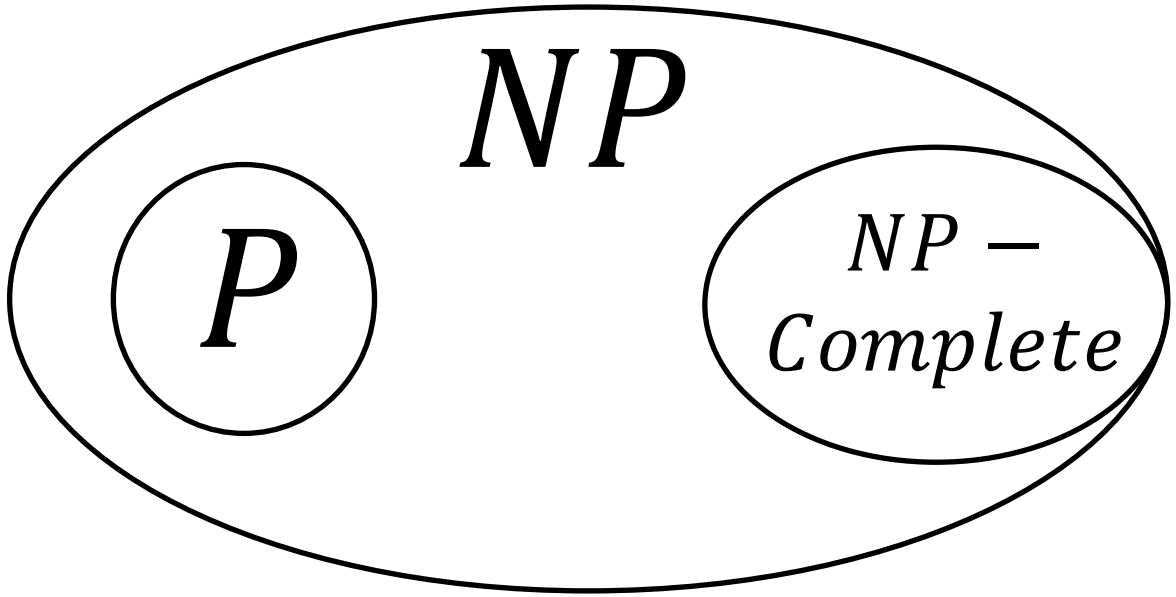
Approximation Algorithms
CSCI 532

NP Complete



Problems in NP whose algorithms can solve any other problem in NP with polynomial extra time.

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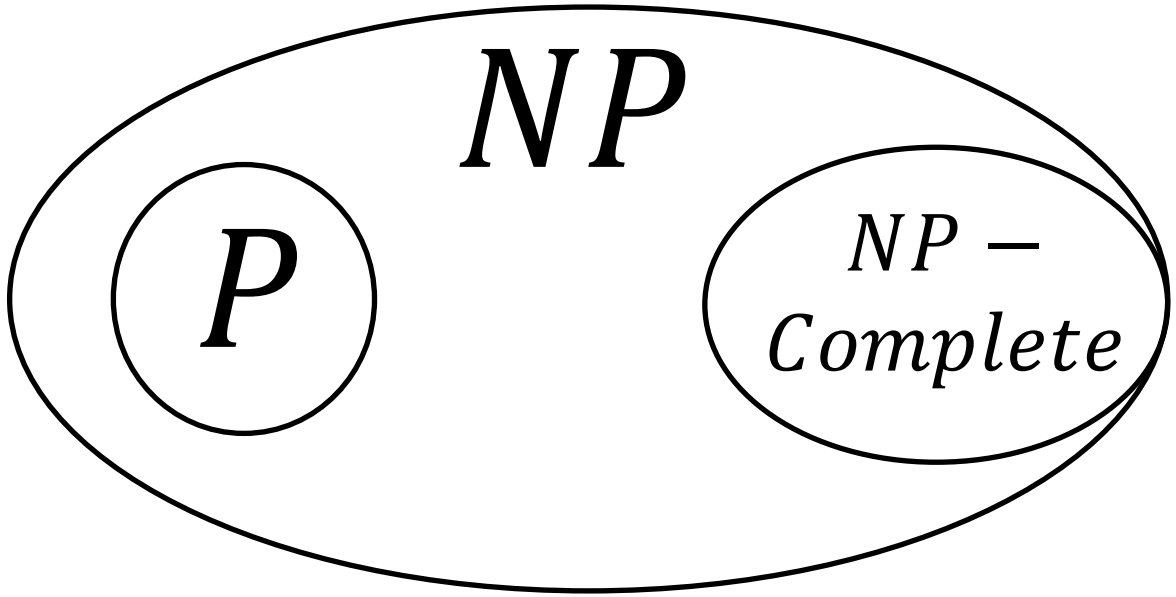
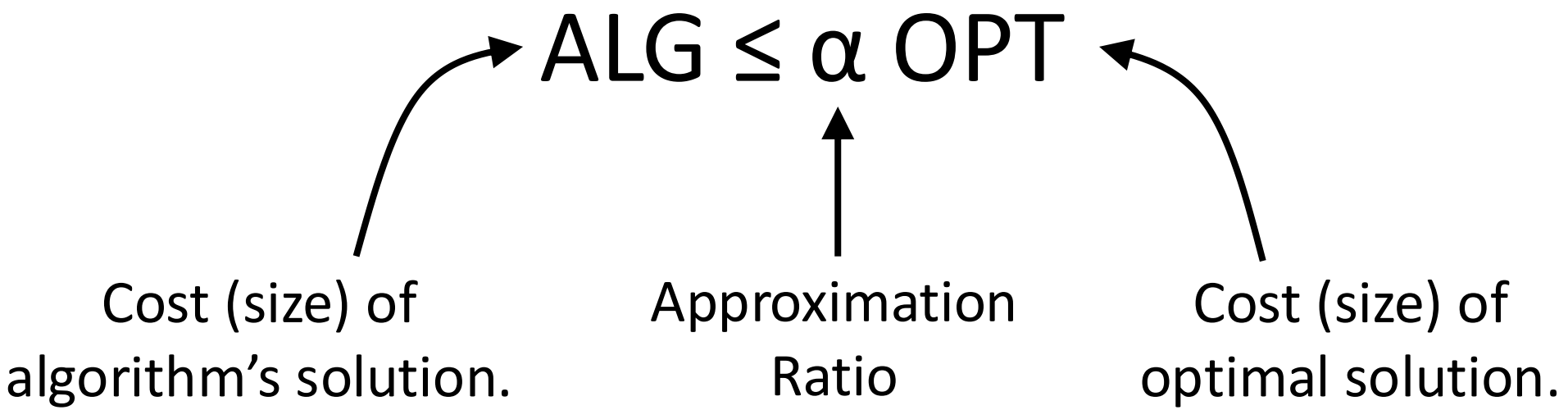


Problems in NP whose algorithms can solve any other problem in NP with polynomial extra time.

