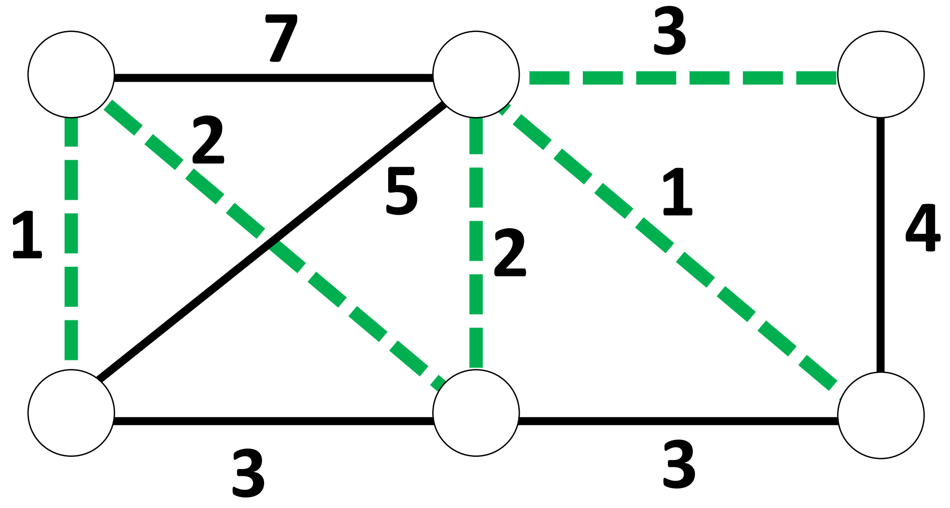
Closest Pair of Points

CSCI 532

Kruskal's MST Algorithm

<u>Algorithm:</u> Add the edge with smallest weight, that does not create a cycle.



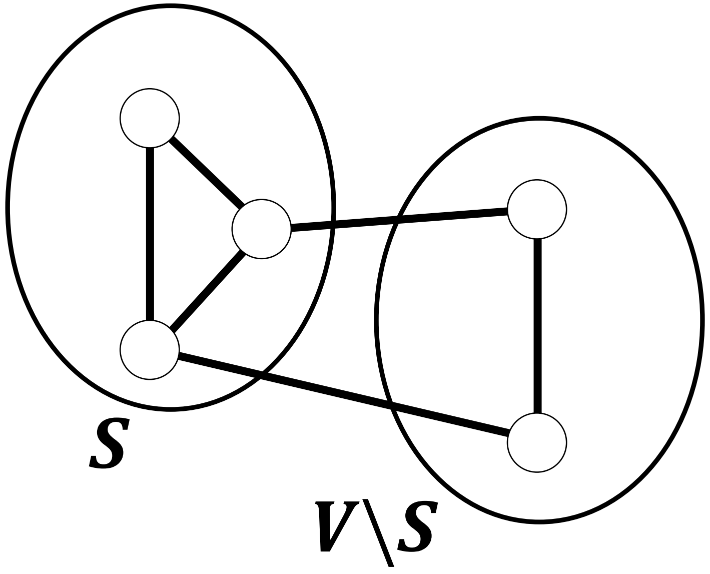
What are some questions we may have about the algorithm?

1. ~~Is the solution valid? (Does it actually find a spanning tree?)~~
2. ~~What is the running time?~~
3. Is the solution optimal?

MST Cut Property

Lemma: Suppose that S is a subset of nodes from $G = (V, E)$. Then, the cheapest edge e between S and $V\backslash S$ is part of every MST.

Proof:



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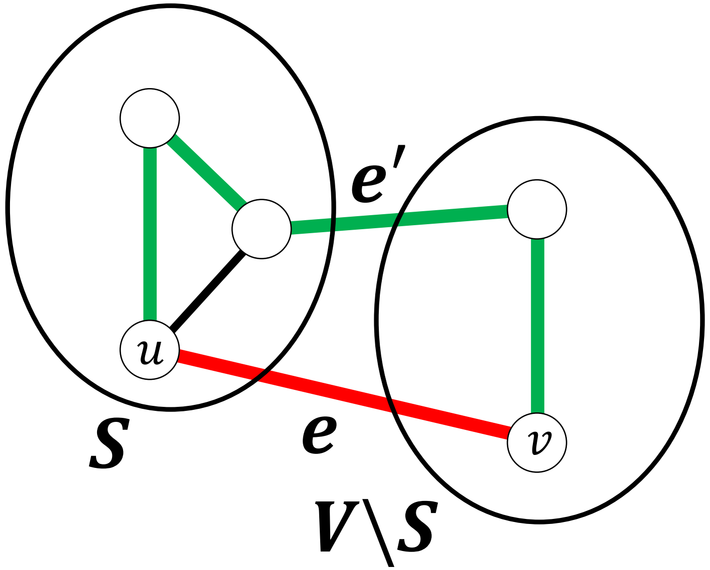

