Inapproximability
CSCI 532

Test 3 Logistics

1. During class on Thursday 11/20.
2. You can bring your book and any notes you would like, but no electronic devices.
3. You may assume anything proven in class or on homework.
4. Three questions (10 points):
   1) Approximation algorithm (5 points).
   2) Special case (2 points).
   3) Tightness example (3 points).

Project Logistics

1. Report due 12/2 (first day of presentations).
2. Presentation schedule posted.
3. 15 minutes <u>maximum</u> presentation time. Leave a couple minutes for questions.
4. Attend other peoples’ presentations. If attendance plummets, I may have to factor attendance into project grades.

Project Logistics

1. Report due 12/2 (first day of presentations).
2. Presentation schedule posted.
3. 15 minutes maximum presentation time. Leave a couple minutes for questions.
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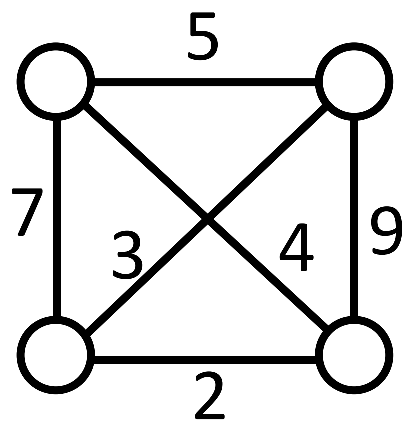
Also, please do the course evaluation:
https://faculty.campuslabs.com/eval-home/direct/8521795

Travelling Salesman Problem

TSP: Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city once and returns to the origin city?

$\in$ NP-Complete

Example:



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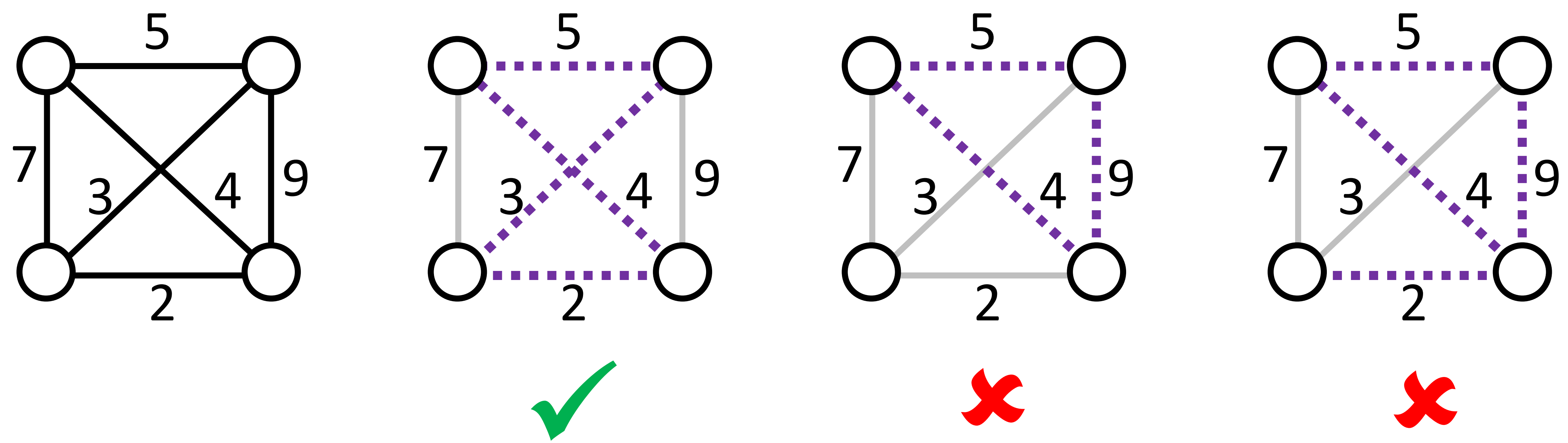

Hamiltonian Cycle Problem

Hamiltonian Cycle: Given a graph, find a cycle that visits each vertex exactly once.

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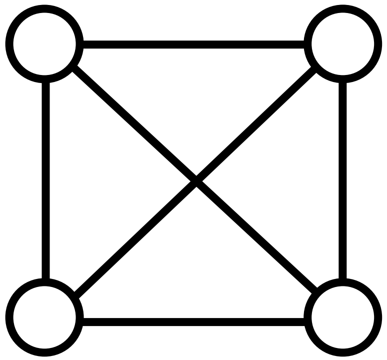

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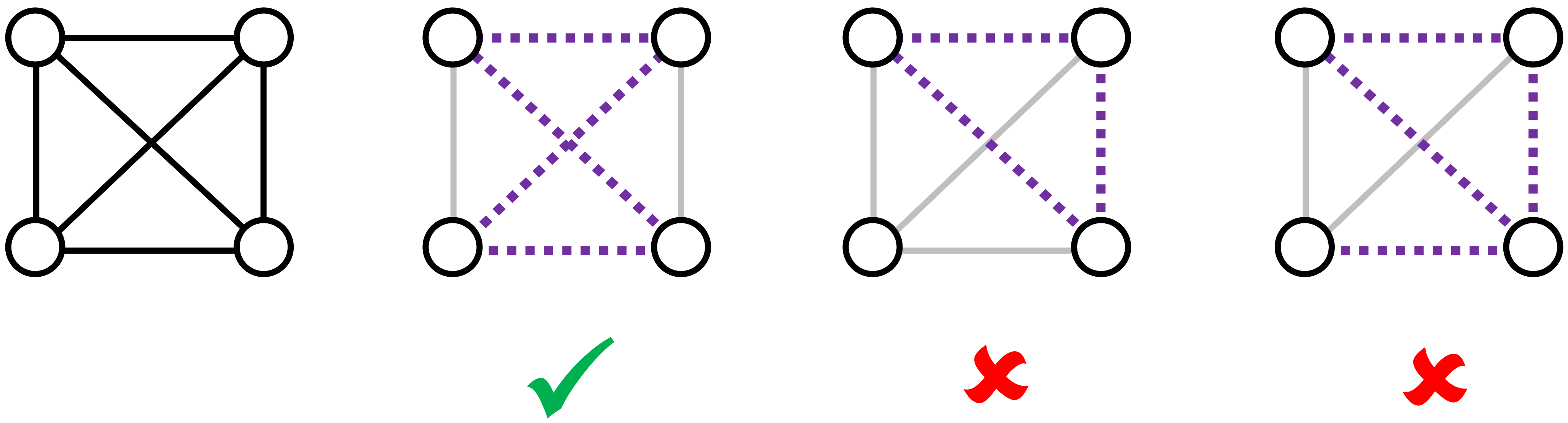

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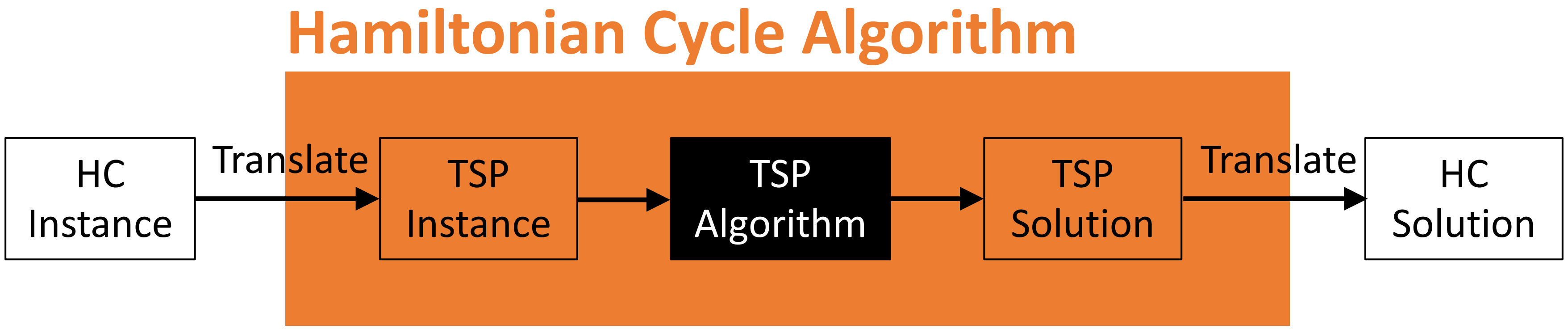

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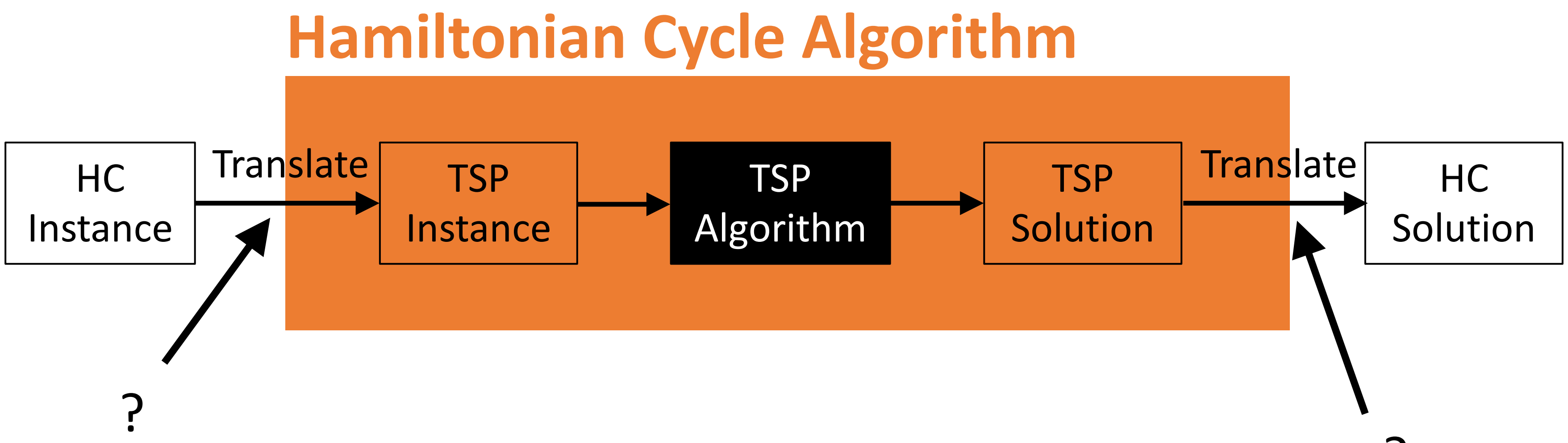