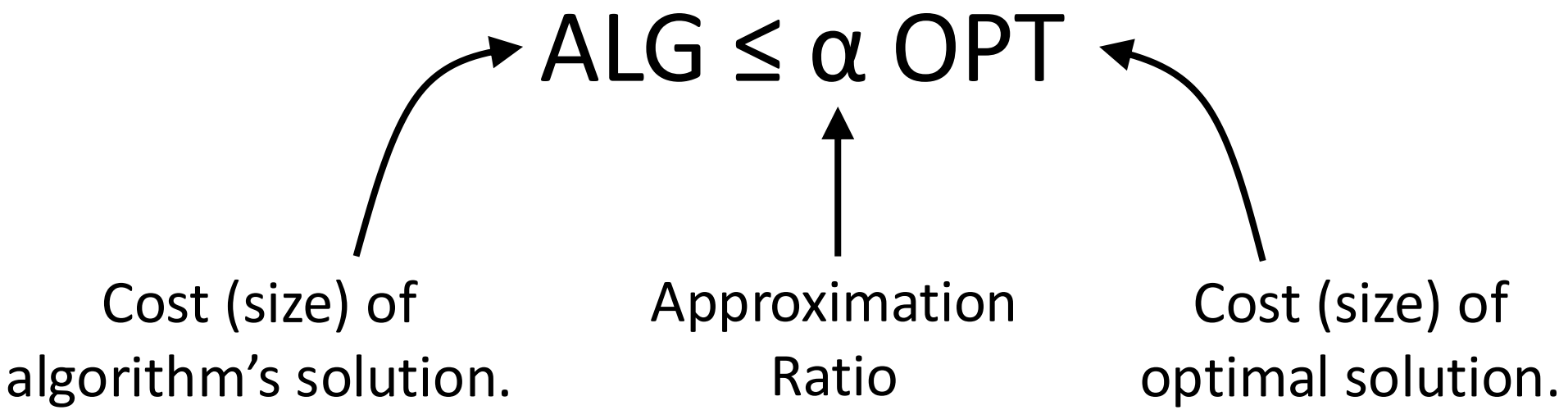
Techniques to handle NP-Complete problems:

1. Brute Force (i.e. Exponential Time).
2. Heuristics.
3. Approximation Algorithms.
4. Fixed-parameter Tractable Algorithms.

Approximation Algorithms

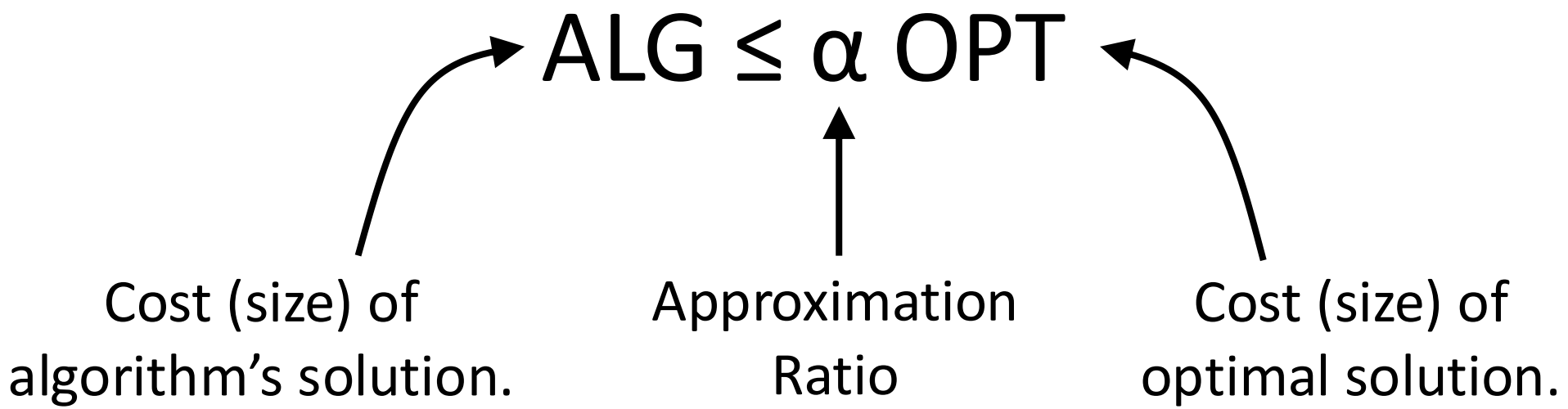


Approximation Algorithms



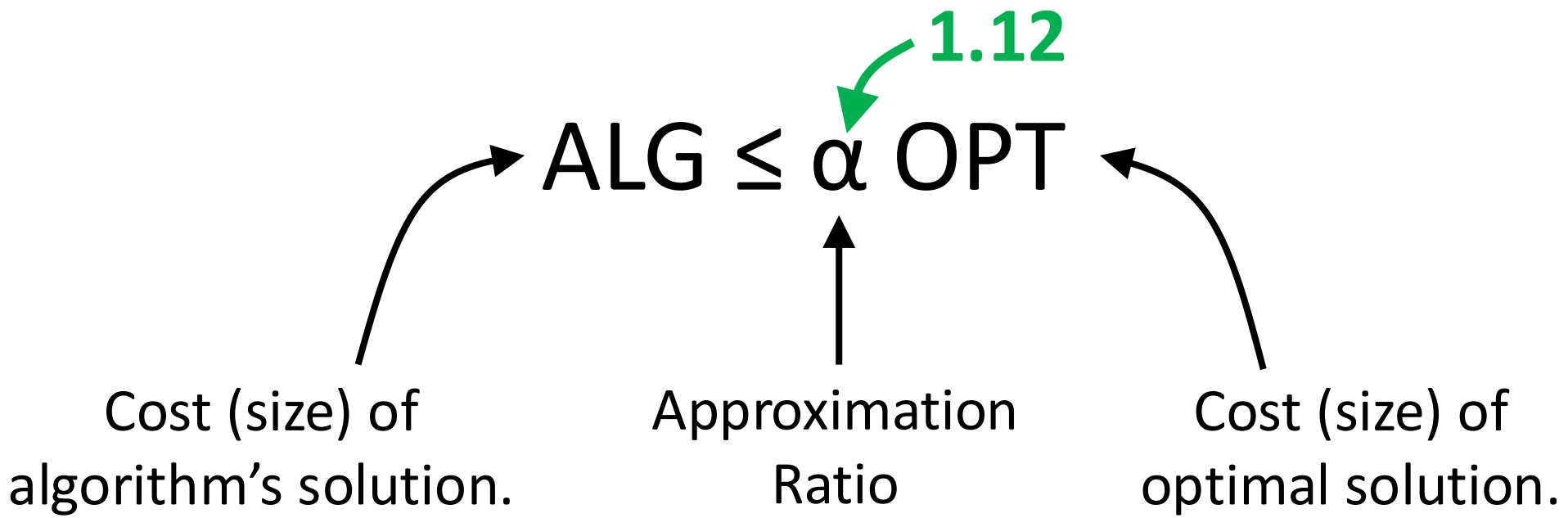
if problem is a maximization problem, $ALG \geq \frac{1}{\alpha} OPT$