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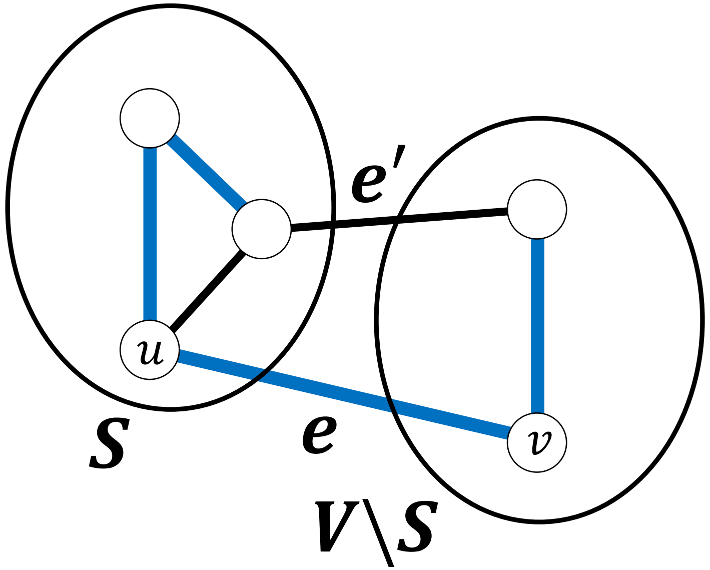
Remove e' to form $\boldsymbol{T'} = T \cup \{e\} \setminus \{e'\}$.



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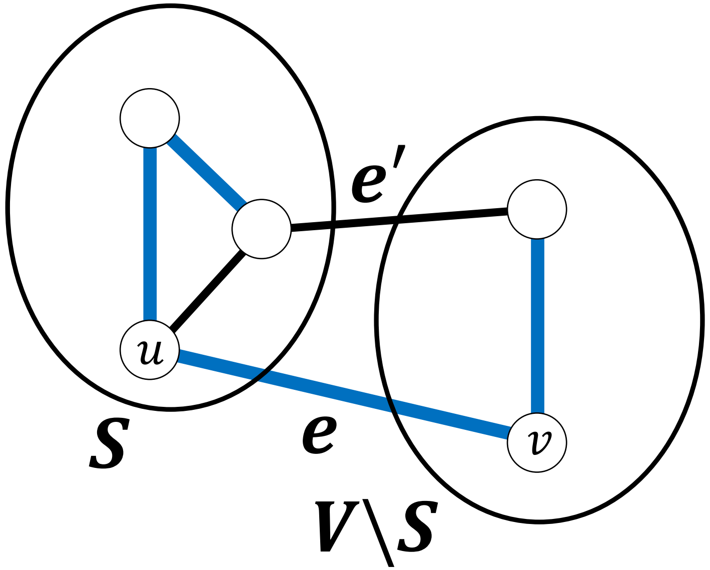

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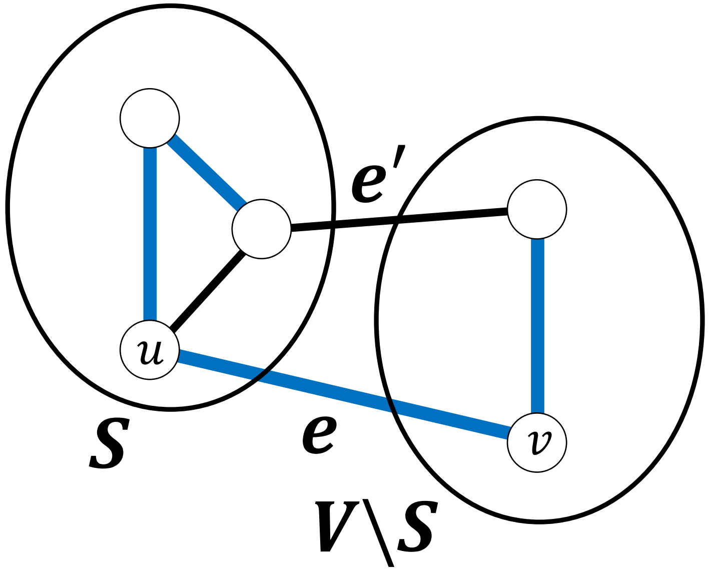

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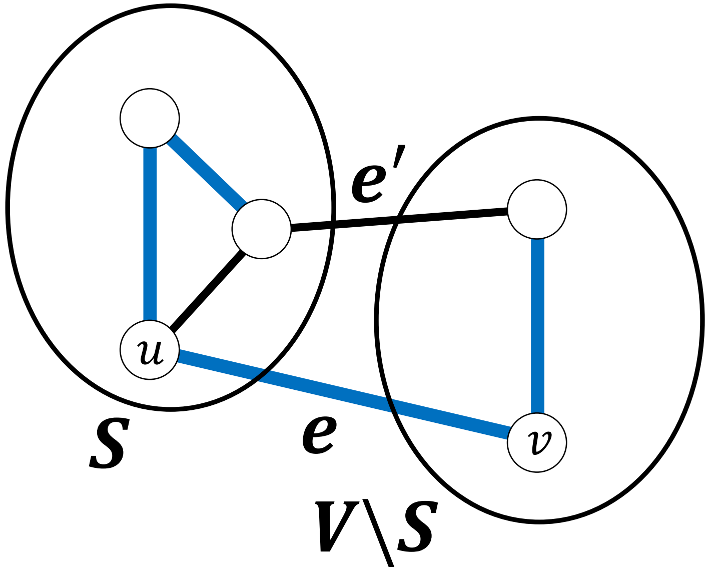

