Greedy Algorithms
CSCI 432

Greedy Change Making (USD)

Greedy decision: Select the highest denomination coin that evenly subtracts from the remaining balance (i.e. max quarters + max dimes + max nickels + max pennies).

Proof of optimality:

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What can we say about q_{ALG} vs q_{OPT}?

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If $q_{OPT} < q_{ALG}, V - 25\, q_{OPT} \geq 25$.

Why is this a problem?

Contradict the fact that T_{OPT} is optimal.

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I.e. $V - 25\, q_{OPT} = 10\, d_{OPT} + 5\, n_{OPT} + p_{OPT}$

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$$\begin{aligned} \text{I.e. } V - 25\, q_{OPT} &= 10\, d_{OPT} + 5\, n_{OPT} + p_{OPT} \\ &= 25 + 10(d_{OPT} - 2) + 5(n_{OPT} - 1) + p_{OPT} \end{aligned}$$

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$= 25 + 10(d_{OPT} - 2) + 5(n_{OPT} - 1) + p_{OPT}$

Technically we don't know if there are enough dimes and nickels for this, but if there are not, then it will require even more coins to account for the 25-cents.

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$$\Rightarrow V = 25(q_{OPT} + 1) + 10(d_{OPT} - 2) + 5(n_{OPT} - 1) + p_{OPT}$$

$$\Rightarrow \text{Num coins} = q_{OPT} + d_{OPT} + n_{OPT} + p_{OPT} - 2$$

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$\Rightarrow V = 25(q_{OPT} + 1) + 10(d_{OPT} - 2) + 5(n_{OPT} - 1) + p_{OPT}$

$\Rightarrow$ Num coins $= q_{OPT} + d_{OPT} + n_{OPT} + p_{OPT} - 2 < T_{OPT}$. Contradiction.

Thus, $q_{OPT} = q_{ALG}$.

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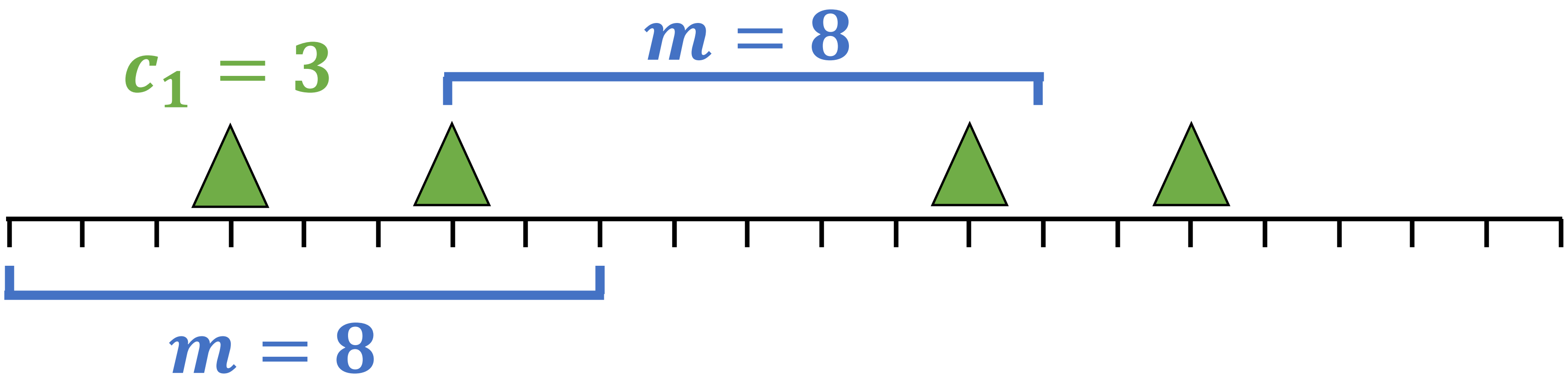
Thus, $q_{OPT} = q_{ALG}$.

