

ILP-Based Approximations

CSCI 532

Vertex Cover ILP

Vertex Cover: Given a graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

$x_i \in \{0,1\}$, for each vertex i

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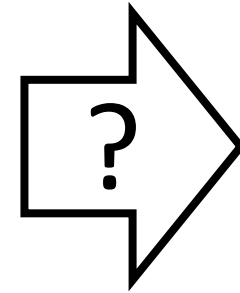
∈ P

LP Relaxation: Remove all integrality constraints to turn ILP into LP.

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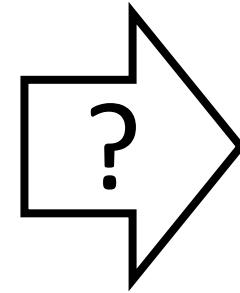


Vertex
Selection

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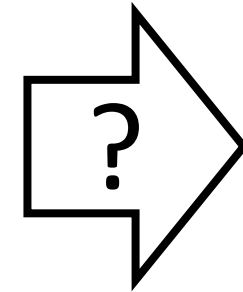
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If $x_i = 1$, what should we do with vertex i ?

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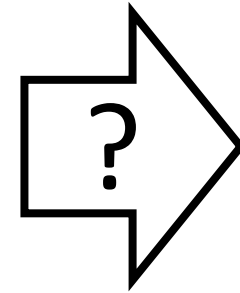
If $x_i = 1$, what should we do with vertex i ? Add to subset S

If $x_i = 0$, what should we do with vertex i ?

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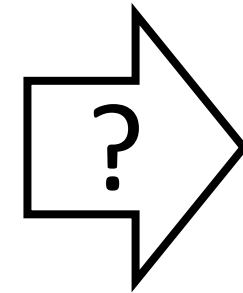
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If $x_i = 1$, what should we do with vertex i ? Add to subset S

If $x_i = 0$, what should we do with vertex i ? Don't add to subset S

If $x_i = \frac{126}{337}$, what should we do with vertex i ?

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+ If $x_i \geq \frac{1}{2}$, add vertex i
to our subset S .

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Is S a vertex cover?

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Yes. For every edge, $x_i + x_j \geq 1$.

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Is S a vertex cover?

Yes. For every edge, $x_i + x_j \geq 1$. Thus, at least one of x_i or $x_j \geq \frac{1}{2}$. So for every edge, at least one of its vertices will be in S .

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What is the relationship between $ALG = |S|$ and OPT ?

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Can we bound OPT from below?

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Let x_{ILP} and x_{LP} be the set of x values found by the ILP and LP

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Let x_{ILP} and x_{LP} be the set of x values found by the ILP and LP

Claim: $\sum x_{\text{LP}} \leq \text{OPT}$.

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Can we bound OPT from below?

Let x_{ILP} and x_{LP} be the set of x values found by the ILP and LP

Claim: $\sum x_{\text{LP}} \leq \text{OPT}$.

Proof: $\text{OPT} = \sum x_{\text{ILP}}$, where $x_i \in \{0,1\}$. When x_i is relaxed so that $0 \leq x_i \leq 1$, this gives more possibilities to further decrease $\sum_i x_i$. Thus, $\sum x_{\text{LP}} \leq \text{OPT}$.

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How does $\sum x_{LP}$ relate to ALG?

$$\sum x_{LP} = \sum_{x_i \in X_{LP}} x_i \geq \sum_{x_i \in X_{LP}: x_i \geq \frac{1}{2}} x_i, \text{ because...?}$$

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What is the relationship between ALG and OPT?

$$\sum x_{LP} \geq \frac{1}{2} \text{ALG and } \sum x_{LP} \leq \text{OPT}$$

$$\text{ALG} \leq 2 \text{OPT}$$

Set Cover ILP

Set Cover: Given a universe of elements U and sets S , find the smallest subset of S such that every element in U is in some selected subset.

$$U = \{1, 4, 7, 8, 10\}$$

$$S = \left\{ \begin{array}{l} \{1, 7, 8\}, \{1, 4, 7\}, \\ \{7, 8\}, \{4, 8, 10\} \end{array} \right\}$$

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 $x_s \in \{0,1\}$, for each set s

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Example:

Objective: $\min x_1 + x_2 + x_3 + x_4$
Subject to: $x_1 + x_2 \geq 1$
 $x_2 + x_4 \geq 1$
 $x_1 + x_2 + x_3 \geq 1$
 $x_1 + x_3 + x_4 \geq 1$
 $x_4 \geq 1$
 $x_1, x_2, x_3, x_4 \in \{0,1\}$

$U = \{1, 4, 7, 8, 10\}$

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If $x_s \geq \frac{1}{2}$, add set s
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Could this lead to an invalid solution?