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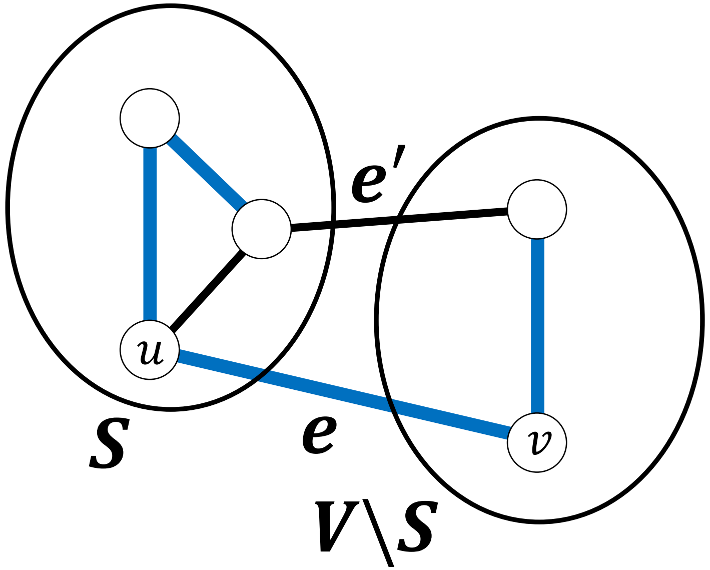

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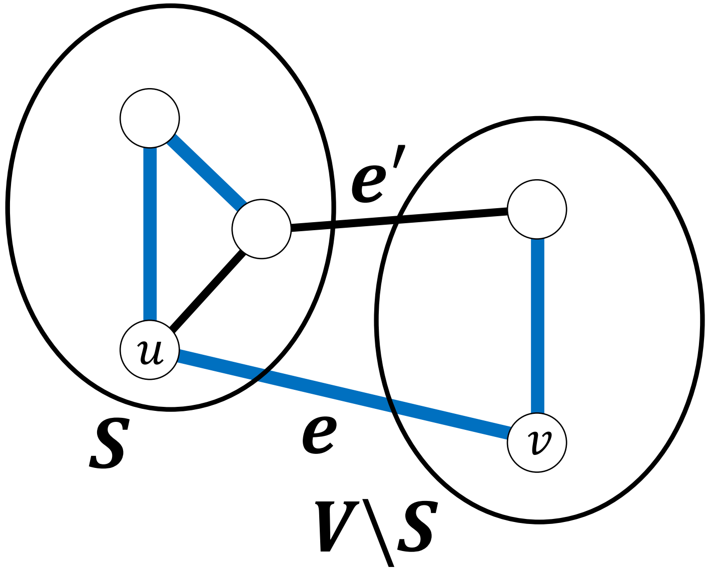

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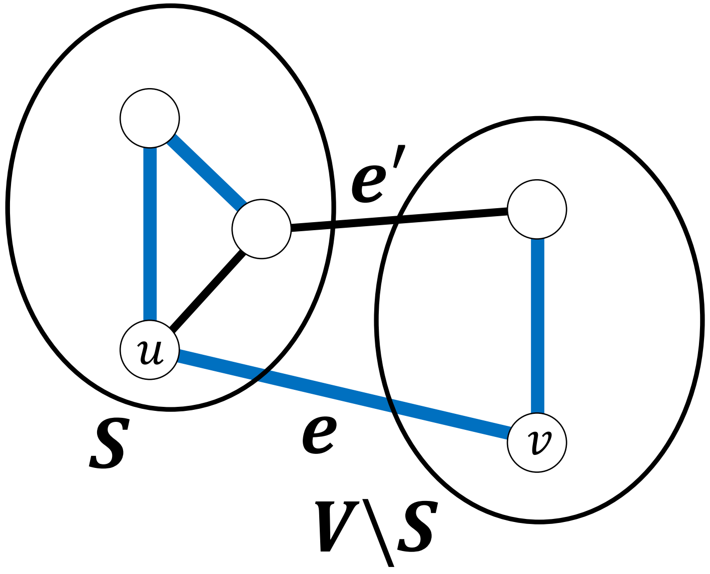
⇒ Every MST must include e.



Kruskal’s MST Algorithm

Algorithm: Add the edge with smallest weight, that does not create a cycle.

Proof of optimality: Let $G = (V, E)$, and $T \subseteq E$ be the set of edges resulting from Kruskal’s algorithm.