Approximation Algorithms



Example:

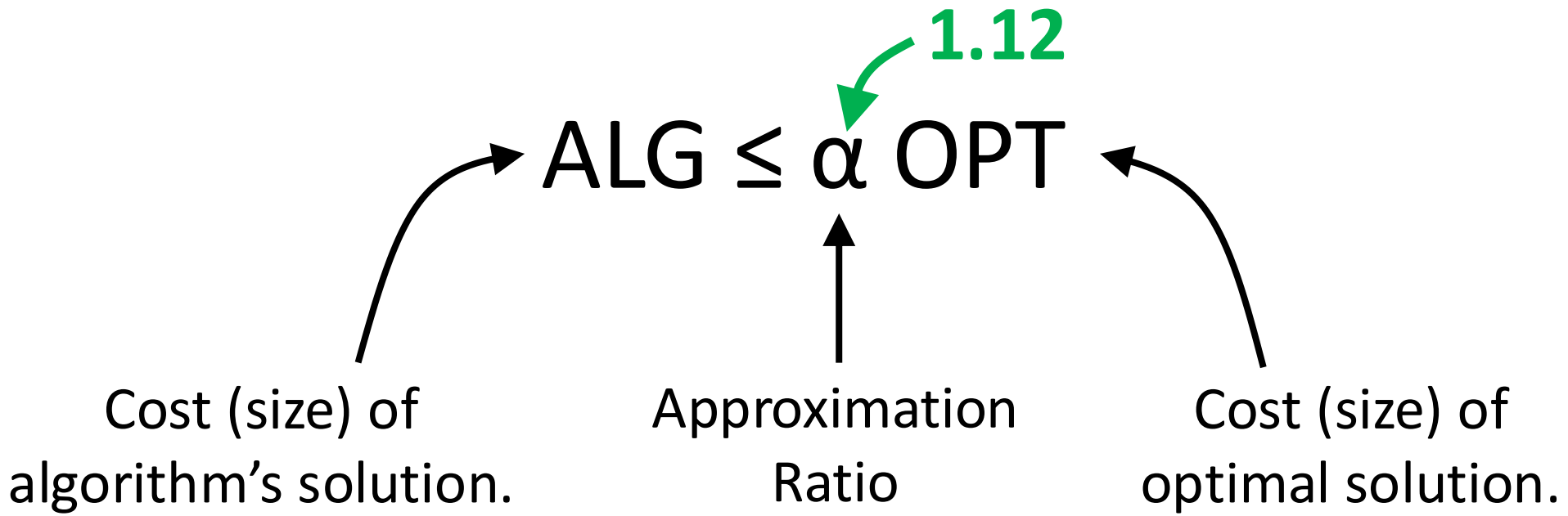
Approximation Algorithms



Example:

- Suppose I know my algorithm is a 1.12-approximation algorithm.

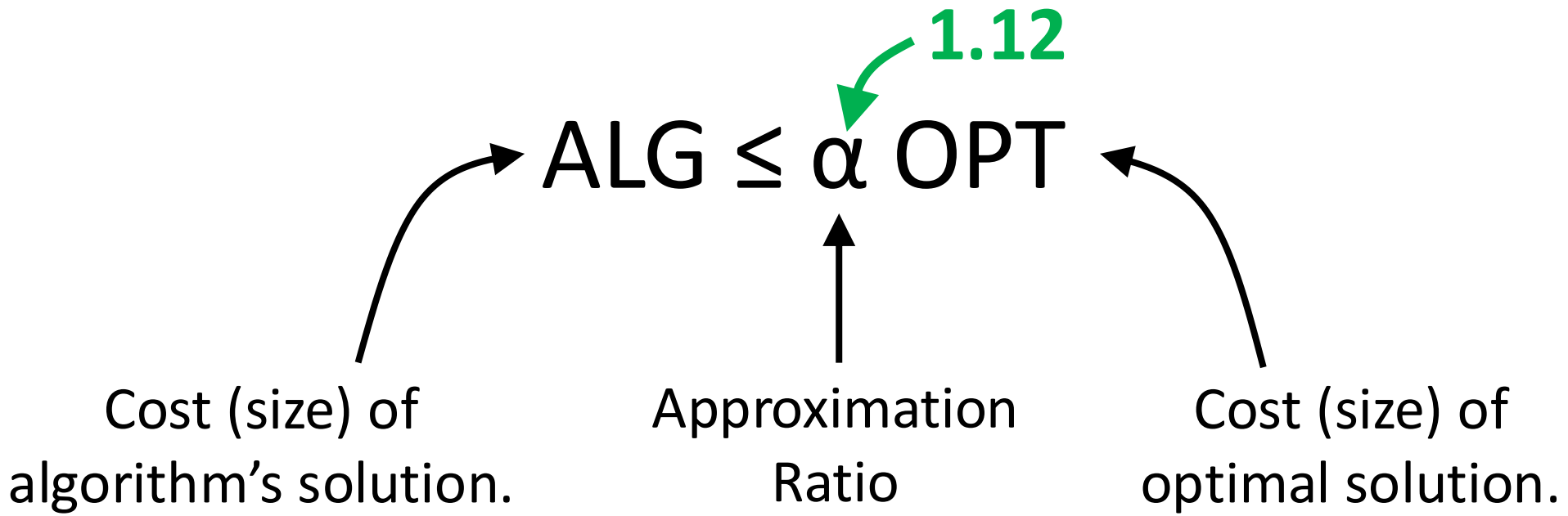
Approximation Algorithms



Example:

- Suppose I know my algorithm is a 1.12-approximation algorithm.
- Suppose my algorithm returns a solution of cost 746.125.

Approximation Algorithms



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Then, I know that $746.125 \leq 1.12$ OPT