Repeat argument to show that $d_{OPT} = d_{ALG}$, $n_{OPT} = n_{ALG}$, and $p_{OPT} = p_{ALG}$. Thus, $T_{OPT} = T_{ALG}$ and T_{ALG} is optimal.

Problem Statement

Suppose you are going on a long backpacking trip on some trail. You can hike up to m miles a day. There are designated campsites along the trail at miles $(c_1, c_2, \ldots, c_n)$. You need to stay in a designated campsite each night. Your goal is to hike the whole trail in as few days as possible.



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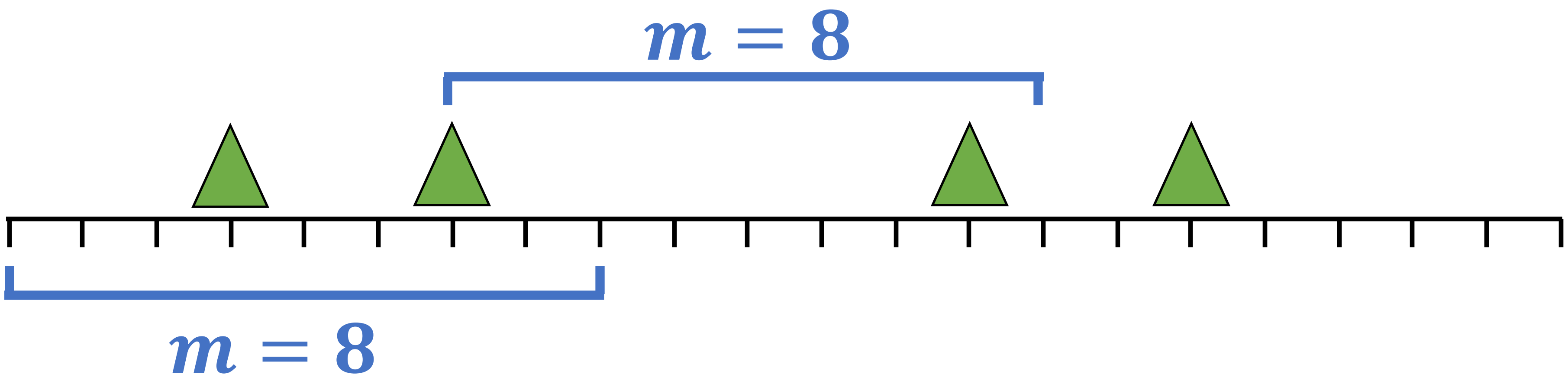
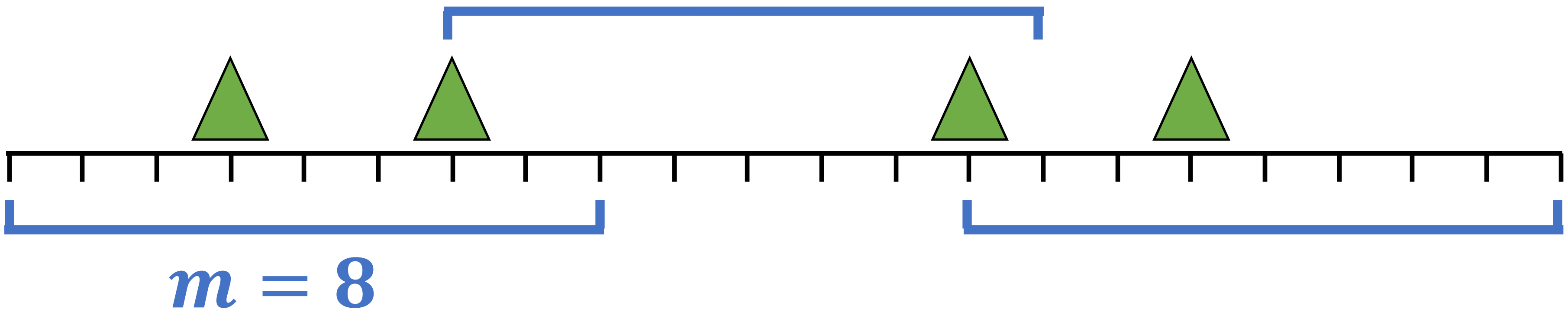


Come up with a greedy algorithm that will provide a valid solution (i.e. don't worry about it being optimal right now).