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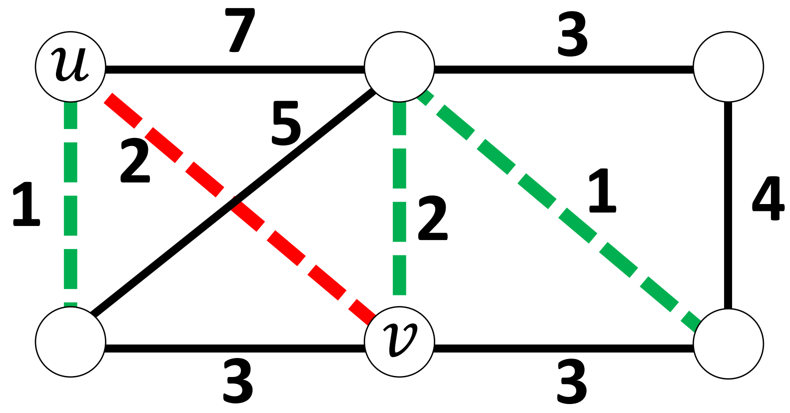

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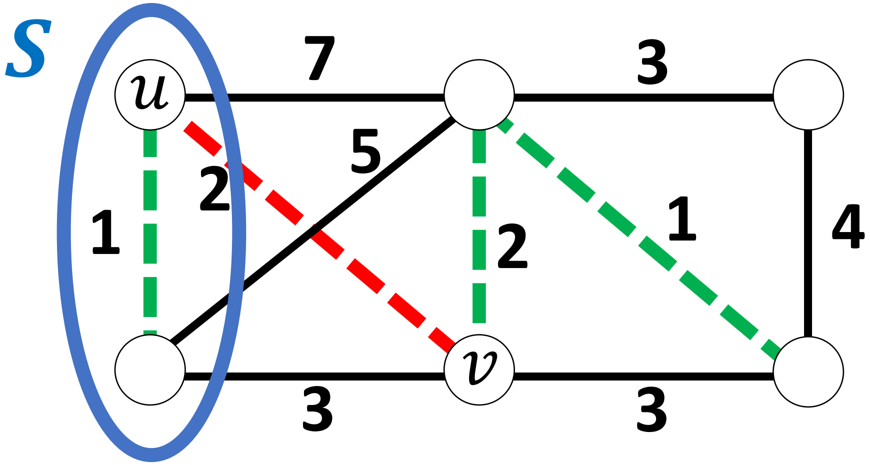

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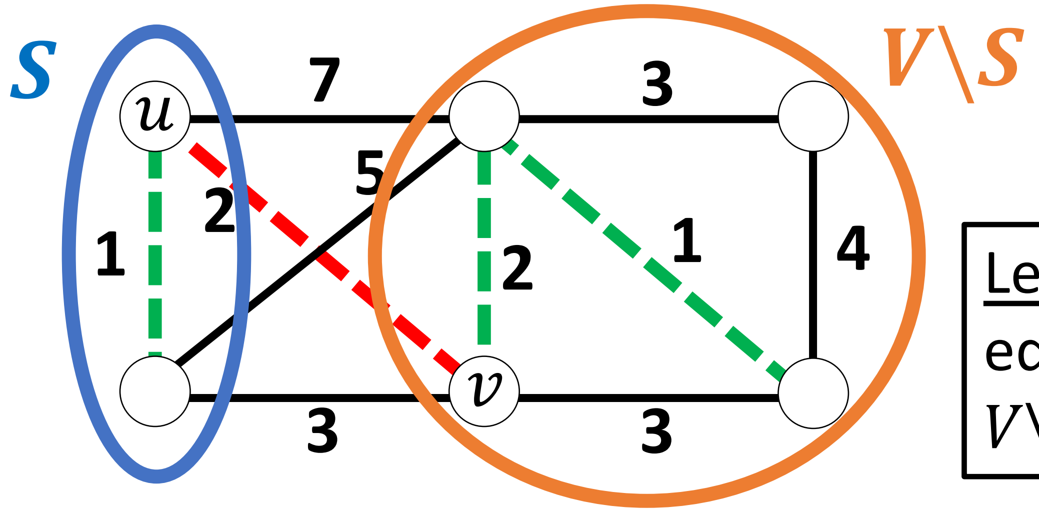

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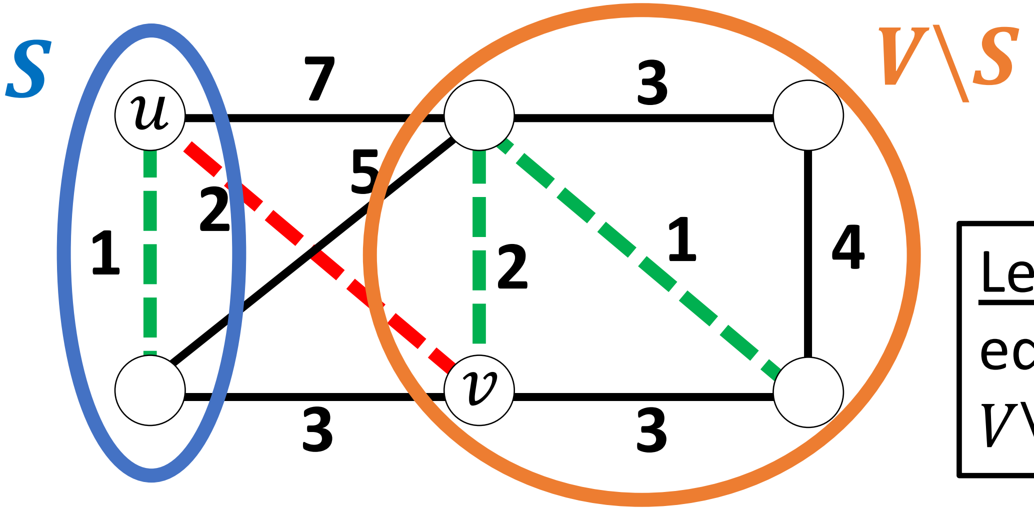