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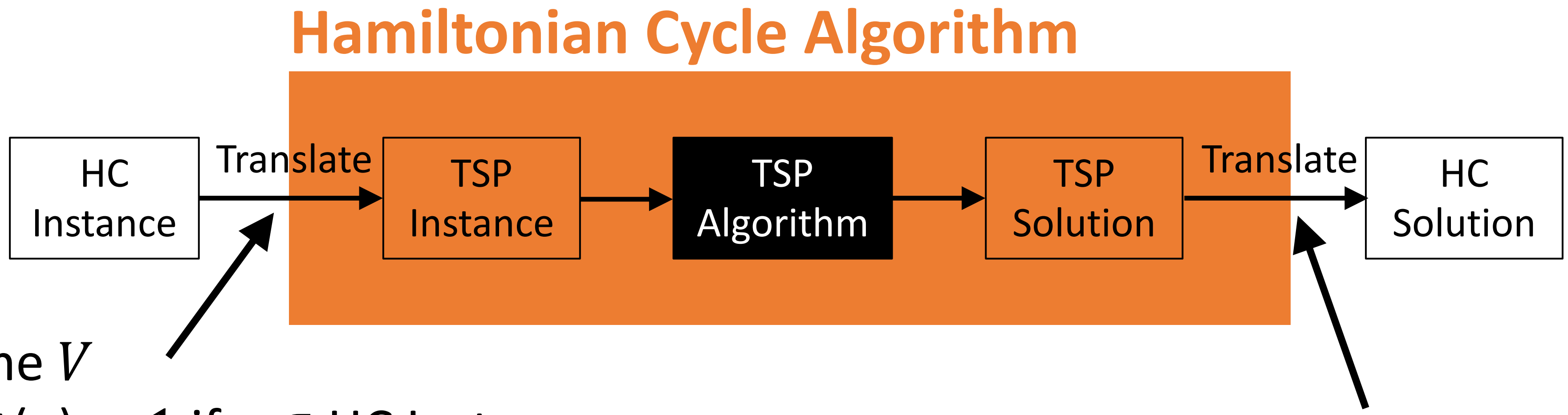

Same V
$\text{cost}(e) = 1$ if $e \in$ HC Instance
$\text{cost}(e) = \infty$ if $e \notin$ HC Instance

HC $\Longleftrightarrow$ TSP

Same selected edges

TSP ∈ NP-Complete

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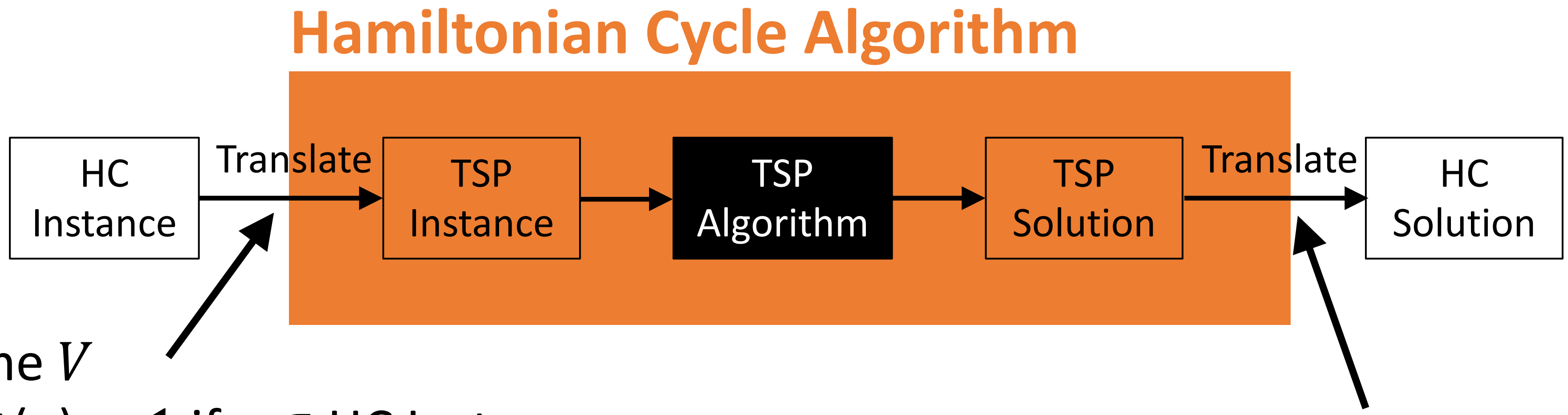

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Same selected edges

TSP Approximation Algorithm

Translation: cost$(e) = 1$ if $e \in$ HC

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TSP Approximation Algorithm

Translation: cost(e) $= 1$ if e ∈ HC
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If $G = (V, E)$ has a Hamiltonian Cycle, $\text{OPT}_{\text{TSP}} =$?

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Let A be an α-approximation algorithm for TSP (i.e. ALG $\leq \alpha$ OPT)
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$\text{ALG}_A \leq$?

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Let $G', \beta = \alpha\,|V|$ be input to TSP and run A:

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$\therefore \nexists$ poly time approx alg for TSP, unless P = NP

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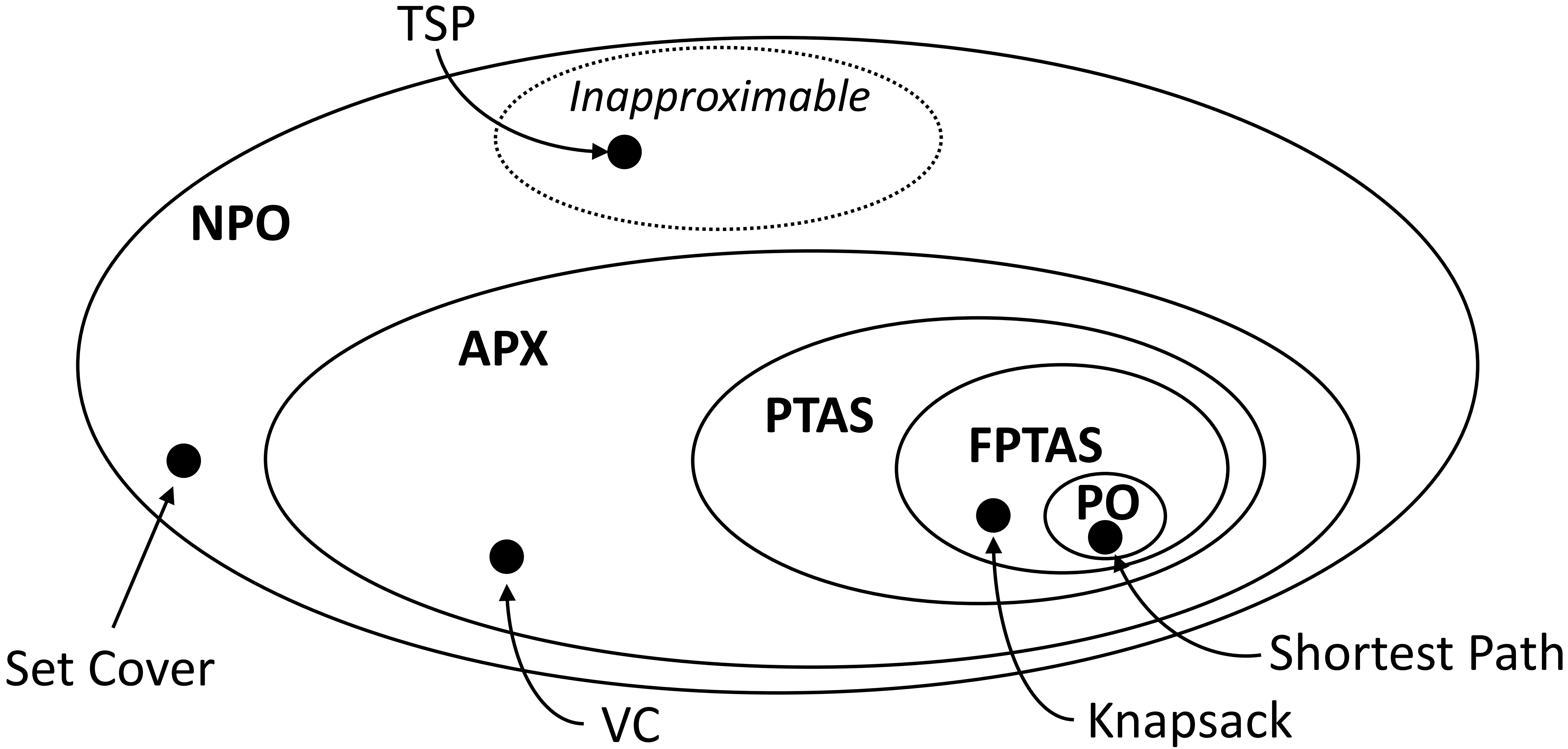
Why is this the case?

Any approx alg for TSP is required to be valid, and a valid output is a Hamiltonian Cycle.

Approximability Hierarchy



Integer Linear Programming

Suppose one could write an ILP for TSP:

$$x_{ij} = \begin{cases} 1 & \text{the path goes from city } i \text{ to city } j \\ 0 & \text{otherwise} \end{cases}$$

For $i = 1, \ldots, n$, let u_i be a dummy variable, and finally take c_{ij} to be the distance from city i to city j.

$$\begin{aligned}
\min \sum_{i=1}^{n} \sum_{j \neq i, j=1}^{n} & c_{ij} x_{ij} \colon && \\
& x_{ij} \in \{0,1\} && i,j = 1,\ldots,n; \\
& u_i \in \mathbf{Z} && i = 2,\ldots,n; \\
& \sum_{i=1, i \neq j}^{n} x_{ij} = 1 && j = 1,\ldots,n; \\
& \sum_{j=1, j \neq i}^{n} x_{ij} = 1 && i = 1,\ldots,n; \\
& u_i - u_j + n x_{ij} \leq n - 1 && 2 \leq i \neq j \leq n; \\
& 0 \leq u_i \leq n - 1 && 2 \leq i \leq n.
\end{aligned}$$

From: Wikipedia

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What could you conclude?