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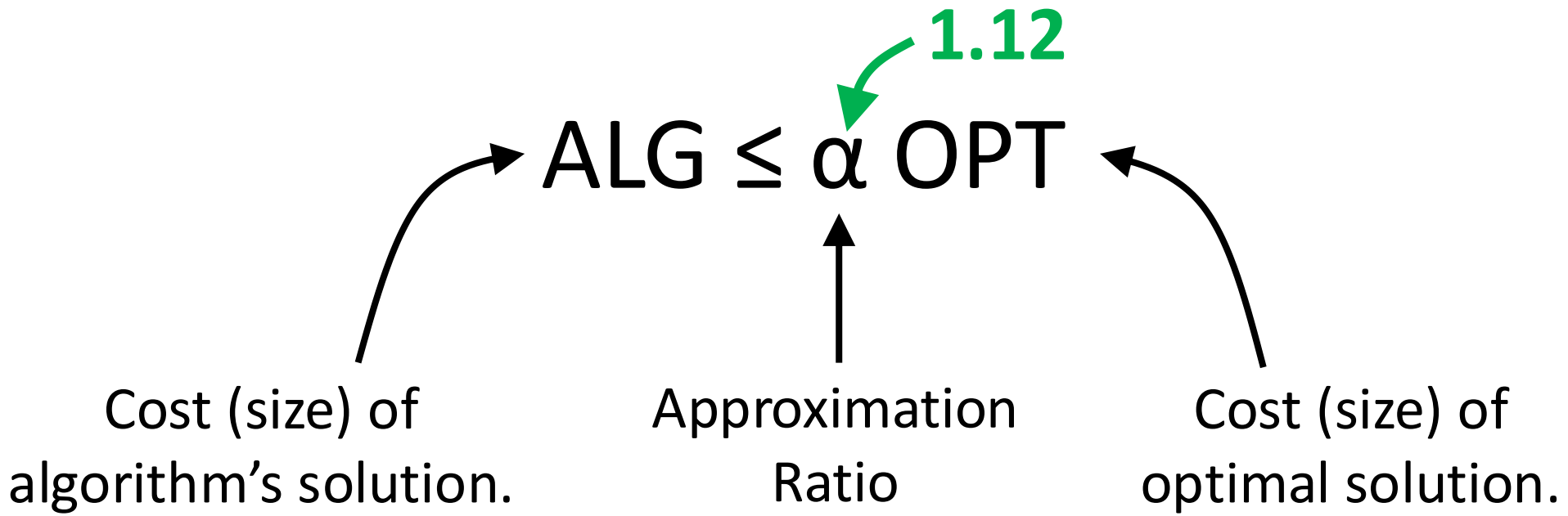

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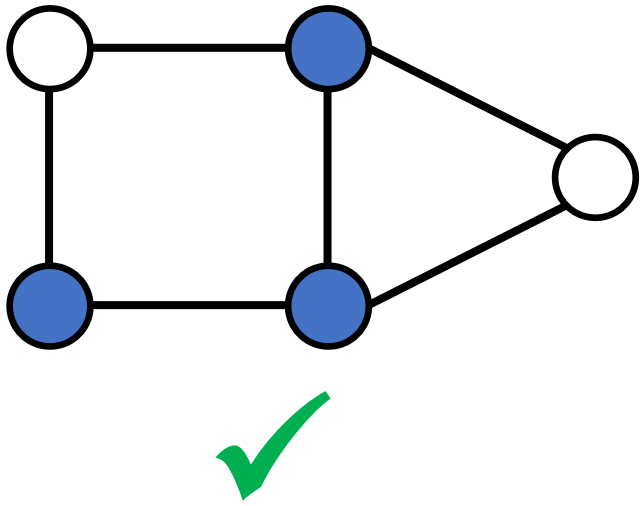
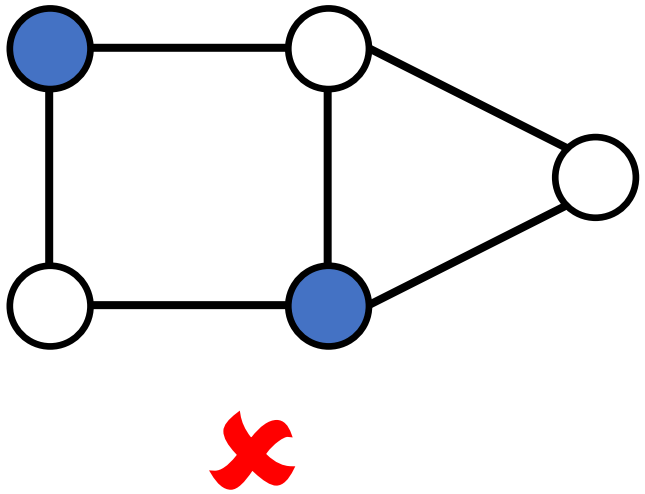
$$\Rightarrow \frac{746.125}{1.12} = 666.183 \leq \text{OPT} \leq 746.125$$

Vertex Cover

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.





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Algorithm:

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while uncovered edge exists
   select both of its vertices
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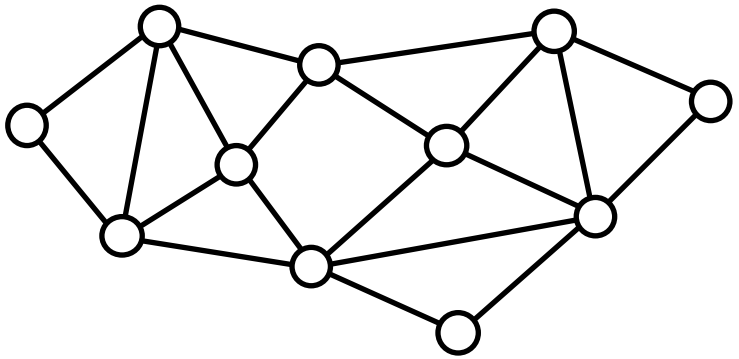

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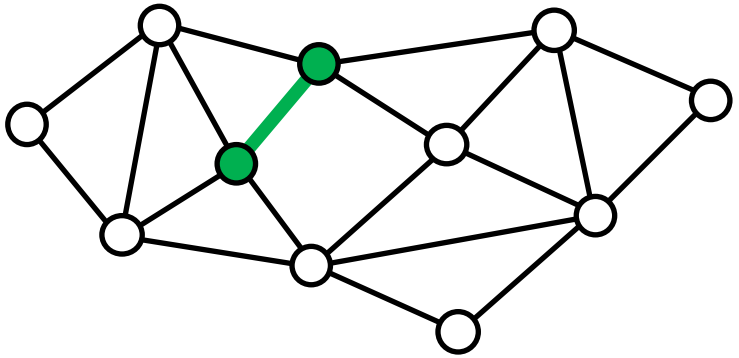

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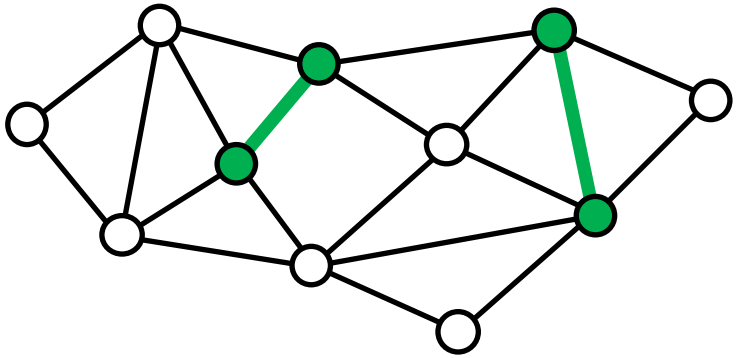

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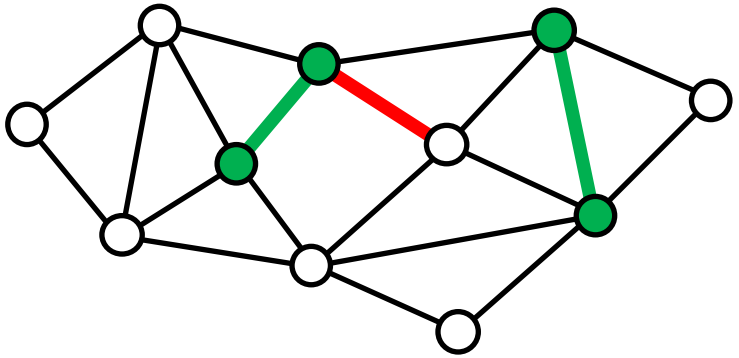