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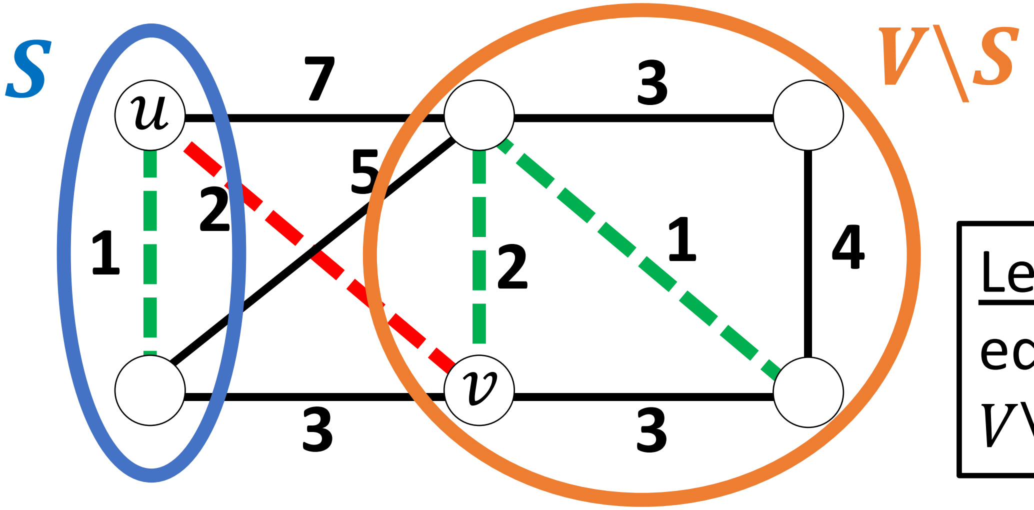

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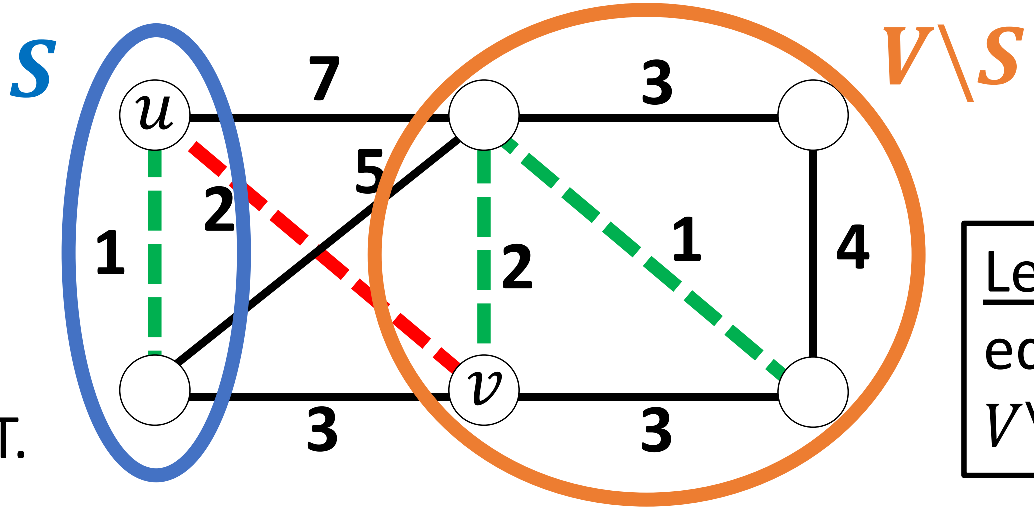
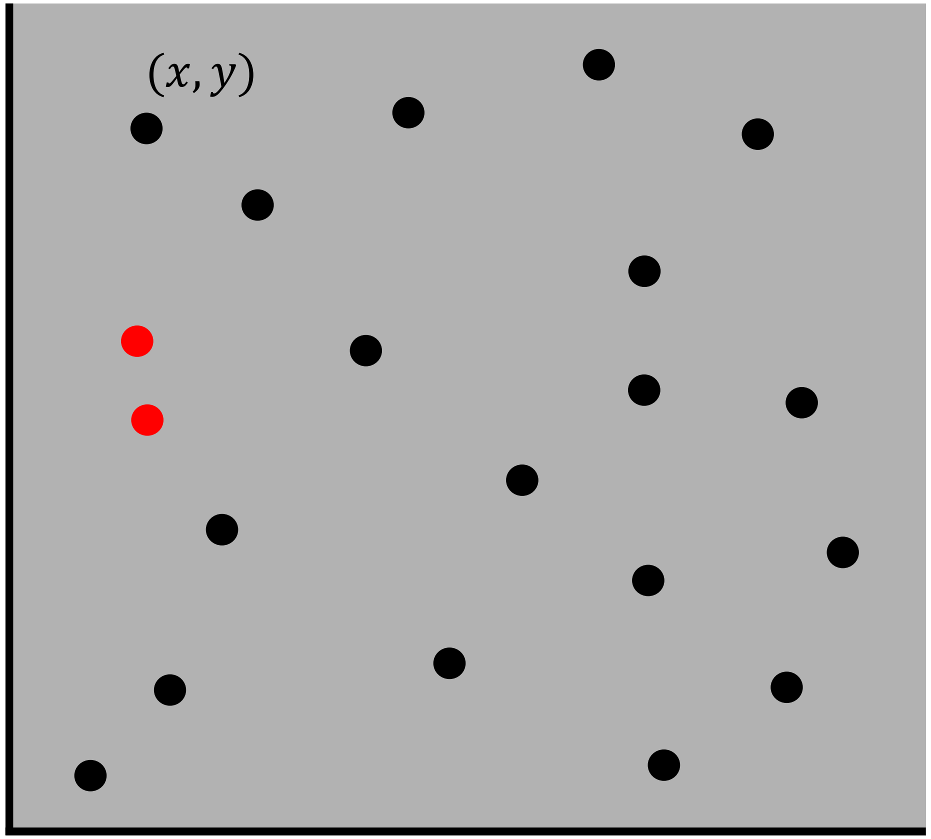
Thus, every edge found by Kruskal's algorithm is part of the MST, and since the edges found form a spanning tree, it is the MST.



