Randomized Rounding
CSCI 532
Set Cover ILP

Set Cover: Given a universe of elements $U$ and sets $S$, find the smallest subset of $S$ such that every element in $U$ is in some selected subset.

$U = \{1, 4, 7, 8, 10\}$

$S = \{\{1, 7, 8\}, \{1, 4, 7\}, \{7, 8\}, \{4, 8, 10\}\}$
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Objective: $\min \sum_s x_s$
Subject to: $\sum_{s: u \in s} x_s \geq 1$, for each $u \in U$
$x_s \in \{0,1\}$, for each set $s$

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Example:

Objective: \( \min x_1 + x_2 + x_3 + x_4 \)
Subject to: \( x_1 + x_2 \geq 1 \)
\( x_2 + x_4 \geq 1 \)
\( x_1 + x_2 + x_3 \geq 1 \)
\( x_1 + x_3 + x_4 \geq 1 \)
\( x_4 \geq 1 \)
\( x_1, x_2, x_3, x_4 \in \{0,1\} \)

\( U = \{1, 4, 7, 8, 10\} \)
\( S = \{\{1, 7, 8\}, \{1, 4, 7\}, \{7, 8\}, \{4, 8, 10\}\} \)
Set Cover ILP

Objective: \( \min \sum_s x_s \)
Subject to: \( \sum_{s: u \in s} x_s \geq 1, \) for each \( u \in U \)
\( 0 \leq x_s \leq 1, \) for each set \( s \)
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Objective: \[ \min \sum_s x_s \]
Subject to: \[ \sum_{s: u \in s} x_s \geq 1, \text{ for each } u \in U \]
\[ 0 \leq x_s \leq 1, \text{ for each set } s \]

+ If \( x_s \geq \frac{1}{2} \), add set \( s \) to our subset \( S_{ALG} \).

Could this lead to an invalid solution?
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\( x_1, x_2, x_3, x_4 \in [0,1] \)

\[ U = \{1, 2, 3, 4\} \]
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If \( x_s \geq \frac{1}{2} \), add set \( s \) to our subset \( S_{ALG} \).
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\( x_2 + x_3 + x_4 \geq 1 \)
\( x_1, x_2, x_3, x_4 \in [0,1] \)

Yes, in this case \( x_s = \frac{1}{3} \), \( \forall s \Rightarrow \) No sets are selected (invalid solution).

If \( x_s \geq \frac{1}{2} \), add set \( s \) to our subset \( S_{ALG} \).

\[ U = \{1, 2, 3, 4\} \]
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**Objective:** \[ \min \sum_s x_s \]

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Add set \( s \) to our subset \( S_{ALG} \) with probability of \( x_s \).
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Objective: \( \min \sum_s x_s \)
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+ Add set \( s \) to our subset \( S_{ALG} \) with probability of \( x_s \).

1. What is the size of the solution?
2. What is the probability the solution is valid?
Set Cover ILP

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1. What is the size of the solution?
   Let \( x_s^* \) be the optimal solutions to the LP relaxation.
Set Cover ILP

Objective: \[ \min \sum_s x_s \]
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1. What is the size of the solution?
   Let \( x_s^* \) be the optimal solutions to the LP relaxation.
   Define random variable \( X_s = \begin{cases} 
1, & s \in S_{ALG} \\
0, & \text{otherwise} 
\end{cases} \)
Set Cover ILP

<table>
<thead>
<tr>
<th>Objective:</th>
<th>( \min \sum_s x_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subject to:</td>
<td>( \sum_{s: u \in s} x_s \geq 1 ), for each ( u \in U )</td>
</tr>
<tr>
<td></td>
<td>( 0 \leq x_s \leq 1 ), for each set ( s )</td>
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</table>

1. What is the size of the solution?

Let \( x_s^* \) be the optimal solutions to the LP relaxation.