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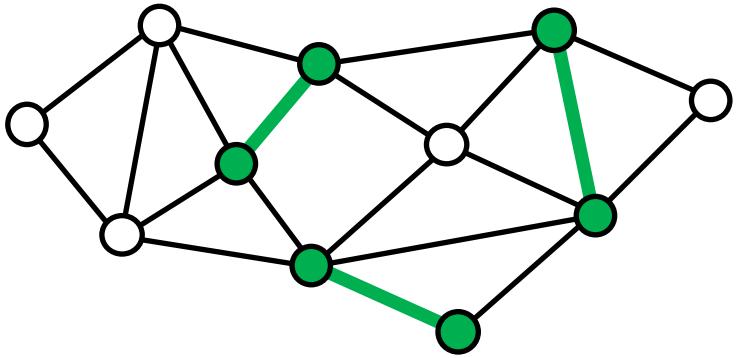

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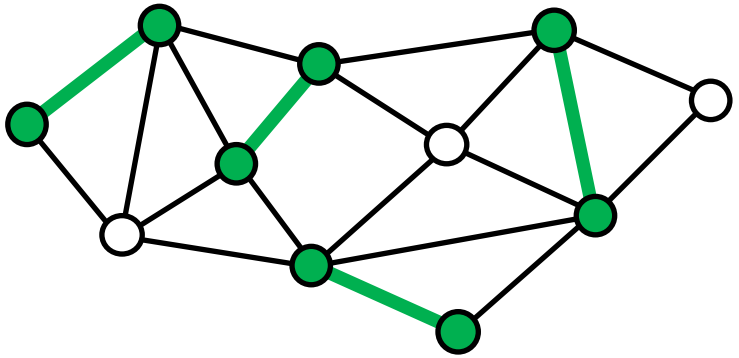

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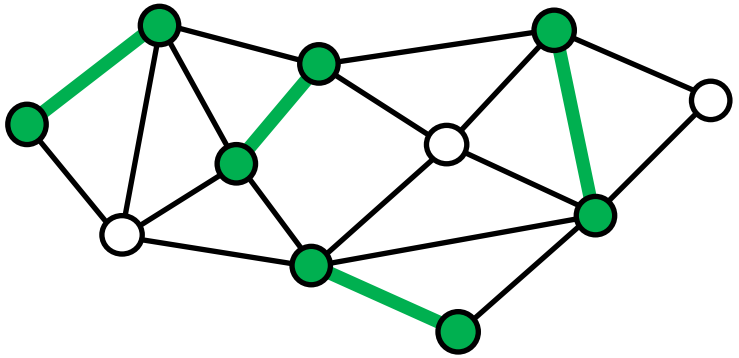
Consider a set of edges, $E' \subset E$, that do not share vertices.



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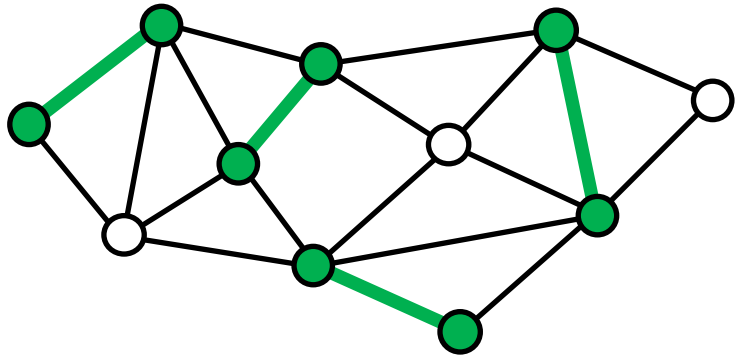
Consider a set of edges, $E' \subset E$, that do not share vertices. Is there a relationship between the minimum vertex cover and $|E'|$?



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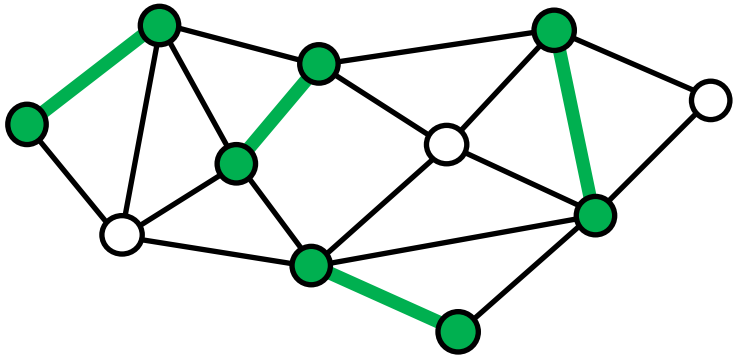
Consider a set of edges, $E' \subset E$, that do not share vertices. Is there a relationship between the minimum vertex cover and $|E'|$?

$$|E'| \leq \text{OPT}$$

Size of actual smallest vertex cover.

If we selected fewer than one vertex per edge, we would not have a vertex cover, because that edge would not be covered!



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$$\text{ALG} = 2\,|E'|$$

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We cannot find optimal vertex covers in poly time unless $P = NP$, but this algorithm is at worst 2-times optimal.

Does t
do not share vertices?