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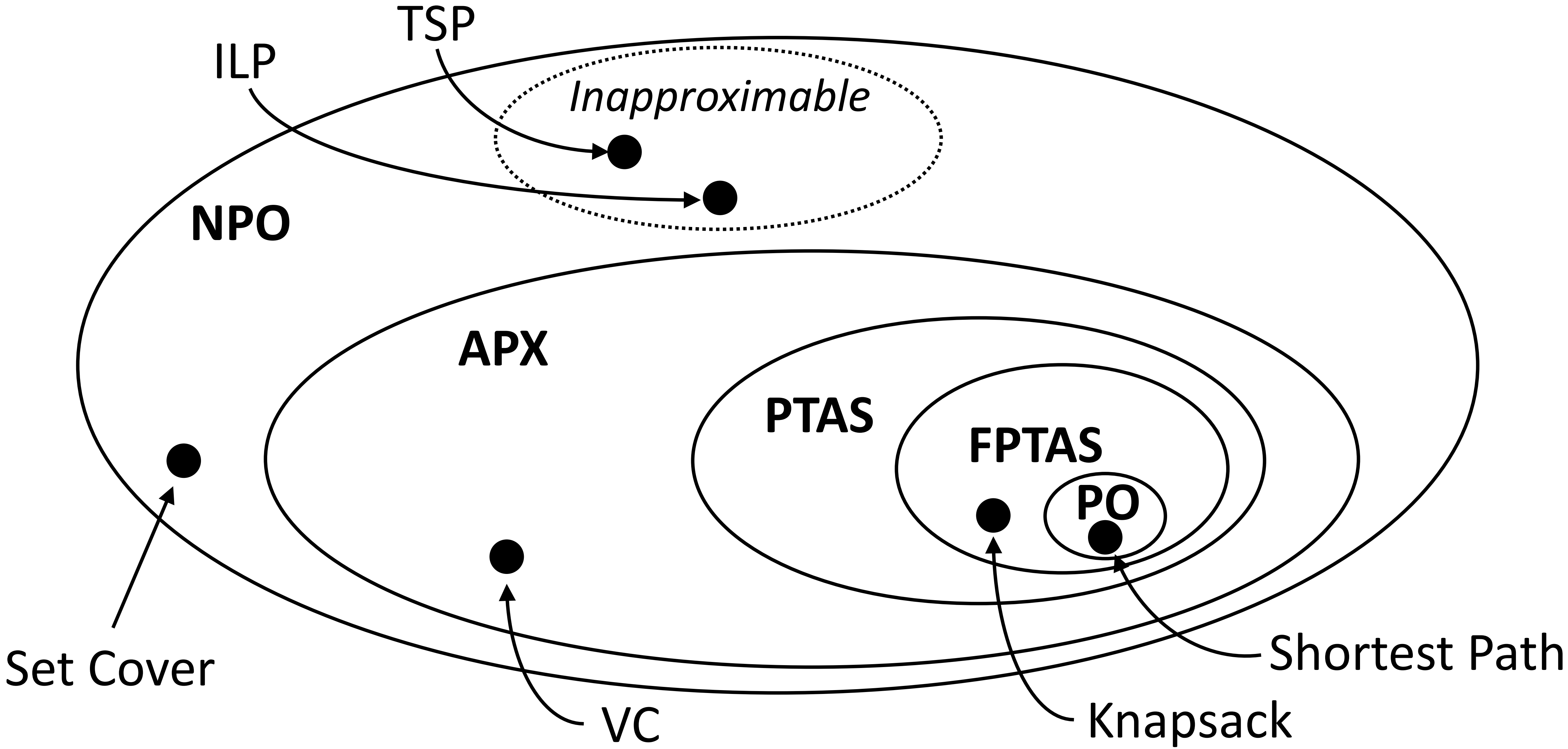
What could you conclude?

If TSP in inapproximable, solving ILPs must be too!

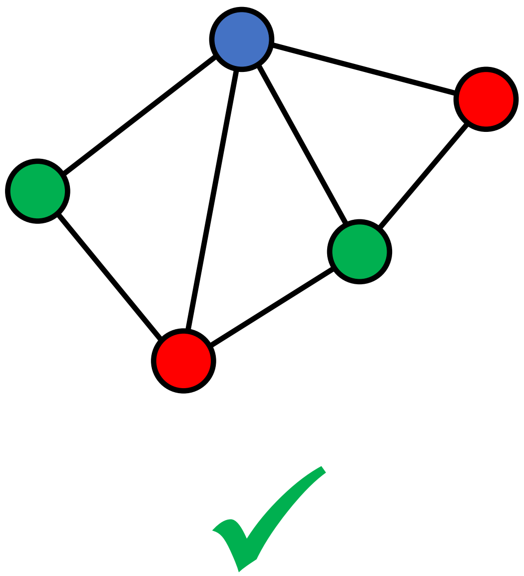
(Otherwise an approximation algorithm for ILPs would provide an approximation algorithm for TSP.)

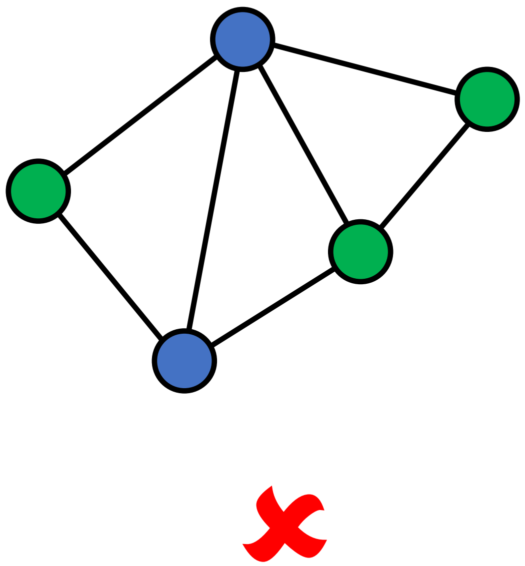
Approximability Hierarchy



Graph Coloring

Minimum Graph Coloring: Given graph, find the smallest number of colors such that each vertex can be assigned a color and neighboring vertices do not share colors.





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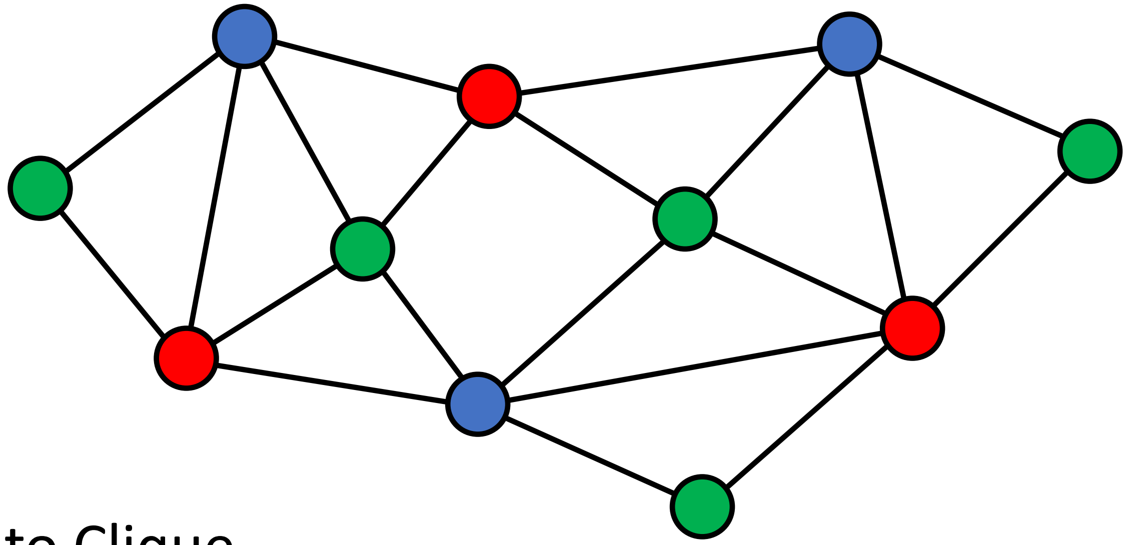

- Related to Clique.
- Influenced by vertex degree.

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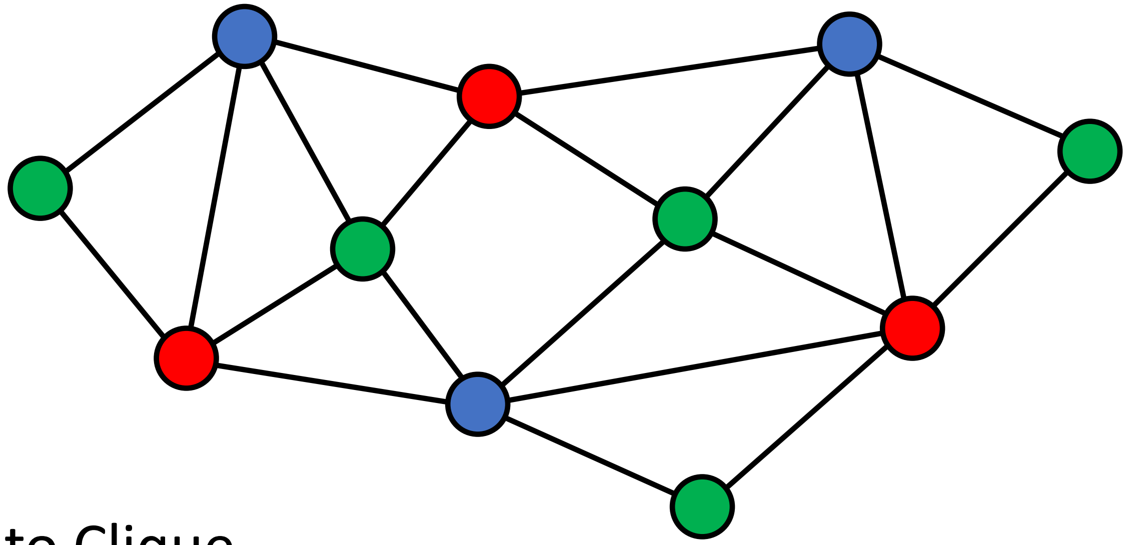

n	$\chi(n)$
0	?