Define random variable \( X_s = \begin{cases} 1, & s \in S_{ALG} \\ 0, & \text{otherwise} \end{cases} \)

\[
E[ALG] = E[\sum_{s \in S_{ALG}} X_s]
\]
Set Cover ILP

Objective: \( \min \sum_s x_s \)
Subject to: \( \sum_{s: u \in s} x_s \geq 1, \) for each \( u \in U \)
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\[ E[ALG] = E[\sum_{s \in S_{ALG}} X_s] = E[\sum_{s \in S} X_s], \] since the rest of the \( X_s = 0 \)
Set Cover ILP

Objective: \( \min \sum_s x_s \)
Subject to: \( \sum_{s: u \in s} x_s \geq 1 \), for each \( u \in U \)
\( 0 \leq x_s \leq 1 \), for each set \( s \)

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1. What is the size of the solution?

Let \( x^*_s \) be the optimal solutions to the LP relaxation.

Define random variable \( X_s = \begin{cases} 1, \ s \in S_{ALG} \\ 0, \text{otherwise} \end{cases} \)

\[ E[ALG] = E[\sum_{s \in S_{ALG}} X_s] = E[\sum_{s \in S} X_s], \text{since the rest of the } X_s = 0 \]
\[ = \sum_{s \in S} E[X_s], \text{by linearity of expectation} \]
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Objective: \( \min \sum_s x_s \)
Subject to: \( \sum_{s: u \in s} x_s \geq 1, \text{ for each } u \in U \)
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= \sum_{s \in S} E[X_s], \text{ by linearity of expectation}
= \sum_{s \in S} x_s^*, \text{ since } x_s^* \text{ is probability } X_s = 1
\]

Add set \( s \) to our subset \( S_{ALG} \) with probability of \( x_s \).
Set Cover ILP

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= \sum_{s \in S} x_s^*, \text{ since } x_s^* \text{ is probability } X_s = 1 \\
= OPT_{LP} + \text{ Add set } s \text{ to our subset } S_{ALG} \text{ with probability of } x_s.
Set Cover ILP

Objective: \( \min \sum_s x_s \)
Subject to: \( \sum_{s: \mu \in s} x_s \geq 1, \text{ for each } \mu \in U \)
\( 0 \leq x_s \leq 1, \text{ for each set } s \)

1. What is the size of the solution?

Let \( x_s^* \) be the optimal solutions to the LP relaxation.

Define random variable \( X_s = \begin{cases} 1, & s \in S_{ALG} \\ 0, & \text{otherwise} \end{cases} \)

\( E[ALG] = E[\sum_{s \in S_{ALG}} X_s] = E[\sum_{s \in S} X_s], \) since the rest of the \( X_s = 0 \)
\( = \sum_{s \in S} E[X_s], \) by linearity of expectation
\( = \sum_{s \in S} x_s^*, \) since \( x_s^* \) is probability \( X_s = 1 \)
\( = OPT_{LP} \leq OPT \)
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Add set \( s \) to our subset \( S_{ALG} \) with probability of \( x_s \).

2. What is the probability solution is valid?
Set Cover ILP

Objective: $\min \sum_s x_s$
Subject to: $\sum_{s: u \in s} x_s \geq 1$, for each $u \in U$
            $0 \leq x_s \leq 1$, for each set $s$

2. What is the probability solution is valid?

Add set $s$ to our subset $S_{ALG}$ with probability of $x_s$.

Recall: $x_s^* = \text{optimal solutions to LP relaxation}$

$$X_s = \begin{cases} 1, & s \in S_{ALG} \\ 0, & \text{otherwise} \end{cases}$$
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2. What is the probability solution is valid?
   Let \( S_u \) be sets of \( S \) that contain element \( u \).

Add set \( s \) to our subset \( S_{ALG} \) with probability of \( x_s \).

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Let \( S_u \) be sets of \( S \) that contain element \( u \).

\( u \) is covered by \( S_{ALG} \) \( \iff \sum_{S_u} X_s \geq 1 \).