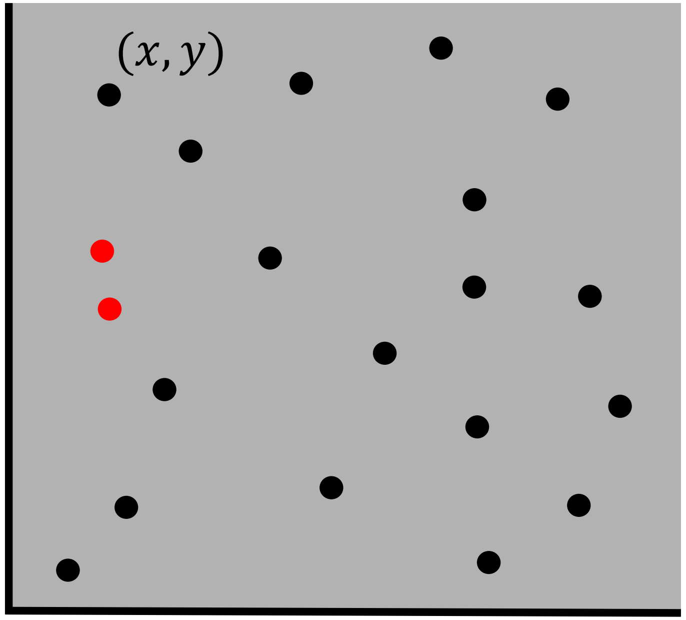
Lemma: The cheapest edge between $S \subseteq V$ and $V\backslash S$ is part of every MST.

Closest Pair Problem



Given n points, find a pair of points with the smallest distance between them.

Closest Pair Problem



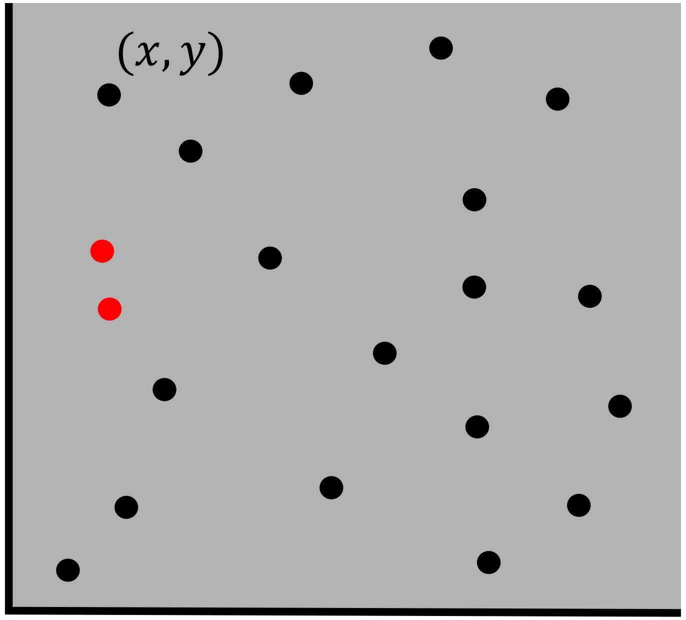
	P_1	P_2	…	P_n
P_1	/	$d_{1,2}$	…	$d_{1,n}$
P_2	$d_{2,1}$	/	…	$d_{2,n}$
…	…	…	…	…
P_n	$d_{n,1}$	$d_{n,2}$	…	/

Simple solution:

1. Compute distance for each pair.
2. Select smallest.

Running Time = ?