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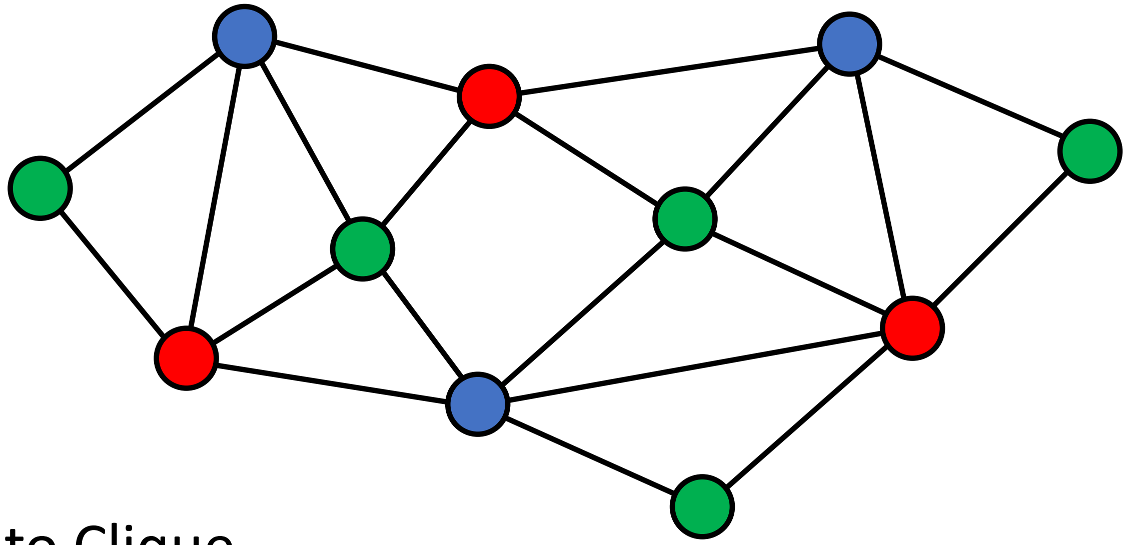

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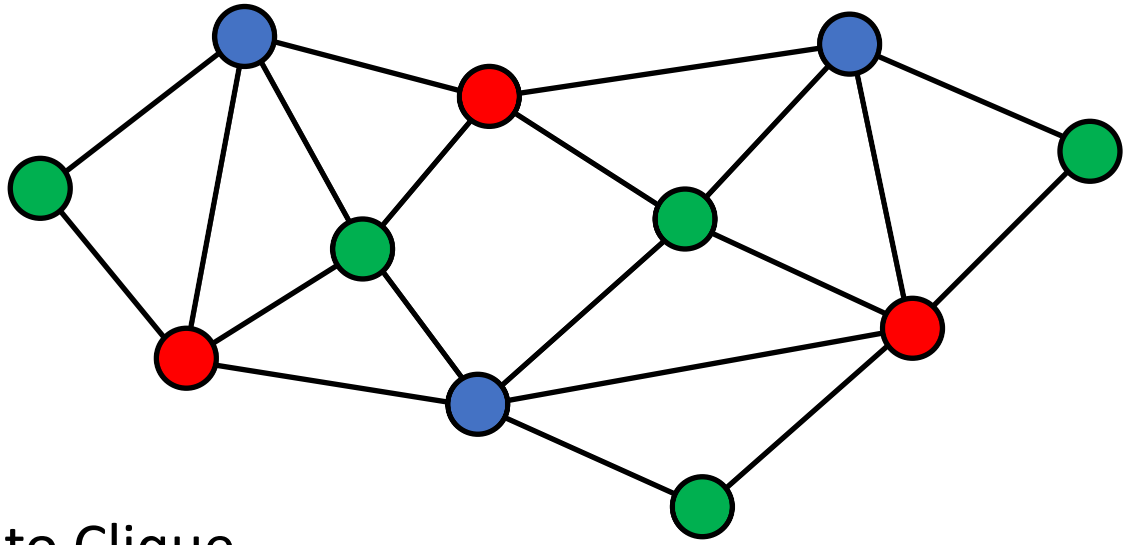

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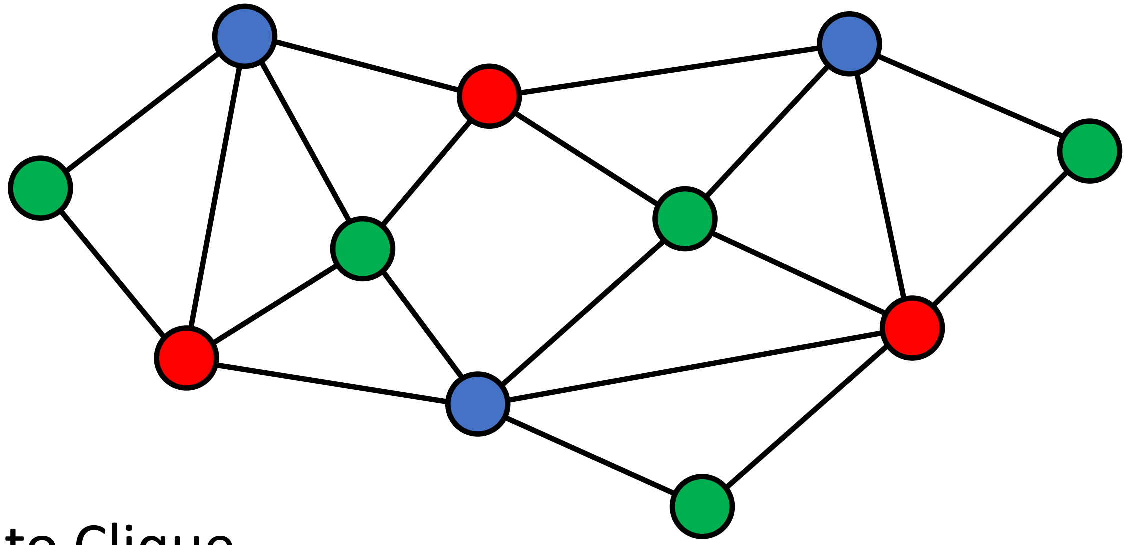

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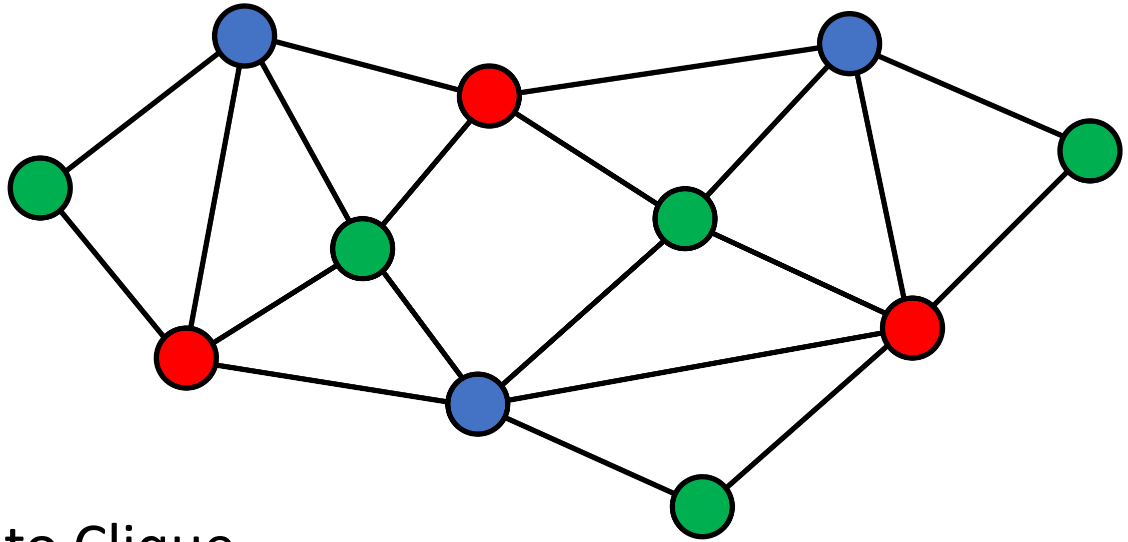

n	$\chi(n)$
0	✓
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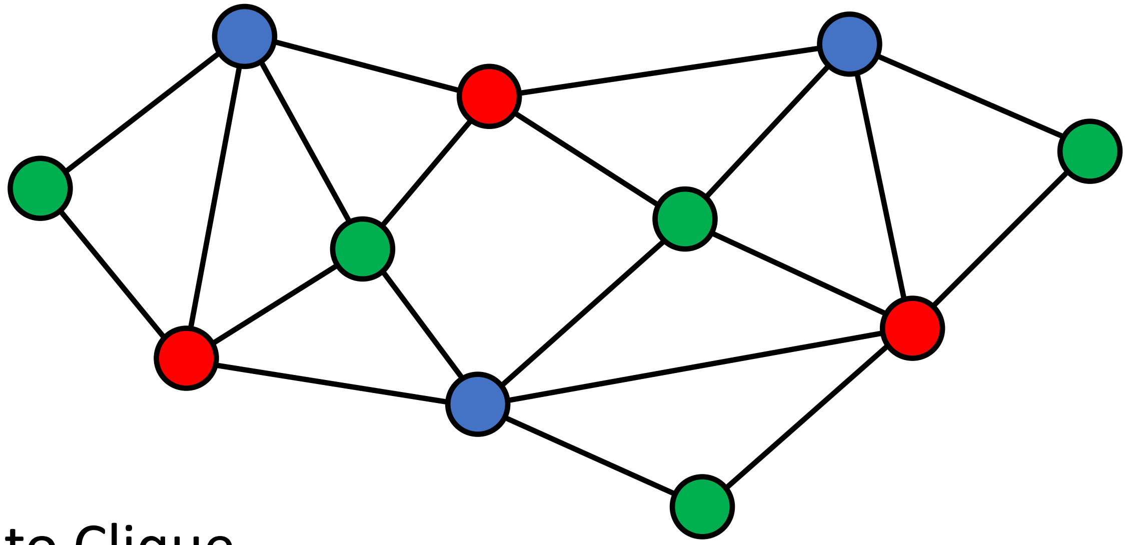