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Let \( S_u \) be sets of \( S \) that contain element \( u \).
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\( \Pr[u \ not \ covered \ by \ S_{ALG}] \)

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\( u \) is covered by \( S_{ALG} \) if \( \sum_{s \in S_u} X_s \geq 1 \).

\[ \Pr[u \text{ not covered by } S_{ALG}] = \prod_{S_u} (1 - x^*_s), \] since \( X_s \) independent

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Let \( S_u \) be sets of \( S \) that contain element \( u \).
\( u \) is covered by \( S_{ALG} \) \( \iff \sum_{S_u} X_s \geq 1 \).
\( \Pr[u \) not covered by \( S_{ALG} ] = \Pi_{S_u} (1 - x_s^*), \) since \( X_s \) independent
\[ \leq \Pi_{S_u} e^{-x_s^*}, \] since \( 1 + y \leq e^y \) for all \( y \in \mathbb{R} \)

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   \[ = e^{-\sum_{S_u} x^*_s}, \] because of algebra
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     \( = e^{-\sum_{S_u} x_s^*}, \) because of algebra
     \( \leq \frac{1}{e} \approx 0.37, \) since \( \sum_{S_u} x_s^* \geq 1 \)
   Add set \( s \) to our subset \( S_{ALG} \) with probability of \( x_s \).
   Recall: \( x_s^* = \) optimal solutions to LP relaxation
   \[ X_s = \begin{cases} 1, & s \in S_{ALG} \\ 0, & \text{otherwise} \end{cases} \]
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\[ \Pr[u \text{ not covered by } S_{ALG}] = \prod_{S_u} (1 - x_s^*), \text{ since } X_s \text{ independent} \]
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\[ = e^{-\sum_{S_u} x_s^*}, \text{ because of algebra} \]
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Highly unlikely \( S_{ALG} \) will cover all \( U \), but each individual element has good likelihood of being covered.

Add set \( s \) to our subset \( S_{ALG} \) with probability of \( x_s \).

Recall: \( x_s^* = \) optimal solutions to LP relaxation

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Suppose \( T \geq \ln 4n \), where \( |U| = n \).

Recall: \( x_s^* \) = optimal solutions to LP relaxation
\( X_s = \begin{cases} 1, & s \in S_{ALG} \\ 0, & \text{otherwise} \end{cases} \)

\[ \begin{align*}
1 - & \text{Add set } s \text{ to our subset } S_{ALG} \text{ with probability of } x_s. \\
2 - & \text{Repeat step 1 } T \text{-times, while adding sets to } S_{ALG}.
\end{align*} \]
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1. What is the size of the solution?

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1. What is the size of the solution?
   \[ E[ALG] = E\left[ \sum_{s \in S_{ALG}} X_s \right] \]

2 - Repeat step 1 \( T \)-times, while adding sets to \( S_{ALG} \).

Recall: \( x^*_s = \) optimal solutions to LP relaxation
\[ X_s = \begin{cases} 1, & s \in S_{ALG} \\ 0, & \text{otherwise} \end{cases} \]
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1. What is the size of the solution?
   \[
   E[ALG] = E\left[ \sum_{s \in S_{ALG}} X_s \right] \\
   \leq E\left[ \sum_{t \leq T} \sum_{s \in S_{ALG_t}} X_s \right], \text{ since multiple iterations may select } s
   \]

Recall: \( x_s^* = \) optimal solutions to LP relaxation
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E[ALG] = E\left[ \sum_{s \in S_{ALG}} X_s \right] \\
\leq E\left[ \sum_{t \leq T} \sum_{s \in S_{ALG_t}} X_s \right], \text{ since multiple iterations may select } s \\
= E\left[ \sum_{s \in S} x_s^* \right], \text{ by previous bound } E\left[ \sum_{s \in S_{ALG}} X_s \right] = \sum_{s \in S} x_s^*
\]

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X_s = \begin{cases} 
1, & s \in S_{ALG} \\
0, & \text{otherwise}
\end{cases}
\]

3. Repeat step 1 \( T \)-times, while adding sets to \( S_{ALG} \).
Set Cover ILP

| Objective: | min \( \sum_s x_s \) |
| Subject to: | \( \sum_{s: u \in s} x_s \geq 1 \), for each \( u \in U \) |
|            | 0 \( \leq x_s \leq 1 \), for each set \( s \) |

Suppose \( T \geq \ln 4n \), where \( |U| = n \).