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$$\begin{array}{ll}\text{Objective:} & \min x_1 + x_2 + x_3 + x_4 \\ \text{Subject to:} & x_1 + x_2 + x_3 \geq 1 \\ & x_1 + x_2 + x_4 \geq 1 \\ & x_1 + x_3 + x_4 \geq 1 \\ & x_2 + x_3 + x_4 \geq 1 \\ & x_1, x_2, x_3, x_4 \in [0,1]\end{array}$$

$$U = \{1, 2, 3, 4\}$$

$$S = \left\{ \{1, 2, 3\}, \{1, 2, 4\}, \right. \\ \left. \{1, 3, 4\}, \{2, 3, 4\} \right\}$$

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Yes, in this case $x_s = \frac{1}{3}, \forall s \Rightarrow$ No sets are selected (invalid solution).

Set Cover ILP

Where each
element appears
in at most d sets.



Set Cover: Given a universe of elements U and sets S , find the smallest subset of S such that every element in U is in some selected subset.

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Valid? 

Approximation Ratio?

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$$\begin{aligned} \sum x_{LP} &= \sum_{x_i \in x_{LP}} x_i \geq \sum_{x_i \in x_{LP}: x_i \geq \frac{1}{2}} x_i, \text{ because it's a subset of } x_{LP} \\ &\geq \sum_{x_i \in x_{LP}: x_i \geq \frac{1}{2}} \frac{1}{2}, \text{ because each } x_i \text{ is at least } \frac{1}{2} \\ &= \frac{1}{2} \left| \left\{ x_i \in x_{LP}: x_i \geq \frac{1}{2} \right\} \right| = \frac{1}{2} \text{ALG} \end{aligned}$$

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Valid? 

Approximation Ratio?

$$ALG \leq d \text{ OPT}$$

Knapsack

Knapsack: Given a set of n items with values $v_1, \dots, v_n$ and weights $w_1, \dots, w_n$, select the most valuable combination with total weight $\leq W$.

Example:

$$W = 11$$

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

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Example:

$$W = 11$$

Item	Value	Weight	Ratio
1	1	1	1
2	6	2	3
3	18	5	3.6
4	22	6	3.67
5	28	7	4

Knapsack

Knapsack: Given a set of n items with values $v_1, \dots, v_n$ and weights $w_1, \dots, w_n$, select the most valuable combination with total weight $\leq W$.

Example:

$$W = 11$$

Item	Value	Weight	Ratio
1	1	1	1
2	6	2	3
3	18	5	3.6
4	22	6	3.67
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Algorithm: Select highest value-weight items until no space is left.

Approximation Ratio?

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Are there circumstances where this will do very poorly?

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$$ALG = 2 + (W - 1)\varepsilon$$

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**Algorithm: Select highest value-weight items until no space is left
OR select the single highest valued item.**

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Knapsack: Given a set of n items with values $v_1, \dots, v_n$ and weights $w_1, \dots, w_n$, select the most valuable combination with total weight $\leq W$.

Suppose we could select fractional items.

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$$OPT \leq OPT_{frac}$$

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 **Need to show.**

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Work Scheduling

Given a set of n tasks and w identical workers, assign all tasks such that the longest work time is minimized.

Algorithm: Assign each task to the worker with the smallest current work time.

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This holds when task i is assigned and at the end of the schedule, since $ALG - t_i$ does not change as more tasks are scheduled, but T_w 's may get larger.

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Does this
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Can we get a $\frac{1}{w}$ over here?

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What happens if we end up with something like this:

$$ALG \leq OPT + t_i$$

Max Cut

Given an edge weighted undirected graph, find a cut such that the sum of edge weights that cross the cut is maximized.

Algorithm: Assigned each vertex to set A with probability $\frac{1}{2}$. Let $B = V \setminus A$.

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 - 3. $u \in B, v \in A$
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- $\left. \begin{array}{l} 1. \\ 2. \\ 3. \\ 4. \end{array} \right\} \frac{1}{2}$ **probability that e crosses the cut.**

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Let random variable $X_e = \begin{cases} 1 & \text{if } e \text{ crosses cut} \\ 0 & \text{if not} \end{cases}$

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Value of Cut = $\sum_e$?

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Value of Cut = $\sum_e w_e X_e$

$E[\text{Value of Cut}] = \dots$

Use fact: $E[X_e] = \frac{1}{2}$ and $\sum_e w_e \geq OPT$