Linear Program Relaxations
CSCI 532
\[ G = (V, E) \]
\[ G' = (V, E') \]
(complement graph)
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\[ G' = (V, E') \]

(complement graph)
Clique of size $k$

$$G = (V, E)$$

Vertex cover of size $|V| - k$

$$G' = (V, E')$$
(complement graph)
Vertex Cover ILP

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.
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Objective: \( \min \sum_i x_i \)

Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
\( x_i \in \{0, 1\} \), for each vertex \( i \)
Vertex Cover ILP

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Subject to: \[ x_i + x_j \geq 1, \text{ for each edge } e = (i, j) \]
\[ x_i \in \{0, 1\}, \text{ for each vertex } i \]

Example:

Objective: \[ \min x_1 + x_2 + x_3 + x_4 \]
Subject to: \[ x_1 + x_2 \geq 1 \]
\[ x_2 + x_3 \geq 1 \]
\[ x_2 + x_4 \geq 1 \]
\[ x_3 + x_4 \geq 1 \]
\[ x_1, x_2, x_3, x_4 \in \{0, 1\} \]
Vertex Cover ILP

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\( \in \text{NP-Complete} \)
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\( \in \text{P} \)
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\( \in \text{NP-Complete} \)

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\( 0 \leq x_i \leq 1 \), for each vertex \( i \)

\( \in \text{P} \)

LP Relaxation: Remove all integrality constraints to turn ILP into LP.
Vertex Cover ILP

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

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Subject to: \[ x_i + x_j \geq 1, \text{ for each edge } e = (i, j) \]
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Vertex Cover ILP

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Subject to: \( x_i + x_j \geq 1, \) for each edge \( e = (i,j) \)
\( 0 \leq x_i \leq 1, \) for each vertex \( i \)

If \( x_i = 1, \) what should we do with vertex \( i? \)
Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Objective: \( \min \sum_i x_i \)
Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
\( 0 \leq x_i \leq 1 \), for each vertex \( i \)

If \( x_i = 1 \), what should we do with vertex \( i \)? Add to subset \( S \)
If \( x_i = 0 \), what should we do with vertex \( i \)?
Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Objective: $\min \sum_{i} x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

$0 \leq x_i \leq 1$, for each vertex $i$

If $x_i = 1$, what should we do with vertex $i$? Add to subset $S$

If $x_i = 0$, what should we do with vertex $i$? Don’t add to subset $S$
Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Objective: \( \min \sum_i x_i \)
Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
\( 0 \leq x_i \leq 1 \), for each vertex \( i \)

If \( x_i = 1 \), what should we do with vertex \( i \)? Add to subset \( S \)
If \( x_i = 0 \), what should we do with vertex \( i \)? Don’t add to subset \( S \)
If \( x_i = \frac{126}{337} \), what should we do with vertex \( i \)?
Vertex Cover ILP

Objective: \[ \min \sum_i x_i \]
Subject to: \[ x_i + x_j \geq 1, \text{ for each edge } e = (i, j) \]
\[ 0 \leq x_i \leq 1, \text{ for each vertex } i \]

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).
Vertex Cover ILP

Objective: \( \min \sum_i x_i \)
Subject to: \( x_i + x_j \geq 1, \text{ for each edge } e = (i, j) \)
\( 0 \leq x_i \leq 1, \text{ for each vertex } i \)

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).

Is \( S \) a vertex cover?
Vertex Cover ILP

Objective: \[ \min \sum_{i} x_i \]
Subject to: \[ x_i + x_j \geq 1, \text{ for each edge } e = (i, j) \]
\[ 0 \leq x_i \leq 1, \text{ for each vertex } i \]

Is \( S \) a vertex cover?
Yes. For every edge, \( x_i + x_j \geq 1 \).

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).
Vertex Cover ILP

Objective: \( \min \sum_i x_i \)
Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
\( 0 \leq x_i \leq 1 \), for each vertex \( i \)

+ If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).

Is \( S \) a vertex cover?
Yes. For every edge, \( x_i + x_j \geq 1 \). Thus, at least one of \( x_i \) or \( x_j \geq \frac{1}{2} \).
Vertex Cover ILP

**Objective:** \( \min \sum_i x_i \)

**Subject to:**
\[
\begin{align*}
  x_i + x_j & \geq 1, \text{ for each edge } e = (i, j) \\
  0 \leq x_i \leq 1, & \text{ for each vertex } i
\end{align*}
\]

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).

Is \( S \) a vertex cover?
Yes. For every edge, \( x_i + x_j \geq 1 \). Thus, at least one of \( x_i \) or \( x_j \geq \frac{1}{2} \). So for every edge, at least one of its vertices will be in \( S \).
Vertex Cover ILP

Objective: \( \min \sum_i x_i \)
Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
\( 0 \leq x_i \leq 1 \), for each vertex \( i \)

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).