1. What is the size of the solution?

\[
E[ALG] = E\left[ \sum_{s \in S_{ALG}} X_s \right] \\
\leq E\left[ \sum_{t \leq T} \sum_{s \in S_{ALG_t}} X_s \right], \text{since multiple iterations may select } s \\
= T \sum_{s \in S} x_s^*, \text{by previous bound } E\left[ \sum_{s \in S_{ALG}} X_s \right] = \sum_{s \in S} x_s^* \\
Thus, Pr[ALG > 4 \ln 4n OPT]
\]

“What is the chance algorithm is worse than \( ALG \leq 4 \ln 4n OPT \)?”
Set Cover ILP

Objective: \( \min \sum_s x_s \)
Subject to: \( \sum_{s: u \in s} x_s \geq 1 \), for each \( u \in U \)
\( 0 \leq x_s \leq 1 \), for each set \( s \)

Suppose \( T \geq \ln 4n \), where \(|U| = n\).

1. What is the size of the solution?

\[
E[ALG] = E \left[ \sum_{s \in S_{ALG}} X_s \right] \\
\leq E \left[ \sum_{t \leq T} \sum_{s \in S_{ALG_t}} X_s \right], \text{ since multiple iterations may select } s \\
= T \sum_{s \in S} x_s^*, \text{ by previous bound } E \left[ \sum_{s \in S_{ALG}} X_s \right] = \sum_{s \in S} x_s^* \\
\]

Thus, \( \Pr[ALG > 4 \ln 4n \ \text{OPT}] \leq \Pr[\sum_{s \in S_{ALG}} X_s > 4T \sum_{s \in S} x_s^*] \)

“What is the chance algorithm is worse than \( ALG \leq 4 \ln 4n \ \text{OPT} \)?”

\[
\text{OPT} \geq \text{OPT}_{LP} = \sum_{s \in S} x_s^* 
\]
Set Cover ILP

Objective: $\min \sum_s x_s$
Subject to: $\sum_{s: u \in s} x_s \geq 1$, for each $u \in U$
$0 \leq x_s \leq 1$, for each set $s$

Suppose $T \geq \ln 4n$, where $|U| = n$.

1. What is the size of the solution?

   $E[ALG] = E\left[\sum_{s \in S_{ALG}} X_s\right]$
   $\leq E\left[\sum_{t \leq T} \sum_{s \in S_{ALG_t}} X_s\right]$, since multiple iterations may select $s$
   $= T \sum_{s \in S} x^*_s$, by previous bound $E\left[\sum_{s \in S_{ALG}} X_s\right] = \sum_{s \in S} x^*_s$

   Thus, $\Pr[ALG > 4 \ln 4n \text{ OPT}] \leq \Pr[\sum_{s \in S_{ALG}} X_s > 4T \sum_{s \in S} x^*_s]$
   $\leq \frac{1}{4}$, by Markov’s inequality:
   $P(X \geq a) \leq \frac{E[X]}{a}$

2 - Add set $s$ to our subset $S_{ALG}$ with probability of $x_s$.

2 - Repeat step 1 $T$-times, while adding sets to $S_{ALG}$.

Recall: $x^*_s = \text{optimal solutions to LP relaxation}$

$X_s = \begin{cases} 
1, & s \in S_{ALG} \\
0, & \text{otherwise} 
\end{cases}$
Set Cover ILP

**Objective:**

\[
\min \sum x_s
\]

**Subject to:**

\[
\sum_{s \in S} x_s \geq 1, \text{ for each } u \in U
\]

\[
0 \leq x_s \leq 1, \text{ for each set } s \in S
\]

Recall:

\[x^*_s = \text{optimal solutions to LP relaxation}
\]

\[X^*_s = \begin{cases} 1, & \text{if } s \in S^* \\ 0, & \text{otherwise} \end{cases}
\]

1. Repeat step 1 \(T\) - times, while adding sets to \(S_{ALG}\).

With probability \(\geq \frac{3}{4}\), \(ALG \leq O(\ln(n))OPT\).

**Good enough?**

What if we run the algorithm twice?