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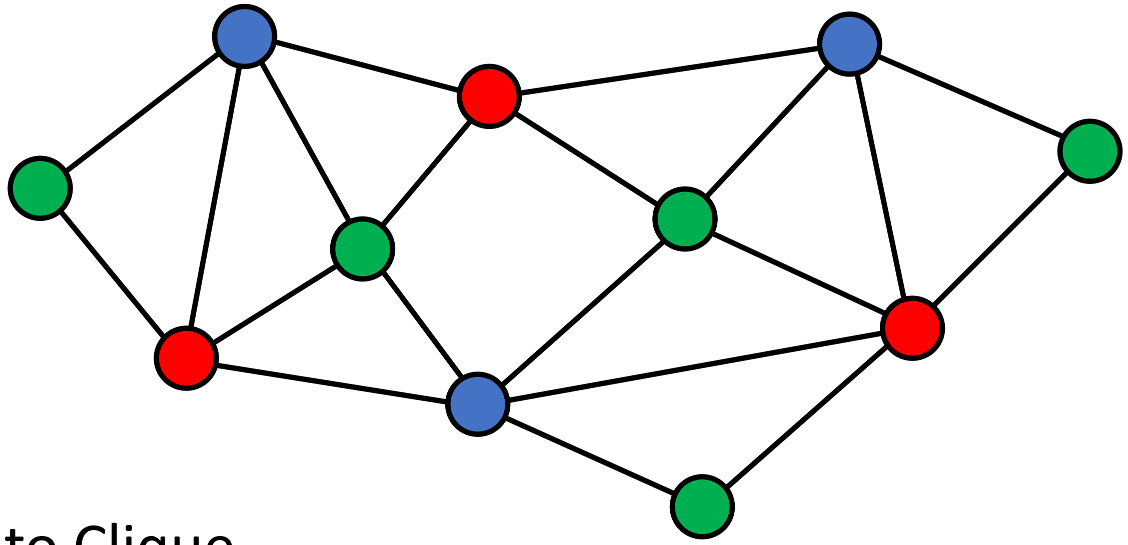

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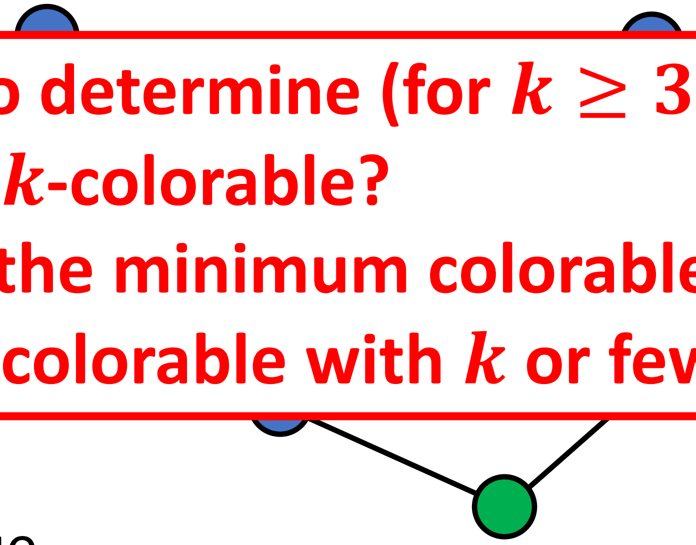
0	✓
1	✓
2	✓
3	✗
4	✗
⋮	⋮

- Related to Clique.
- Influenced by vertex degree.

Graph Coloring

Minimum Graph Coloring: Given graph, find the smallest number of colors such that each vertex can be assigned a color and neighboring vertices do not share colors.



NP-Hard to determine (for $k \geq 3$):
- **Is graph k-colorable?**
- **What is the minimum colorable value?**
- **Is graph colorable with k or fewer colors?**

n	$\chi(n)$
0	✓
1	✓
2	✓
3	✗
4	✗
⋮	⋮

- Related to Clique.
- Influenced by vertex degree.

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Suppose we had a $\left(\frac{4}{3} - \varepsilon\right)$-approximation algorithm for the problem.

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What happens when we run it on a graph with $\chi(G) = 3$?

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$$ALG \leq \left(\frac{4}{3} - \varepsilon\right) OPT$$

Graph Coloring

Minimum Graph Coloring: Given graph, find the smallest number of colors such that each vertex can be assigned a color and neighboring vertices do not share colors.

Suppose we had a $\left(\frac{4}{3}-\varepsilon\right)$-approximation algorithm for the problem.

What happens when we run it on a graph with $\chi(G) = 3$?

$$ALG \leq \left(\frac{4}{3}-\varepsilon\right) OPT = \left(\frac{4}{3}-\varepsilon\right) 3 = 4 - 3\varepsilon$$

Graph Coloring

Minimum Graph Coloring: Given graph, find the smallest number of colors such that each vertex can be assigned a color and neighboring vertices do not share colors.

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What happens when we run it on a graph with $\chi(G) = 3$?

$$ALG \leq \left(\frac{4}{3} - \varepsilon\right) OPT = \left(\frac{4}{3} - \varepsilon\right) 3 = 4 - 3\varepsilon < 4$$

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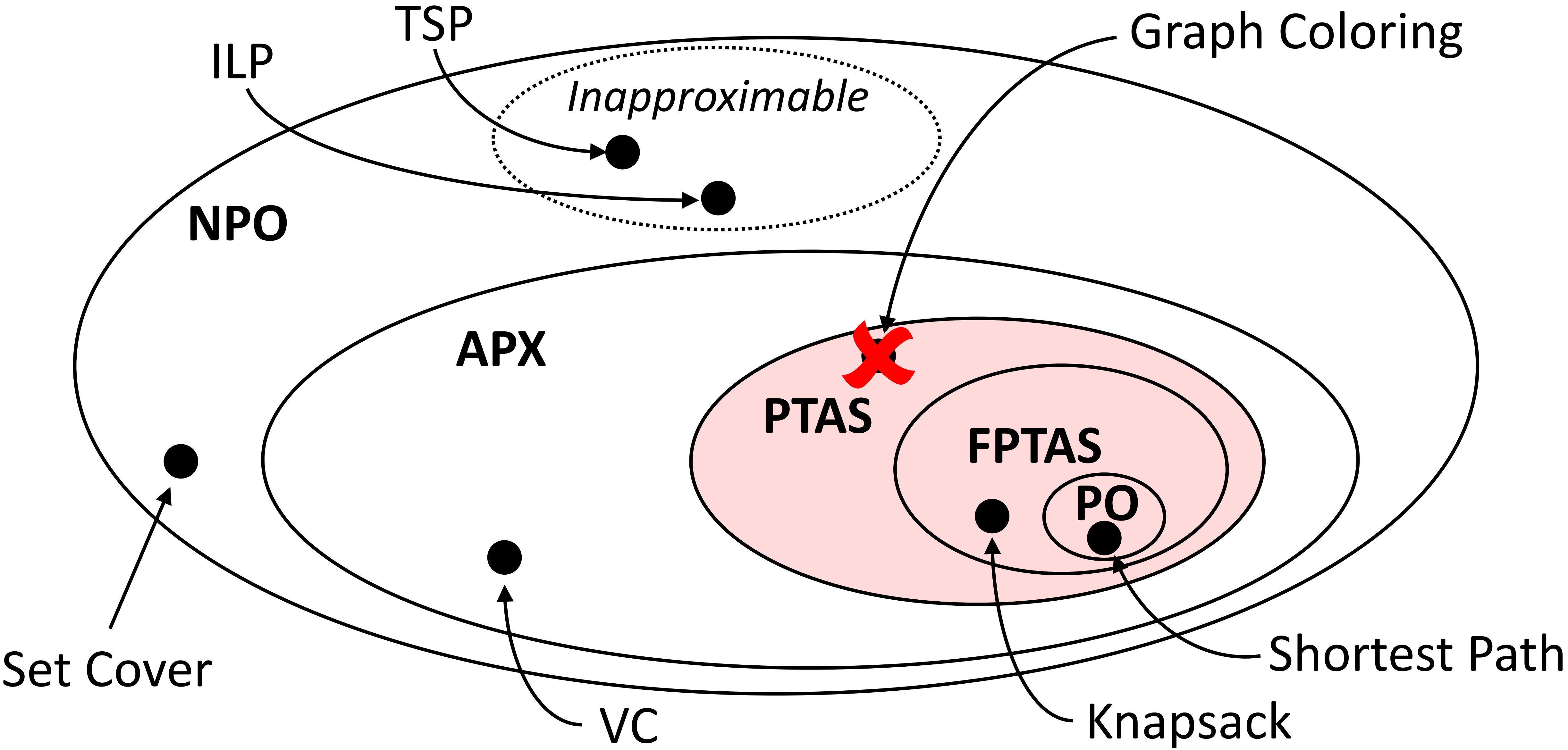
$$ALG \leq \left(\frac{4}{3} - \varepsilon\right) OPT = \left(\frac{4}{3} - \varepsilon\right)\ 3 = 4 - 3\varepsilon < 4 \Rightarrow ALG \leq 3$$

Thus, we could determine if a graph is 3-colorable

$\Rightarrow$ algorithm can't exist.

Approximability Hierarchy



Special Case - Metric TSP

TSP: Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city once and returns to the origin city?

Special Case - Metric TSP

TSP: Given a list of cities and the distances between each pair of cities **(satisfying the triangle inequality)**, what is the shortest possible route that visits each city once and returns to the origin city?