What is the relationship between \( \text{ALG} = |S| \) and \( \text{OPT} \)?
Vertex Cover ILP

Objective: \( \min \sum_i x_i \)
Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
\( 0 \leq x_i \leq 1 \), for each vertex \( i \)

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).

Can we bound \( \text{OPT} \) from below?
Vertex Cover ILP

Objective: \[ \min \sum_i x_i \]
Subject to: \[ x_i + x_j \geq 1, \text{ for each edge } e = (i, j) \]
\[ 0 \leq x_i \leq 1, \text{ for each vertex } i \]

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).

Can we bound \( \text{OPT} \) from below?

Let \( x_{\text{ILP}} \) and \( x_{\text{LP}} \) be the set of \( x \) values found by the ILP and LP.
Vertex Cover ILP

Objective: \( \min \sum_i x_i \)
Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
\( 0 \leq x_i \leq 1 \), for each vertex \( i \)

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).

Can we bound \( \text{OPT} \) from below?

Let \( x_{ILP} \) and \( x_{LP} \) be the set of \( x \) values found by the ILP and LP

Claim: \( \sum x_{LP} \leq \text{OPT} \).
Vertex Cover ILP

Objective: \( \min \sum_i x_i \)
Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
\( 0 \leq x_i \leq 1 \), for each vertex \( i \)

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).

Can we bound \( \text{OPT} \) from below?

Let \( x_{\text{ILP}} \) and \( x_{\text{LP}} \) be the set of \( x \) values found by the ILP and LP

Claim: \( \sum x_{\text{LP}} \leq \text{OPT} \).

Proof: \( \text{OPT} = ? \)
Vertex Cover ILP

Objective: \( \min \sum_i x_i \)
Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
\( 0 \leq x_i \leq 1 \), for each vertex \( i \)

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).

Can we bound OPT from below?

Let \( x_{\text{ILP}} \) and \( x_{\text{LP}} \) be the set of x values found by the ILP and LP

Claim: \( \sum x_{\text{LP}} \leq \text{OPT} \).

Proof: \( \text{OPT} = \sum x_{\text{ILP}} \), where \( x_i \in \{0, 1\} \)
**Vertex Cover ILP**

Objective: \( \min \sum_i x_i \)

Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
\( 0 \leq x_i \leq 1 \), for each vertex \( i \)

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).

Can we bound OPT from below?

Let \( x_{\text{ILP}} \) and \( x_{\text{LP}} \) be the set of x values found by the ILP and LP

Claim: \( \sum x_{\text{LP}} \leq \text{OPT} \).

Proof: \( \text{OPT} = \sum x_{\text{ILP}} \), where \( x_i \in \{0,1\} \). When \( x_i \) is relaxed so that \( 0 \leq x_i \leq 1 \), this gives more possibilities to further decrease \( \sum_i x_i \). Thus, \( \sum x_{\text{LP}} \leq \text{OPT} \).
Vertex Cover ILP

Objective: \[ \min \sum_i x_i \]
Subject to: \[ x_i + x_j \geq 1, \text{ for each edge } e = (i, j) \]
\[ 0 \leq x_i \leq 1, \text{ for each vertex } i \]

Can we bound OPT from below?

**Law of LP Relaxations:**

\[ \text{OPT}_{LP} \leq \text{OPT}_{ILP} \]

*(minimization problem)*

\[ \sum_i x_i \leq \text{OPT} \]

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).

*Objective values, not individual variable values.*
Vertex Cover ILP

Objective: \( \min \sum_i x_i \)

Subject to: 
\[ x_i + x_j \geq 1, \text{ for each edge } e = (i, j) \]
\[ 0 \leq x_i \leq 1, \text{ for each vertex } i \]

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).

How does \( \sum x_{\text{LP}} \) relate to ALG?

\[ \sum x_{\text{LP}} = \sum_{x_i \in \mathcal{X}_{\text{LP}}} x_i \geq \sum_{x_i \in \mathcal{X}_{\text{LP}}: x_i \geq \frac{1}{2}} x_i, \text{ because...?} \]
Vertex Cover ILP

Objective: \( \min \sum_i x_i \)

Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
\( 0 \leq x_i \leq 1 \), for each vertex \( i \)

How does \( \sum x_{LP} \) relate to ALG?
\[
\sum x_{LP} = \sum_{x_i \in X_{LP}} x_i \geq \sum_{x_i \in X_{LP}: x_i \geq \frac{1}{2}} x_i, \text{ because it’s a subset of } x_{LP}
\]

\( + \) If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).
Vertex Cover ILP

Objective: \( \min \sum_{i} x_i \)
Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
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\]
\[
\geq \sum_{x_i \in X_{LP} : x_i \geq \frac{1}{2} \frac{1}{2}, \text{ because...?}}
\]
Vertex Cover ILP

Objective: \( \min \sum_i x_i \)
Subject to: \( x_i + x_j \geq 1, \) for each edge \( e = (i, j) \)
\( 0 \leq x_i \leq 1, \) for each vertex \( i \)

How does \( \sum x_{\text{LP}} \) relate to ALG?