Probability some run gives: \(ALG \leq O(\ln(n))OPT = 1 - \left(\frac{1}{4}\right)^2 \approx 0.94\)

Three times? \(1 - \left(\frac{1}{4}\right)^3 \approx 0.98\)

Four times? \(1 - \left(\frac{1}{4}\right)^4 \approx 0.996\)

Ten times? \(1 - \left(\frac{1}{4}\right)^{10} \approx 0.999999\)

I.e. Exponential improvement \(\left(1 - \left(\frac{1}{4}\right)^t\right)\) for polynomial time work \(t\).
Set Cover ILP

Objective:

\[
\min \sum x_s
\]

Subject to:

\[
\sum x_s \geq 1, \quad \text{for each } u \in U
\]

0 \leq x_s \leq 1, \quad \text{for each set } s \in S

Recall:

\[
x^*_s = \text{optimal solutions to LP relaxation}
\]

Final algorithm:

\[
\begin{align*}
\text{while } S_{ALG} \text{ is not a cover or } ALG > 4 \ln 4n \sum x^*_s \\
\text{for } t \leq \ln 4n \\
\text{add } \frac{1}{1 - \sum x^*_s}, \text{by previous bound } E[\sum x_S] = \sum x^*_s \\
\end{align*}
\]

\[
T \sum x^*_s, \text{ by previous bound } E[\sum x_S] = \sum x^*_s
\]

Thus, \( \Pr[ALG > 4 \ln 4n OPT] \leq \Pr[\sum x_S > 4T \sum x^*_s] \)

\[
\leq \frac{1}{4}, \text{ by Markov’s inequality}
\]
Set Cover ILP

Objective: \( \min \sum_s x_s \)
Subject to: \( \sum_{s: u \in s} x_s \geq 1, \) for each \( u \in U \)
\( 0 \leq x_s \leq 1, \) for each set \( s \)

Suppose \( T \geq \ln 4n, \) where \( |U| = n. \)

2. What is the probability solution is valid?

Recall: \( x_s^* = \) optimal solutions to LP relaxation
\( X_s = \begin{cases} 1, & s \in S_{ALG} \\ 0, & \text{otherwise} \end{cases} \)

\[ \begin{align*}
1 - & \text{ Add set } s \text{ to our subset } S_{ALG} \text{ with probability of } x_s. \\
2 - & \text{ Repeat step 1 } T\text{-times, while adding sets to } S_{ALG}. 
\end{align*} \]
Set Cover ILP

Objective: \( \min \sum_s x_s \)
Subject to: \( \sum_{s: u \in s} x_s \geq 1, \text{ for each } u \in U \)
\( 0 \leq x_s \leq 1, \text{ for each set } s \)

Suppose \( T \geq \ln 4n \), where \( |U| = n \).

2. What is the probability solution is valid?
   Let \( S_u \) be sets of \( S \) that contain element \( u \).

Recall: \( x^*_s = \text{optimal solutions to LP relaxation} \)
\( X_s = \begin{cases} 1, & s \in S_{ALG} \\ 0, & \text{otherwise} \end{cases} \)

1 - Add set \( s \) to our subset \( S_{ALG} \) with probability of \( x_s \).
2 - Repeat step 1 \( T \)-times, while adding sets to \( S_{ALG} \).
Set Cover ILP

Objective: \( \min \sum_{s} x_s \)
Subject to: \( \sum_{s: u \in s} x_s \geq 1 \), for each \( u \in U \)
\( 0 \leq x_s \leq 1 \), for each set \( s \)

Suppose \( T \geq \ln 4n \), where \(|U| = n\).

2. What is the probability solution is valid?

Let \( S_u \) be sets of \( S \) that contain element \( u \).

\[
\Pr[u \text{ not covered by } S_{ALG}] = \prod_{t \leq T} \Pr[u \text{ not covered by } S_{ALG_t}]
\]

Recall: \( x_s^* = \) optimal solutions to LP relaxation

\[
X_s = \begin{cases} 
1, & s \in S_{ALG} \\
0, & \text{otherwise}
\end{cases}
\]
Set Cover ILP

Objective:  \( \min \sum_s x_s \)

Subject to:  \( \sum_{s: u \in s} x_s \geq 1 \), for each \( u \in U \)
\( 0 \leq x_s \leq 1 \), for each set \( s \)

Suppose \( T \geq \ln 4n \), where \( |U| = n \).

