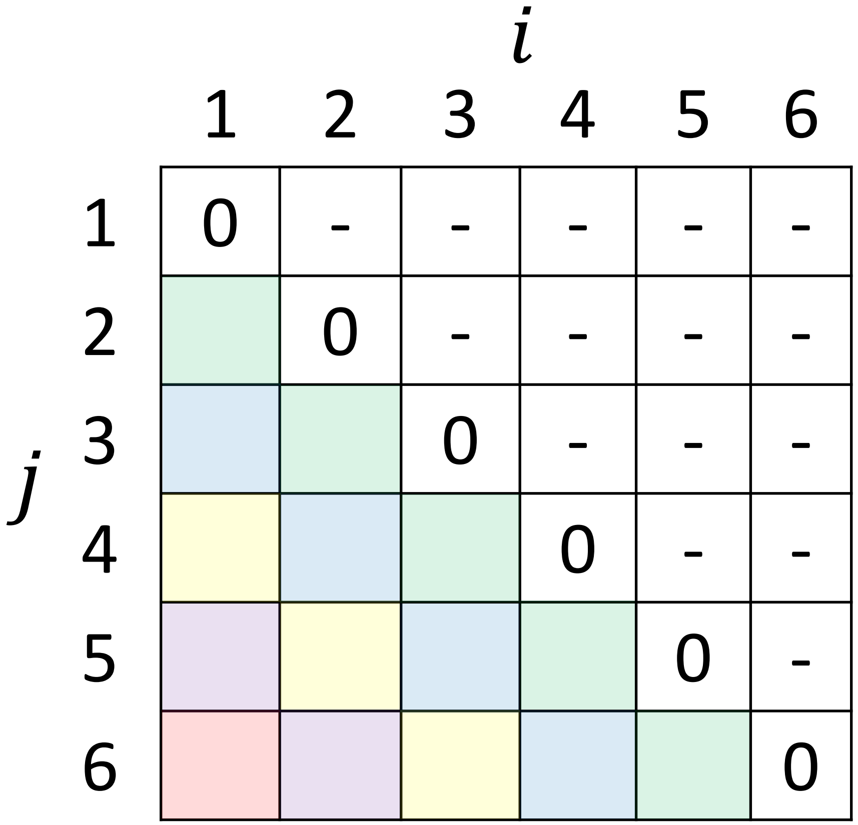
				i			
		1	2	3	4	5	6
	1	0	-	-	-	-	-
	2	🟩	0	-	-	-	-
j	3	🟦	🟩	0	-	-	-
	4	🟨	🟦	🟩	0	-	-
	5	🟪	🟨	🟦	🟩	0	-
	6	🟥	🟪	🟨	🟦	🟩	0

```
for s = 1,...,n - 1
  for i = 1,...,n - s
    j = i + s
    min = ∞
    for k = i,...,j - 1
      a = C[i,k]+C[k+1,j]+$m_k m_{i-1} m_j$
      if a < min
        min = a
    C[i,j] = min
return C[1,n]
```

Running Time?

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)$$



i

		1	2	3	4	5	6
	1	0	-	-	-	-	-
	2	🟩	0	-	-	-	-
j	3	🟦	🟩	0	-	-	-
	4	🟨	🟦	🟩	0	-	-
	5	🟪	🟨	🟦	🟩	0	-
	6	🟥	🟪	🟨	🟦	🟩	0

```
for s = 1,...,n - 1
   for i = 1,...,n - s
      j = i + s
      min = ∞
      for k = i,...,j - 1
         a = C[i,k]+C[k+1,j]+$m_k m_{i-1} m_j$
         if a < min
            min = a
      C[i,j] = min
return C[1,n]
```

Running Time?

$O(n^3)$