Greedy Algorithm

Greedy Campsite Selection (GCS): Camp at the last reachable (i.e. $< m$) campsite each day.

Is GCS valid?



Greedy Algorithm

```
campsite_selection(campsites C, maxDistance m)

                ??
```

Greedy Algorithm

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campsite_selection(campsites C, maxDistance m)
  C.sort() //if needed
  lastSpot = 0
  for i = 1 to C.length
    if lastSpot + m < C[i]
      selected.add(C[i - 1])
      lastSpot = C[i - 1]
  return selected
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Running time?

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Running time?

$n = |C|$

$O(n \log n)$ - If sorting needed

$O(n)$ - If sorting not needed

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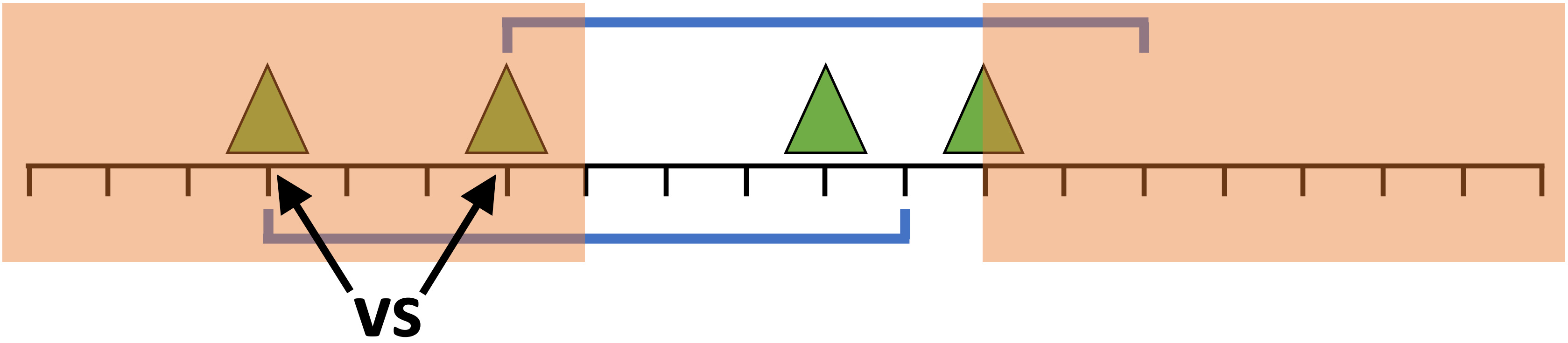

Will this ever not be optimal? Does it ever make sense to end the day early, to help with campsite spacing on a future day?

Theorem: Camping at the last reachable campsite each day is optimal.

Proof:

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Plan?

Greedy Algorithm

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Proof:

Prove this by arguing that you can swap S_{ALG} campsites into S_{OPT} and keep S_{OPT} both valid and optimal.

Theorem: Camping at the last reachable campsite each day is optimal.

Proof: Let $S_{ALG} = (c_{a_1}, \dots, c_{a_k})$ be the campsites chosen by GCS and $S_{OPT} = (c_{o_1}, \dots, c_{o_t})$ be an optimal selection.

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Consider $S'_{OPT} = S_{OPT} \setminus \{c_{o_i}\} \cup \{c_{a_i}\}$.

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So S'_{OPT} is valid (doesn't exceed m-distance restriction) and has the same number of sites as S_{OPT}, so it's optimal.

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We can repeat this by replacing campsites in S_{OPT} with the corresponding site in S_{ALG} without exceeding the m-restriction or increasing the number of sites.

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Thus, S_{ALG} is optimal.