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\sum x_{\text{LP}} = \sum_{x_i \in x_{\text{LP}}} x_i \geq \sum_{x_i \in x_{\text{LP}}: x_i \geq \frac{1}{2}} x_i, \text{ because it’s a subset of } x_{\text{LP}} \\
\geq \sum_{x_i \in x_{\text{LP}}: x_i \geq \frac{1}{2}} \frac{1}{2}, \text{ because each } x_i \text{ is at least } \frac{1}{2}
\]

\[
\sum x_{\text{LP}} \geq \frac{1}{2} \sum_{x_i \in x_{\text{LP}}: x_i \geq \frac{1}{2}} 1
\]

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).
Vertex Cover ILP

**Objective:** \( \min \sum_{i} x_i \)

**Subject to:**
- \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
- \( 0 \leq x_i \leq 1 \), for each vertex \( i \)

+ If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).

How does \( \sum x_{LP} \) relate to ALG?

\[
\sum x_{LP} = \sum_{i \in X_{LP}} x_i \geq \sum_{i \in X_{LP}: x_i \geq \frac{1}{2}} x_i, \text{ because it’s a subset of } x_{LP} \\
\geq \sum_{i \in X_{LP}: x_i \geq \frac{1}{2}} \frac{1}{2}, \text{ because each } x_i \text{ is at least } \frac{1}{2} \\
= \frac{1}{2} \left| \left\{ x_i \in X_{LP}: x_i \geq \frac{1}{2} \right\} \right|
\]
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Subject to: \[ x_i + x_j \geq 1, \text{ for each edge } e = (i, j) \]
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How does \( \sum x_{LP} \) relate to ALG?

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\sum x_{LP} = \sum_{x_i \in X_{LP}} x_i \geq \sum_{x_i \in X_{LP}: x_i \geq \frac{1}{2}} x_i, \text{ because it’s a subset of } X_{LP} \\
\geq \sum_{x_i \in X_{LP}: x_i \geq \frac{1}{2}} \frac{1}{2}, \text{ because each } x_i \text{ is at least } \frac{1}{2} \\
= \frac{1}{2} \left| \left\{ x_i \in X_{LP}: x_i \geq \frac{1}{2} \right\} \right| = ?
\]
Vertex Cover ILP

Objective: \( \min \sum_i x_i \)

Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
\( 0 \leq x_i \leq 1 \), for each vertex \( i \)

How does \( \sum x_{LP} \) relate to ALG?

\[
\sum x_{LP} = \sum_{x_i \in x_{LP}} x_i \geq \sum_{x_i \in x_{LP} : x_i \geq \frac{1}{2}} x_i, \text{ because it’s a subset of } x_{LP}
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\[
\geq \sum_{x_i \in x_{LP} : x_i \geq \frac{1}{2}} \frac{1}{2}, \text{ because each } x_i \text{ is at least } \frac{1}{2}
\]
\[
= \frac{1}{2} \left| \left\{ x_i \in x_{LP} : x_i \geq \frac{1}{2} \right\} \right| = \frac{1}{2} \text{ ALG}
\]

\( + \) If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).
Vertex Cover ILP

Objective: \( \min \sum_i x_i \)
Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
\( 0 \leq x_i \leq 1 \), for each vertex \( i \)

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).

What is the relationship between ALG and OPT?
Vertex Cover ILP

Objective: \( \min \sum_i x_i \)
Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
\( 0 \leq x_i \leq 1 \), for each vertex \( i \)

What is the relationship between ALG and OPT?

\( \sum x_{LP} \geq \frac{1}{2} \) ALG and \( \sum x_{LP} \leq \text{OPT} \)

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).
Vertex Cover ILP

Objective: \( \min \sum_i x_i \)
Subject to: \( x_i + x_j \geq 1 \), for each edge \( e = (i, j) \)
\( 0 \leq x_i \leq 1 \), for each vertex \( i \)

What is the relationship between ALG and OPT?
\[ \sum x_{LP} \geq \frac{1}{2} \text{ALG and } \sum x_{LP} \leq \text{OPT} \]
\[ \text{ALG} \leq 2 \text{OPT} \]

If \( x_i \geq \frac{1}{2} \), add vertex \( i \) to our subset \( S \).
NPO: Roughly, optimization versions of NP decision problems.