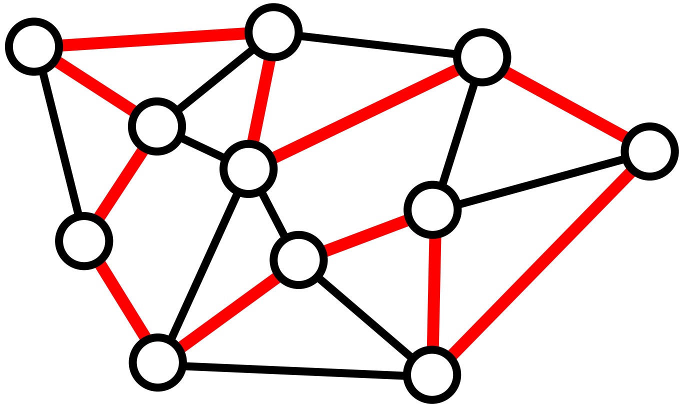
$$\text{dist}(u, v) \leq \text{dist}(u, w) + \text{dist}(w, v)$$

Special Case - Metric TSP

Find some structure that is:

1. Easy to compute.
2. Related to TSP.
3. Lower bound on OPT.

Special Case - Metric TSP



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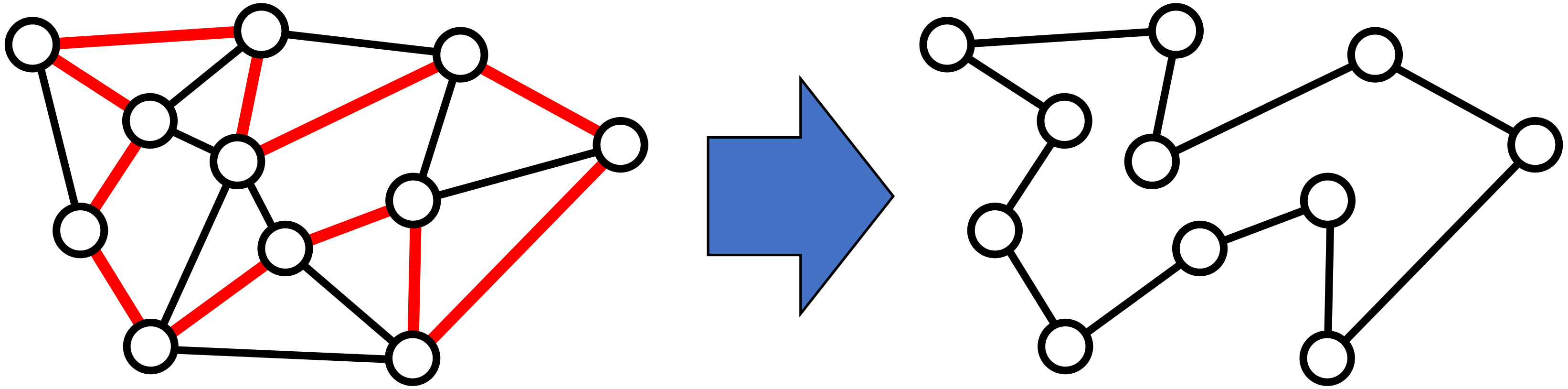

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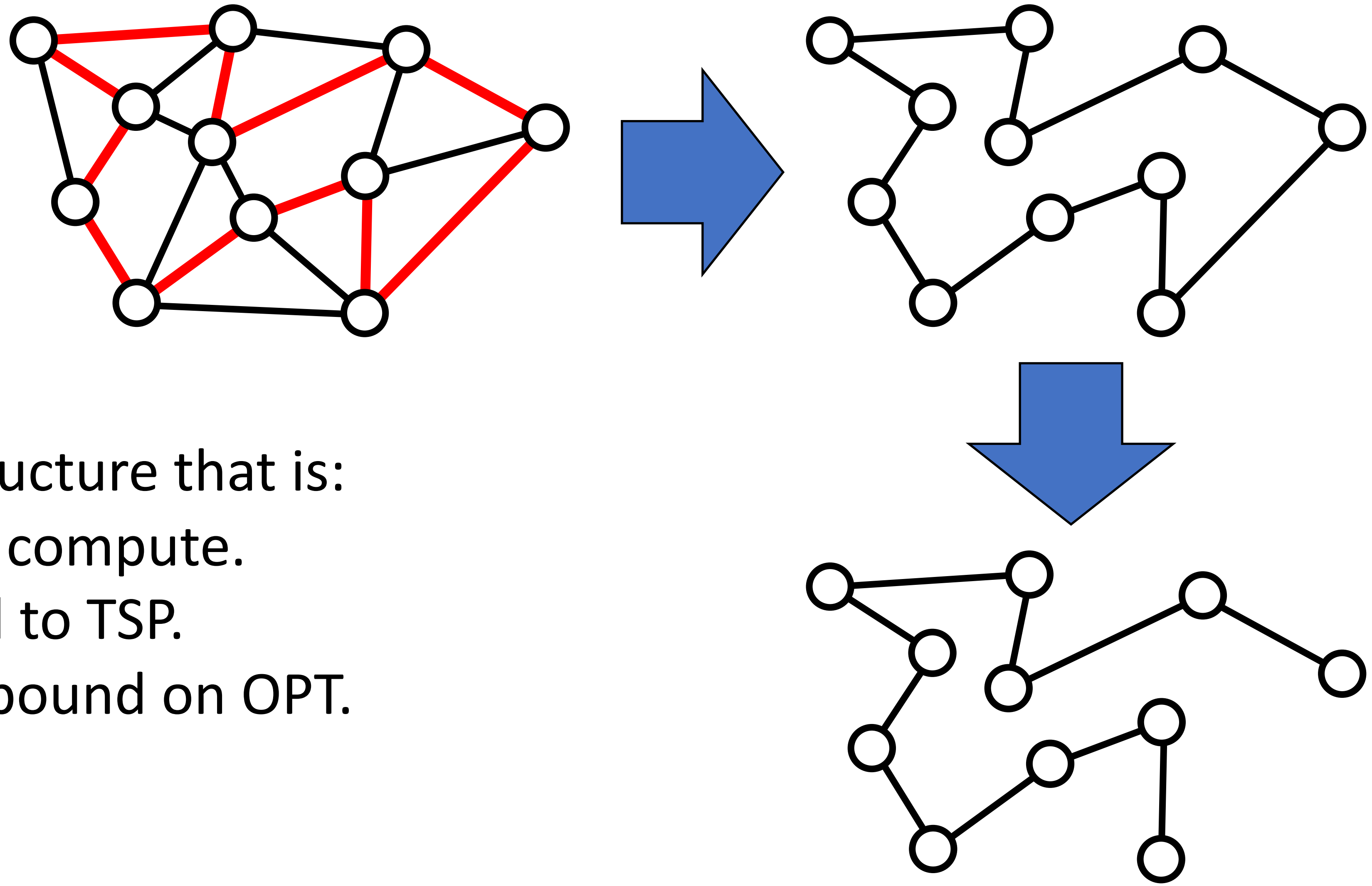

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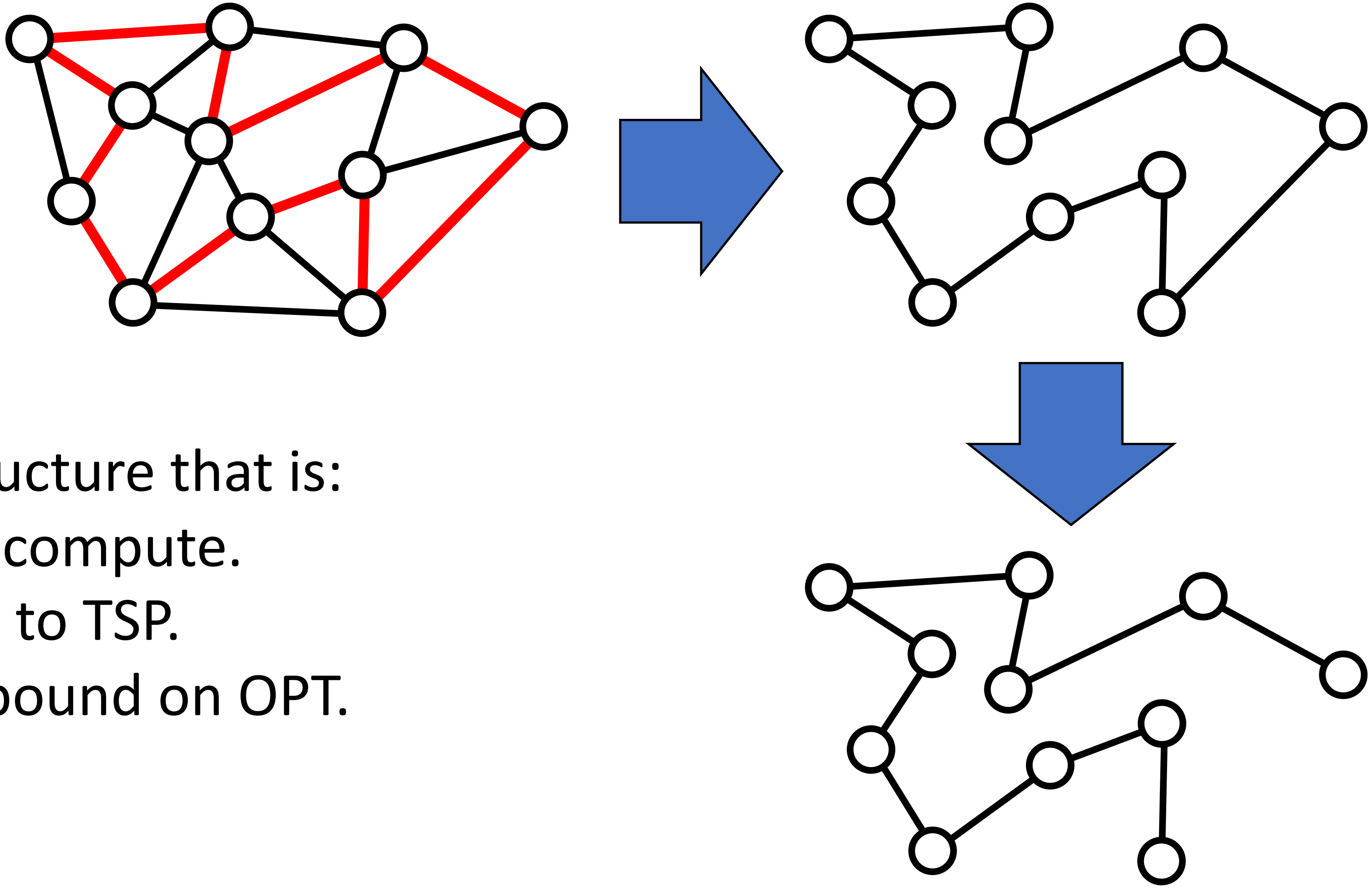
Special Case - Metric TSP