2. What is the probability solution is valid?

Let \( S_u \) be sets of \( S \) that contain element \( u \).

\[
\Pr[u \text{ not covered by } S_{ALG}] = \prod_{t \leq T} \Pr[u \text{ not covered by } S_{ALG_t}]
\leq \prod_{t \leq T} \frac{1}{e}, \text{ by previous bound}
\]

Recall: \( x_s^* = \text{optimal solutions to LP relaxation} \)

\[
X_s = \begin{cases} 
1, & s \in S_{ALG} \\
0, & \text{otherwise}
\end{cases}
\]

1 - Add set \( s \) to our subset \( S_{ALG} \) with probability of \( x_s \).

2 - Repeat step 1 \( T \)-times, while adding sets to \( S_{ALG} \).
Set Cover ILP

Objective: \( \min \sum_S x_S \)
Subject to: \( \sum_{u \in S} x_S \geq 1 \), for each \( u \in U \)
\( 0 \leq x_S \leq 1 \), for each set \( s \)

Suppose \( T \geq \ln 4n \), where \( |U| = n \).

2. What is the probability solution is valid?
   Let \( S_u \) be sets of \( S \) that contain element \( u \).
   \[
   \Pr[u \text{ not covered by } S_{ALG}] = \prod_{t \leq T} \Pr[u \text{ not covered by } S_{ALG_t}]
   \leq \prod_{t \leq T} \frac{1}{e^t}, \text{ by previous bound }
   = \frac{1}{e^T} \leq \frac{1}{4n}, \text{ by plugging in } T \geq \ln 4n
   \]

Recall: \( x_S^* = \text{optimal solutions to LP relaxation} \)
\[
X_S = \begin{cases} 
1, & s \in S_{ALG} \\
0, & \text{otherwise}
\end{cases}
\]

1 - Add set \( s \) to our subset \( S_{ALG} \) with probability of \( x_S \).
2 - Repeat step 1 \( T \)-times, while adding sets to \( S_{ALG} \).
Set Cover ILP

Objective: \( \min \sum_s x_s \)
Subject to: \( \sum_{s:u \in s} x_s \geq 1, \text{ for each } u \in U \)
\( 0 \leq x_s \leq 1, \text{ for each set } s \)

Suppose \( T \geq \ln 4n \), where \( |U| = n \).

2. What is the probability solution is valid?
   Let \( S_u \) be sets of \( S \) that contain element \( u \).
   \[ \Pr[u \text{ not covered by } S_{ALG}] = \prod_{t \leq T} \Pr[u \text{ not covered by } S_{ALG_t}] \]
   \[ \leq \prod_{t \leq T} \frac{1}{e}, \text{ by previous bound} \]
   \[ = \frac{1}{e^T} \leq \frac{1}{4n}, \text{ by plugging in } T \geq \ln 4n \]
   Thus, \( \Pr[S_{ALG} \text{ is not a cover}] \leq \sum_u \Pr[u \text{ not covered by } S_{ALG}] \), by union bound

Recall: \( x^*_s = \) optimal solutions to LP relaxation
\[ X_s = \begin{cases} 1, & s \in S_{ALG} \\ 0, & \text{otherwise} \end{cases} \]
**Set Cover ILP**

<table>
<thead>
<tr>
<th>Objective:</th>
<th>$\min \sum_s x_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subject to:</td>
<td>$\sum_{s: u \in s} x_s \geq 1$, for each $u \in U$</td>
</tr>
<tr>
<td></td>
<td>$0 \leq x_s \leq 1$, for each set $s$</td>
</tr>
</tbody>
</table>

Suppose $T \geq \ln 4n$, where $|U| = n$.

2. What is the probability solution is valid?

Let $S_u$ be sets of $S$ that contain element $u$.

$$\Pr[u \text{ not covered by } S_{ALG}] = \prod_{t \leq T} \Pr[u \text{ not covered by } S_{ALG_t}]$$

$$\leq \prod_{t \leq T} \frac{1}{e^T}, \text{ by previous bound}$$

$$= \frac{1}{e^T} \leq \frac{1}{4n}, \text{ by plugging in } T \geq \ln 4n$$

Thus, $\Pr[S_{ALG} \text{ is not a cover}] \leq \sum_u \Pr[u \text{ not covered by } S_{ALG}]$, by union bound

$$\leq n \frac{1}{4n} = \frac{1}{4}$$
Set Cover ILP

Suppose $T \geq \ln 4n$, where $U = n$.