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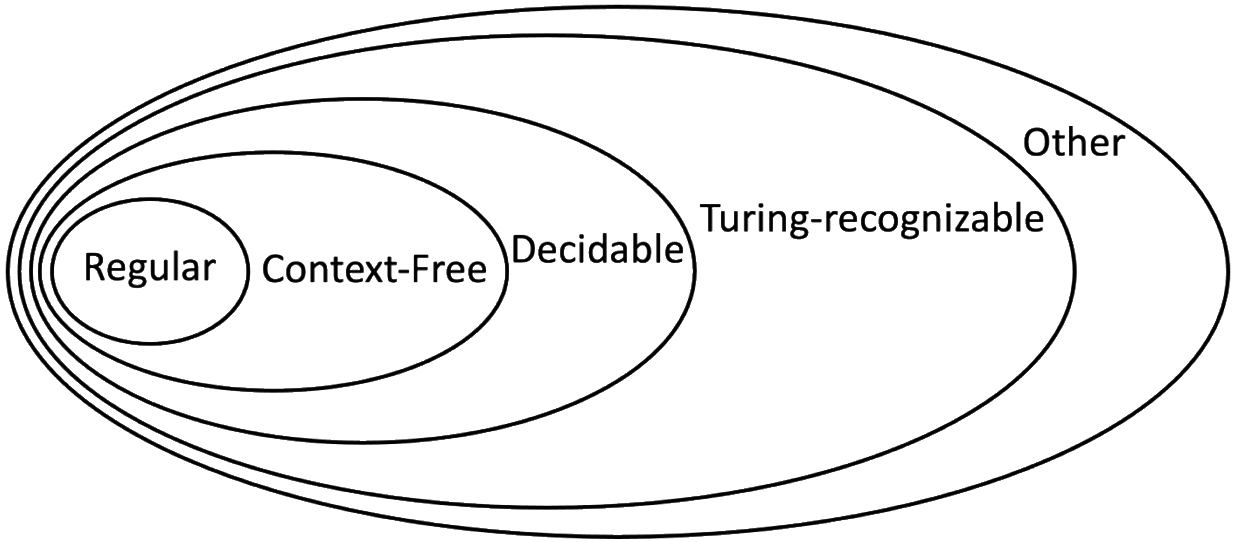
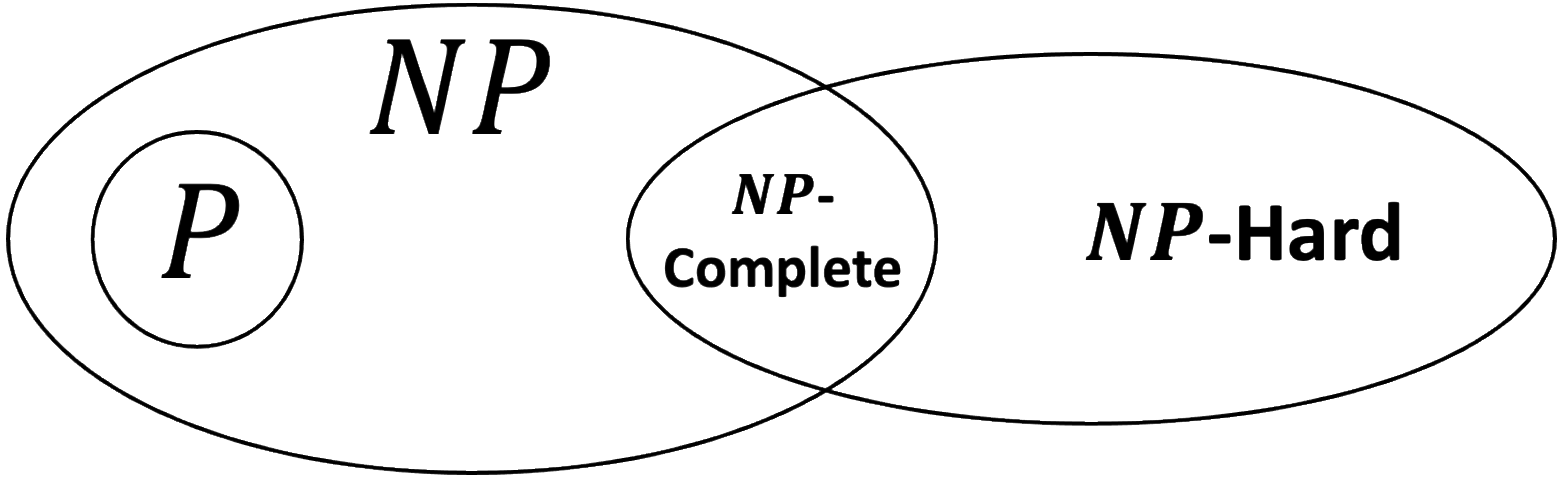
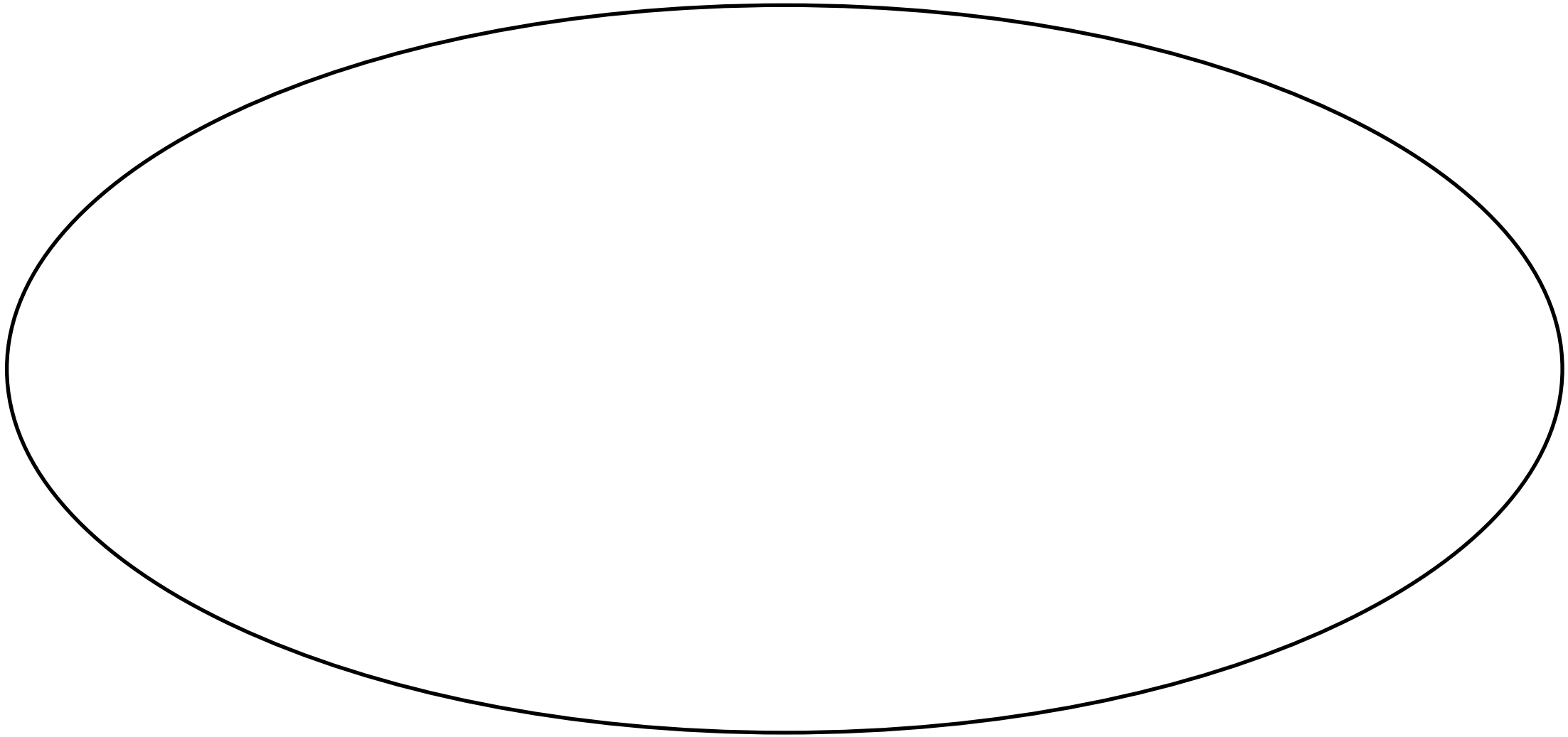