Find some structure that is:

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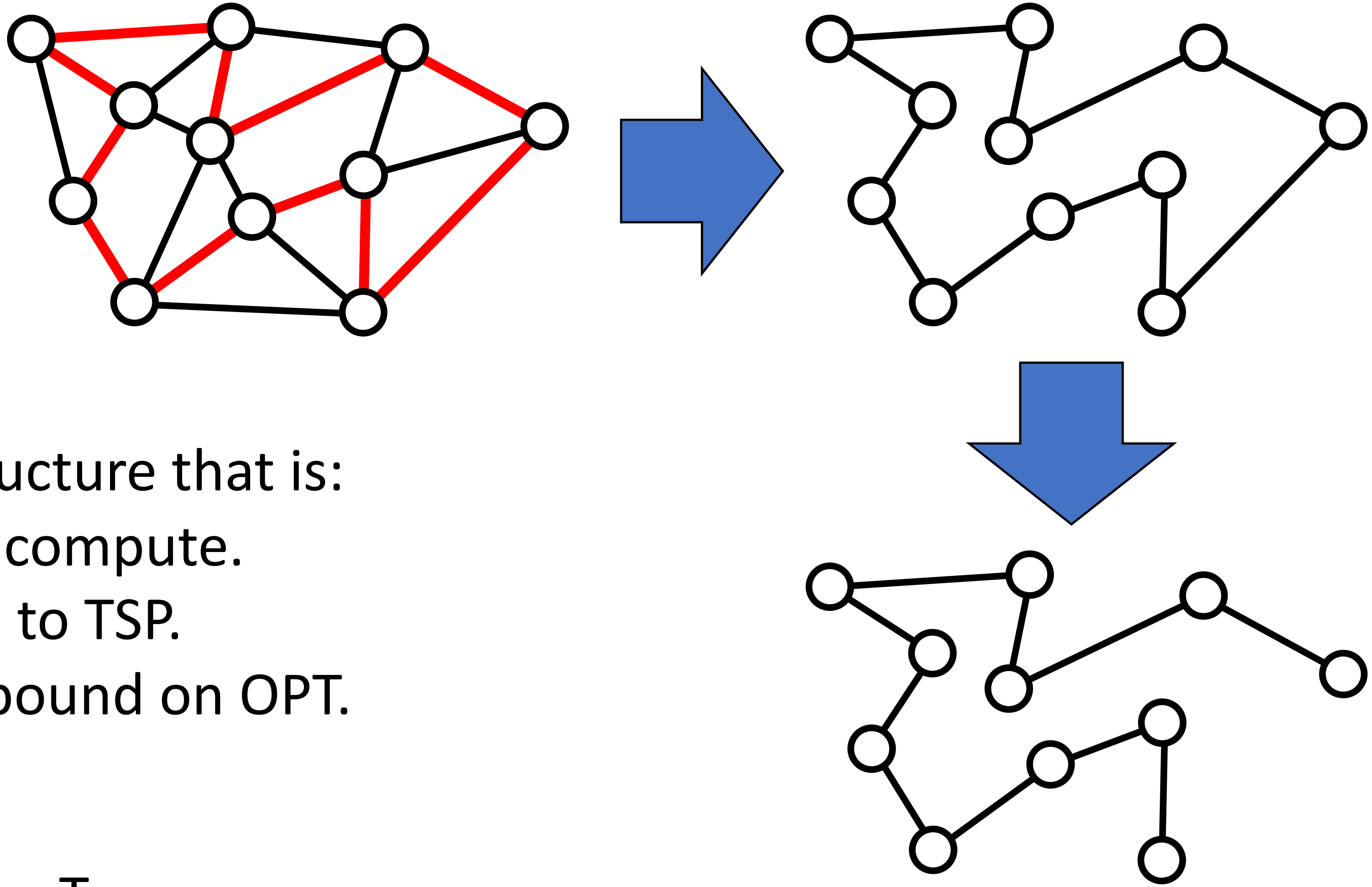
What is this?

Special Case - Metric TSP



Find some structure that is:

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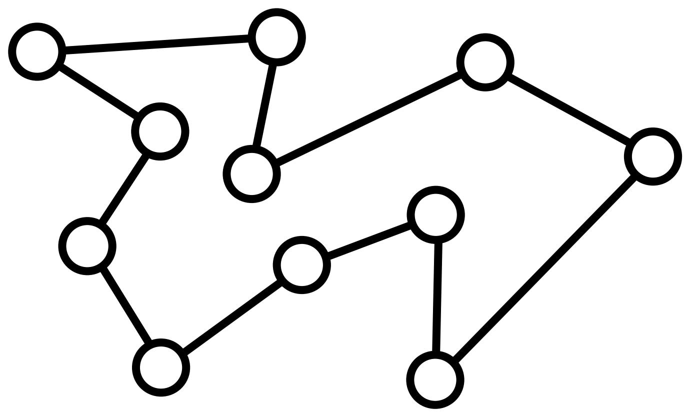
3. Lower bound on OPT.

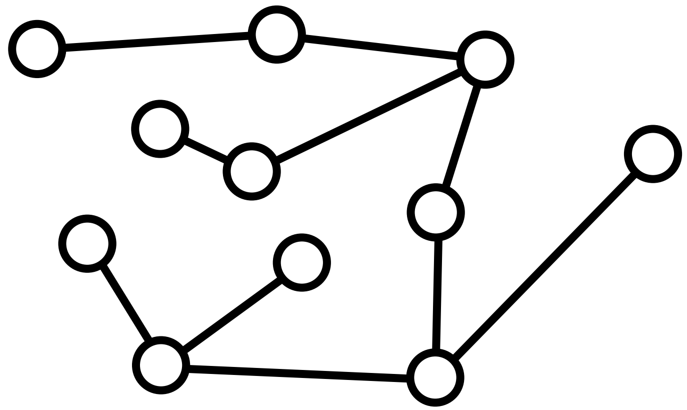
What is this?

Spanning Tree

Special Case - Metric TSP

Relationship between OPT and cost of MST?



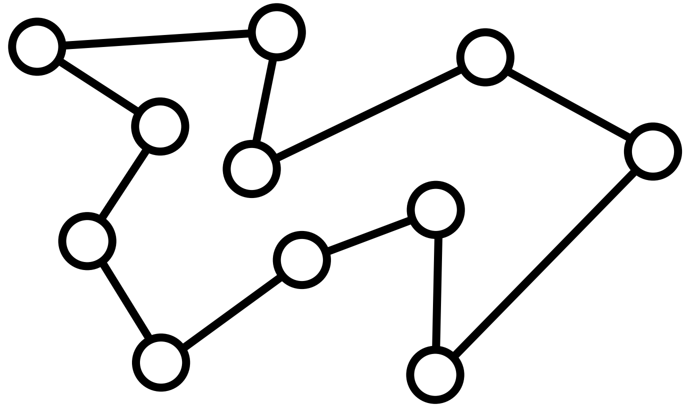


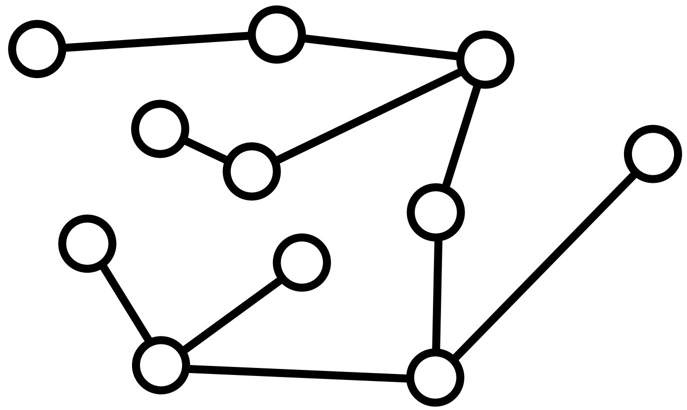
Special Case - Metric TSP

Relationship between OPT and cost of MST?

$\text{OPT} \geq \text{cost(MST)}$

How to turn MST into tour of cities?





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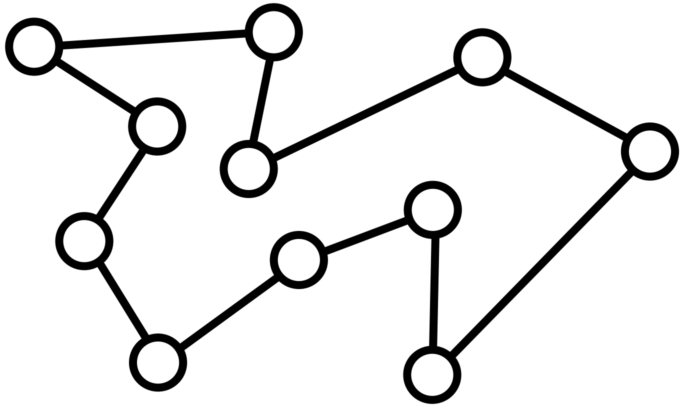
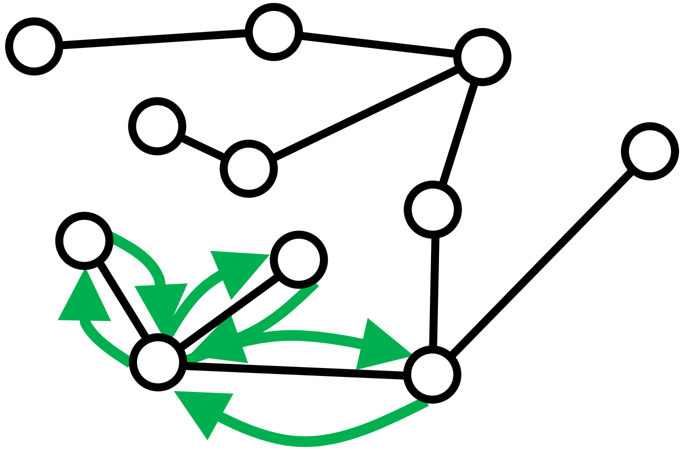
How to turn MST into tour of cities?