2. What is the probability solution is valid?

Let $S_\#$ be sets of $S$ that contain element $u$.

$$\Pr(u \text{ not covered by } S\_(')) = \prod \frac{3}{4} \frac{5}{6} \Pr(u \text{ not covered by } S\_(')(\leq \prod \frac{3}{4} \frac{5}{6} \frac{2}{3}, \text{by previous bound})$$

Thus, $\Pr(S\_\# \text{ is not a cover}) \leq \sum_u \Pr(u \text{ not covered by } S_{ALG})$, by union bound.

$$\leq n \frac{1}{4n} = \frac{1}{4}$$
Set Cover ILP

Objective: $x_S^* = \text{optimal solutions to LP relaxation}$

Subject to:

While $S_{ALG}$ is not a cover or $ALG > 4 \ln 4n \sum_{s \in S} x_s^*$

For $t \leq \ln 4n$

Add $s$ to $S_{ALG}$ with probability $x_s^*$

After each iteration of the while loop:

$Pr[S_{ALG} \text{ is not a cover}] \leq \frac{1}{4}$

$Pr[ALG > 4 \ln 4n OPT] \leq \frac{1}{4}$

Thus, $Pr[S_{ALG} \text{ is not a cover}] \leq \sum_u Pr[u \text{ not covered by } S_{ALG}]$, by union bound

$\leq n \frac{1}{4n} = \frac{1}{4}$
Set Cover ILP

1. Add set $s$ to our subset $S_{ALG}$ with probability of $x_s$.

2. What is the probability solution is valid?

Let $S_{#}$ be sets of $S$ that contain element $u$.

$$\Pr[u \text{ not covered by } S_{#}] = \prod \frac{3}{4} \frac{5}{4} \Pr[u \text{ not covered by } S_{#}] \leq \prod \frac{3}{4} \frac{5}{4} \% \leq \% : F$$

Thus, $\Pr[S_{#}]$ is not a cover $\leq \% : F = \% : F$.

Recall: $x_{#}^{*}$ = optimal solutions to LP relaxation $X_{#} = 6_{1}$. $s \in S_{#}$

$0$ , otherwise

2 - Repeat step 1 $T - 1$ times, while adding sets to $S_{#}$.

1 - Add set $s$ to our subset $S_{#}$ with probability of $x_s$.

After each iteration of the while loop:

$$\Pr[S_{ALG} \text{ is not a cover}] \leq \frac{1}{4}$$

$$\Pr[ALG > 4 \ln 4n OPT] \leq \frac{1}{4}$$

$\Rightarrow$ Probability while loops to second iteration $\leq \frac{1}{2}$

$\Rightarrow E[\# \text{ while loop iterations}] = 2$

Thus, $\Pr[S_{ALG}] \leq n \frac{1}{4n} = \frac{1}{4}$.
Set Cover ILP

**Objective:**

\( x^*_S = \text{optimal solutions to LP relaxation} \)

**Subject to:**

- While \( S_{ALG} \) is not a cover or \( ALG > 4 \ln 4n \sum_{S \subseteq S} x^*_S \)
  - For \( t \leq \ln 4n \)
    - Add set \( s \) to \( S_{ALG} \) with probability \( x^*_S \)

**In Conclusion:** We find a set cover that is at most \( O(\ln n) \)-factor worse than optimal in polynomial expected time.

\[
\operatorname{Pr}[ALG > 4 \ln 4n \cdot \text{OPT}] \leq \frac{1}{4}
\]

\[
\Rightarrow \text{Probability while loops to second iteration} \leq \frac{1}{2}
\]

\[
\Rightarrow E[\# \text{while loop iterations}] = 2
\]

\[
\leq n \frac{1}{4n} = \frac{1}{4}
\]

1. Add set \( s \) to our subset \( S_{ALG} \) with probability of \( x_s \).