Computability Hierarchy



Complexity Hierarchy

Approximability Hierarchy



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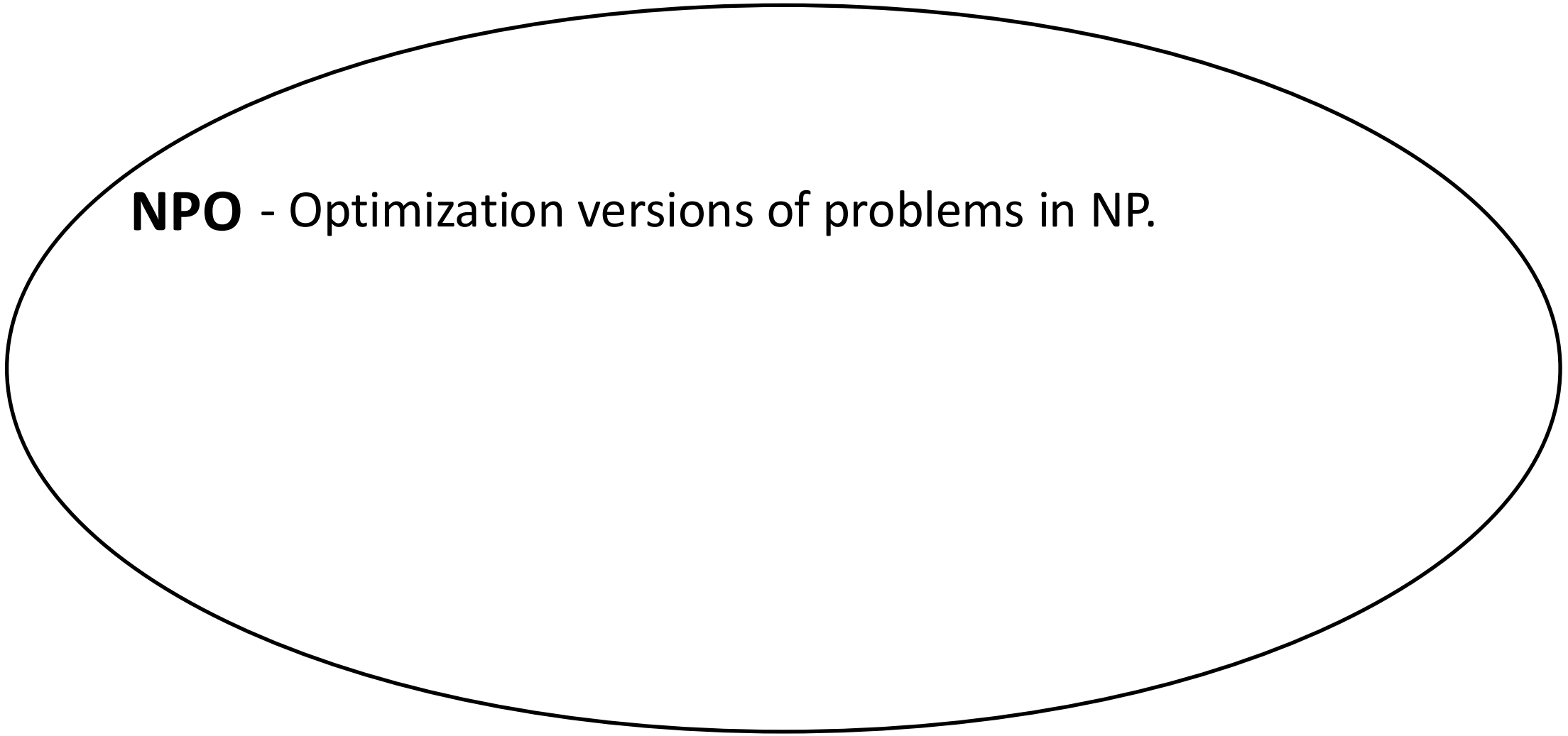

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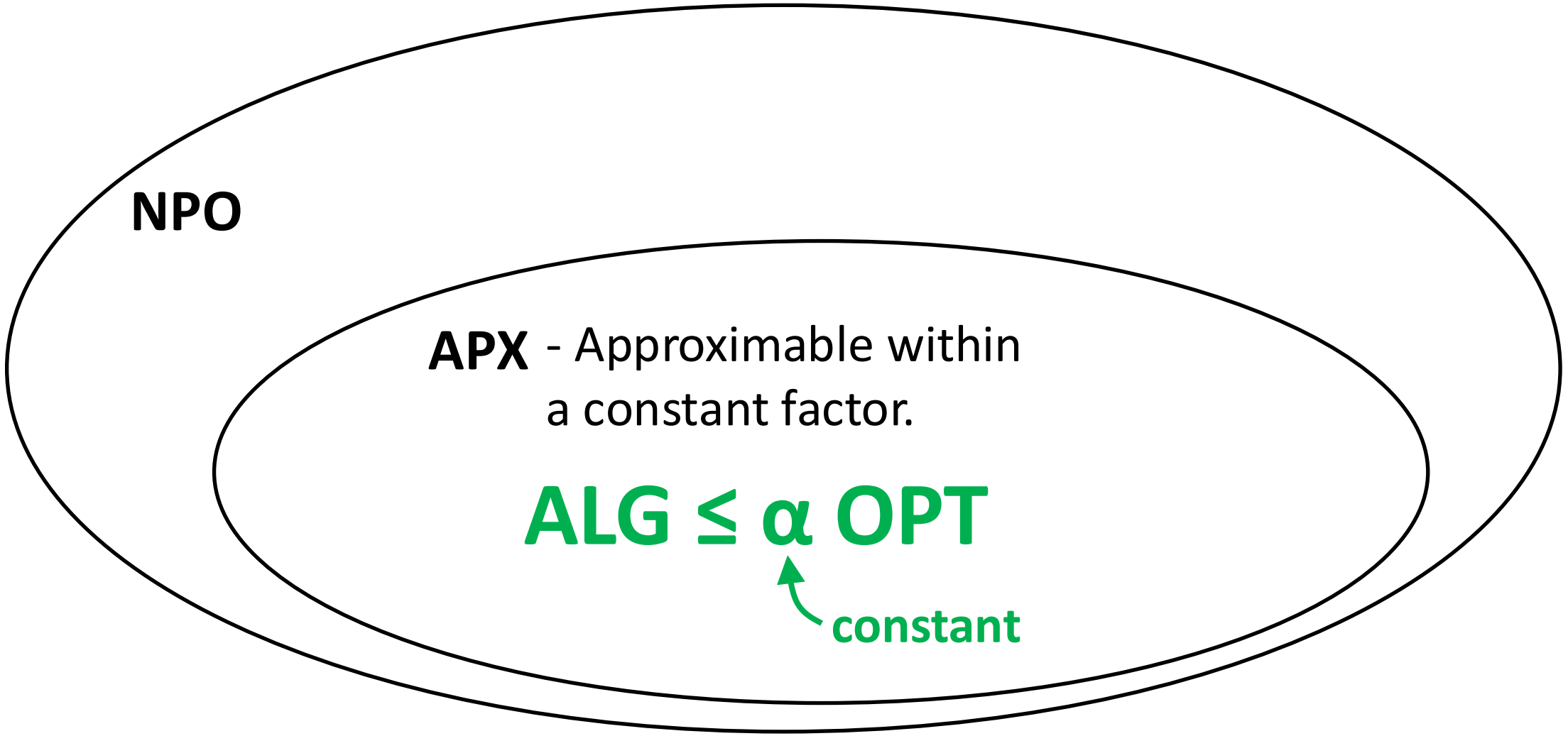