What is the cost of this tour?





Special Case - Metric TSP

Relationship between OPT and cost of MST?

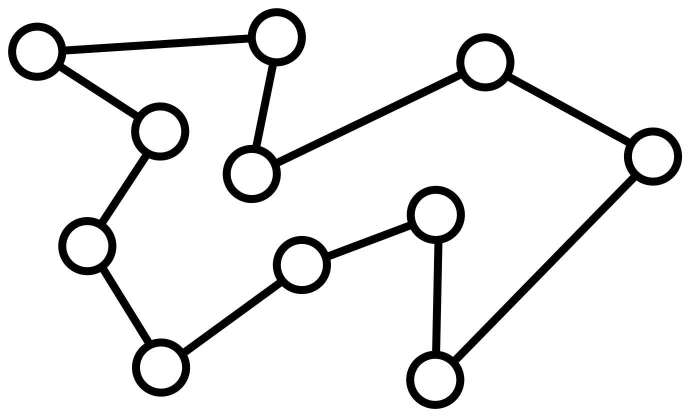
$$\text{OPT} \geq \text{cost}(\text{MST})$$

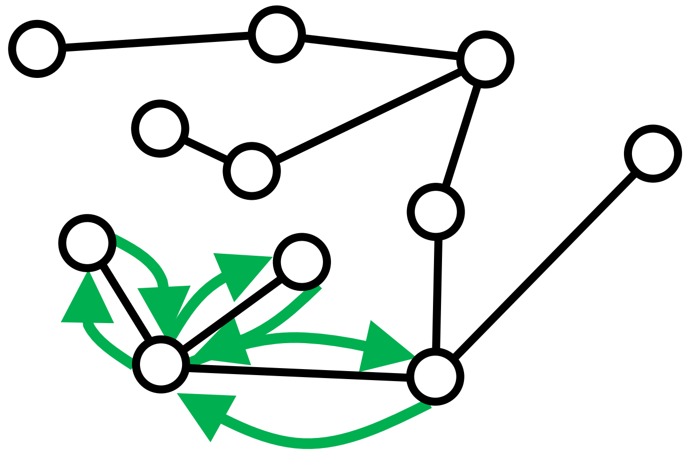
How to turn MST into tour of cities?

What is the cost of this tour?

$$\text{ALG} = 2\ \text{cost}(\text{MST})$$

Relationship between ALG and OPT?





Special Case - Metric TSP

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How to turn MST into tour of cities?

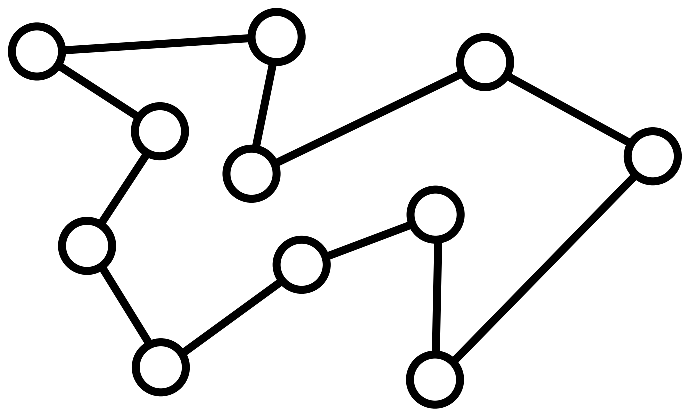
What is the cost of this tour?

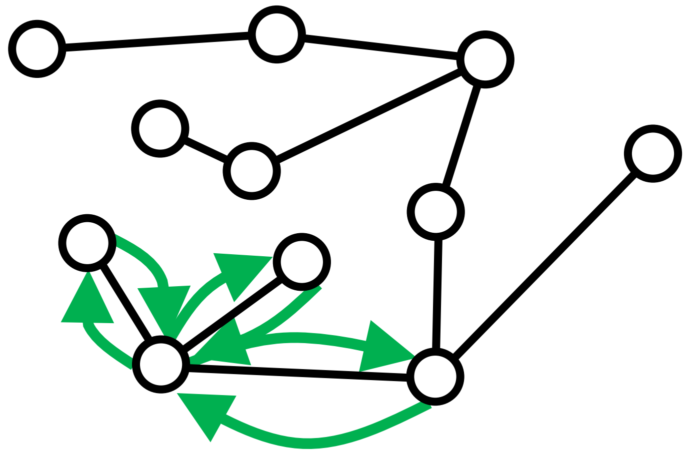
$$\text{ALG} = 2\ \text{cost(MST)}$$

Relationship between ALG and OPT?

$$\text{ALG} = 2\ \text{cost(MST)} \leq 2\ \text{OPT}$$

Any problems?





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Relationship between OPT and cost of MST?

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How to turn MST into tour of cities?

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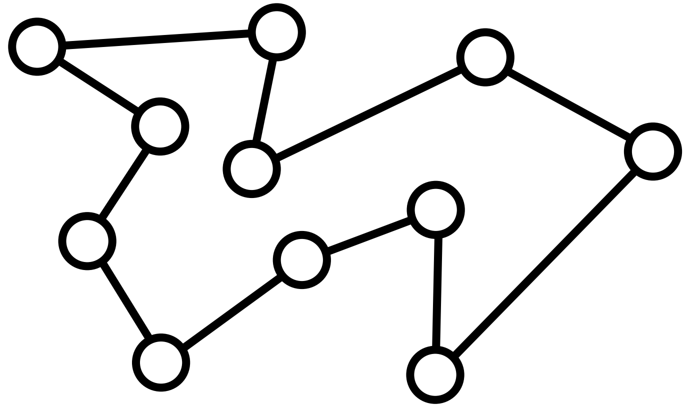
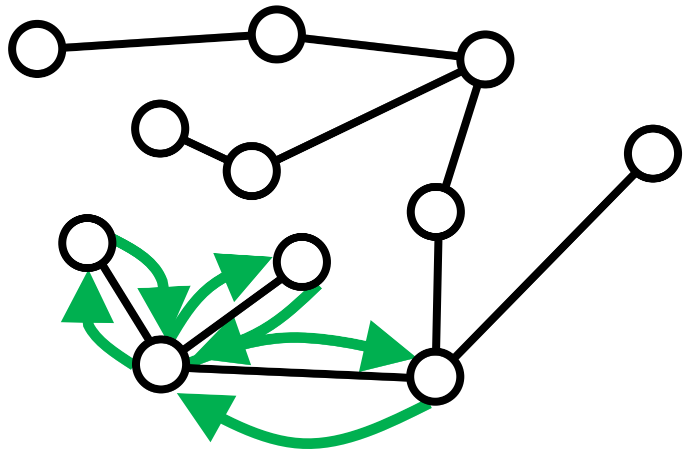
Relationship between ALG and OPT?

$$\text{ALG} = 2\ \text{cost(MST)} \leq 2\ \text{OPT}$$

How can we eliminate double visits (without messing up the cost)?





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How to turn MST into tour of cities?

What is the cost of this tour?

$\mathrm{ALG} = 2\ \mathrm{cost}(\mathrm{MST})$

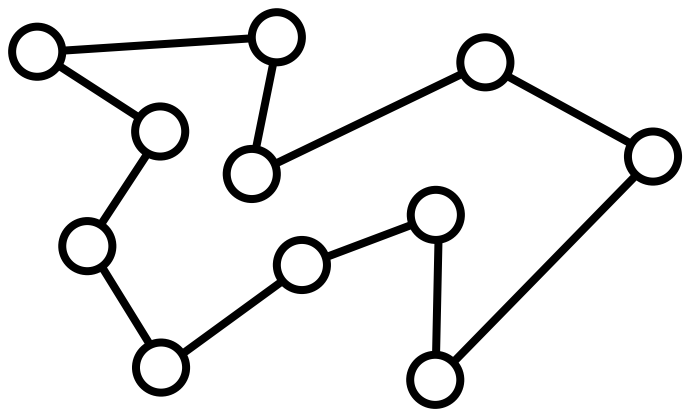
Relationship between ALG and OPT?

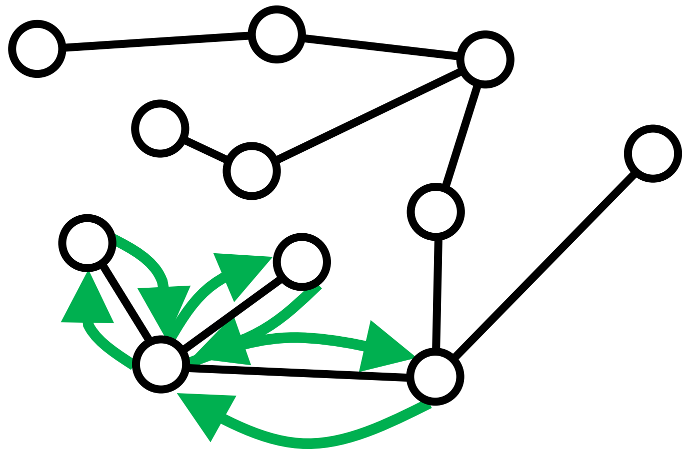
$\mathrm{ALG} = 2\ \mathrm{cost}(\mathrm{MST}) \leq 2\ \mathrm{OPT}$

How can we eliminate double visits (without messing up the cost)?

Skip to next unvisited vertex. Can only decrease cost (triangle inequality).

$\mathrm{dist}(u, v) \leq \mathrm{dist}(u, w) + \mathrm{dist}(w, v)$





Approximability Hierarchy