Closest Pair Problem



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…	…	…	…	…
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Simple solution:

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Running Time = $O(n^2)$

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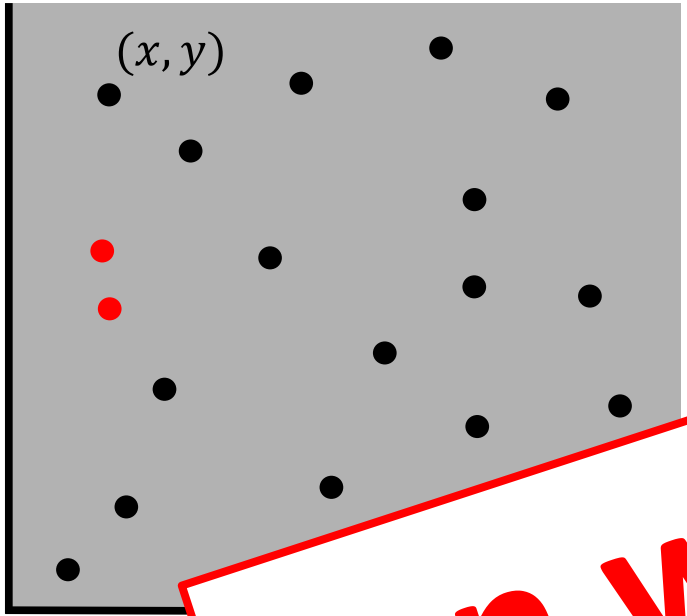

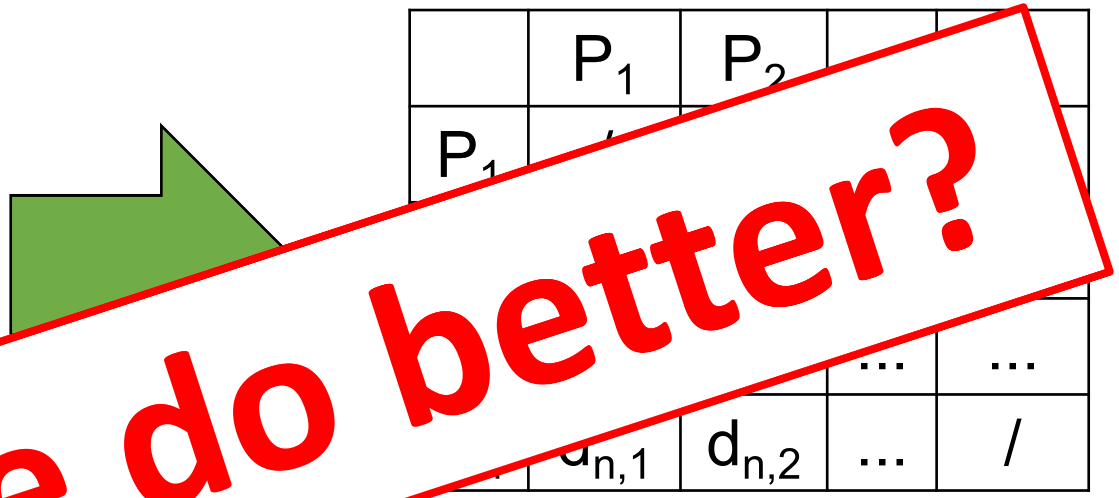

Can we do better?

solution:

1. Compute distance for each pair.
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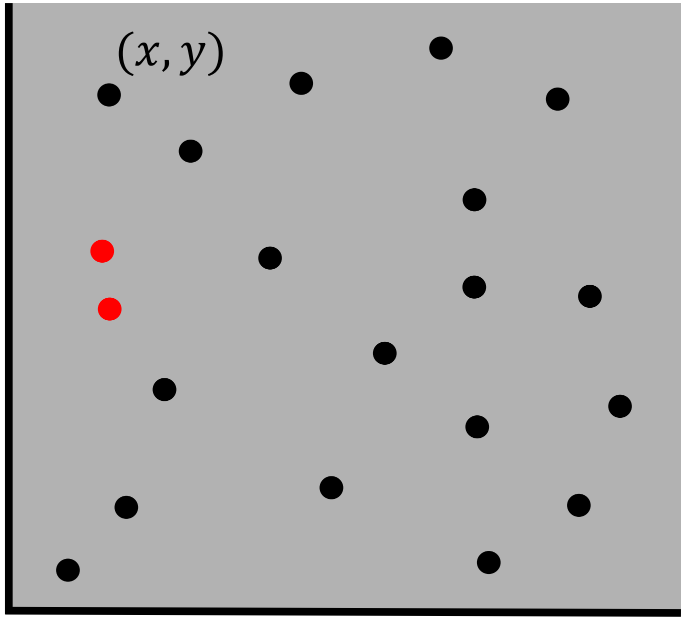
Running Time = $O(n^2)$