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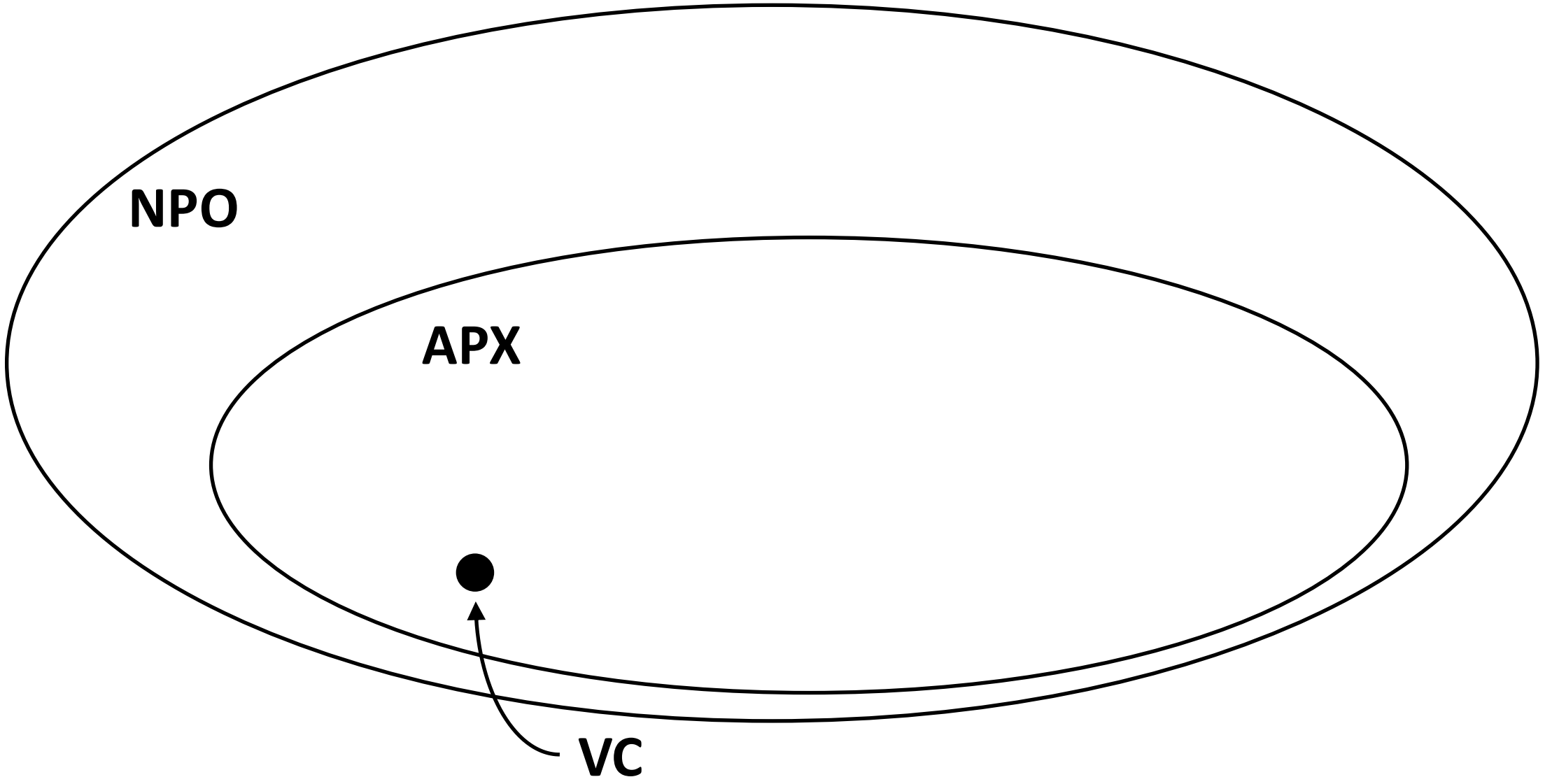

Set Cover

Set Cover: Given a set of elements (the universe), and sets containing those elements, find the smallest number of sets so that every element of the universe is included.

Example:

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Example:

$$U = \{1, 4, 7, 8, 10\}$$

$$S = \{\{1, 7, 8\}, \{1, 4, 7\}, \{7, 8\}, \{4, 8, 10\}\}$$

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$\{\{1, 7, 8\}, \{4, 8, 10\}\}$ ✓

$\{\{1, 4, 7\}, \{7, 8\}\}$ ✗

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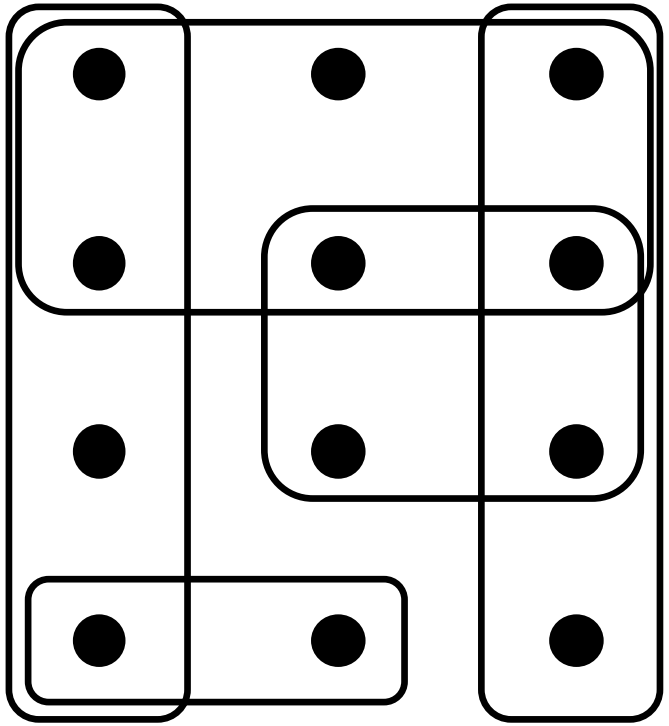

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Algorithm:

?

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Greedy Algorithm:

```
while element of universe not included
   select S_i with largest number of excluded elements.
```

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Suppose the universe contains n elements.

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Game Plan:

Bound the maximum number of sets in ALG…

By bounding the maximum number of iterations of the algorithm…

By bounding the size of each set added by the algorithm.

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What can we say about the first set selected?

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What can we say about the first set selected?

It's the biggest!

At each iteration, we cover the largest number of uncovered elements, and all the elements are uncovered in the first iteration.

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Which do we care about?

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$$? \leq |\text{Biggest Set}| \leq ?$$

Guarantee we do at least this good.

Guarantee we don't do better than this.