Closest Pair Problem



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…	…	…	…	…
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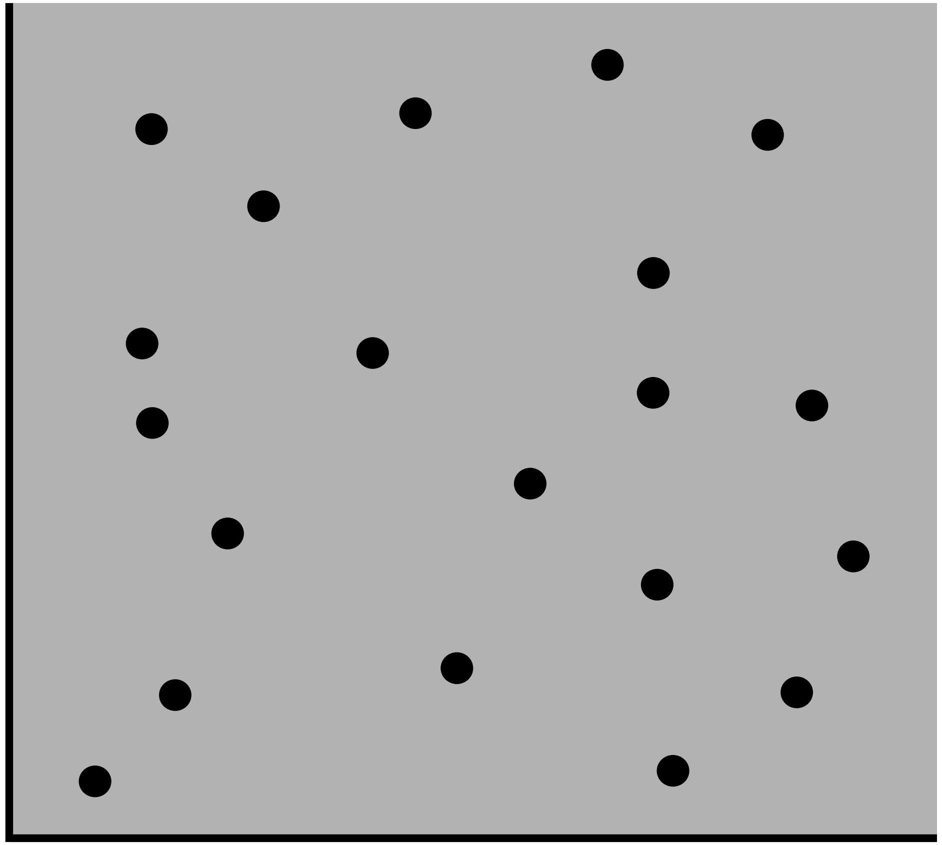
Divide and Conquer Algorithms:

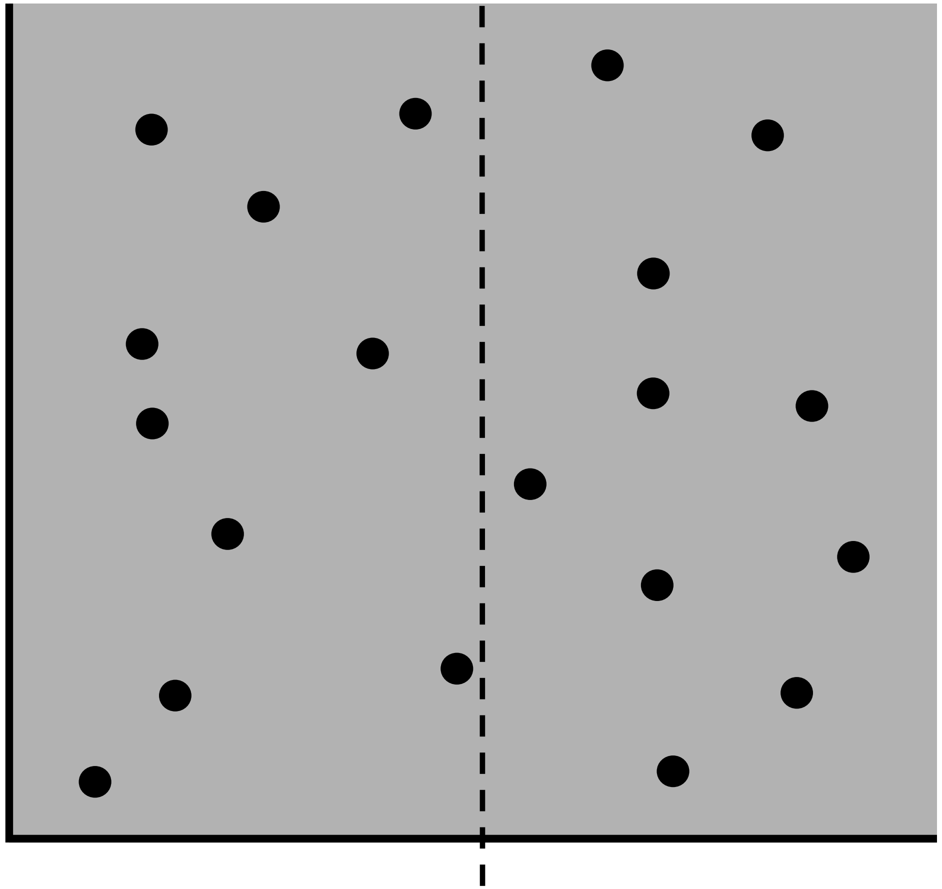
- Divide into subproblems that are smaller instances of the original.
- "Conquer" the subproblems by solving them recursively.
- Combine subproblem solutions into solution for original problem.

Closest Pair Problem – Divide and Conquer



How can we make the problem smaller and easier?

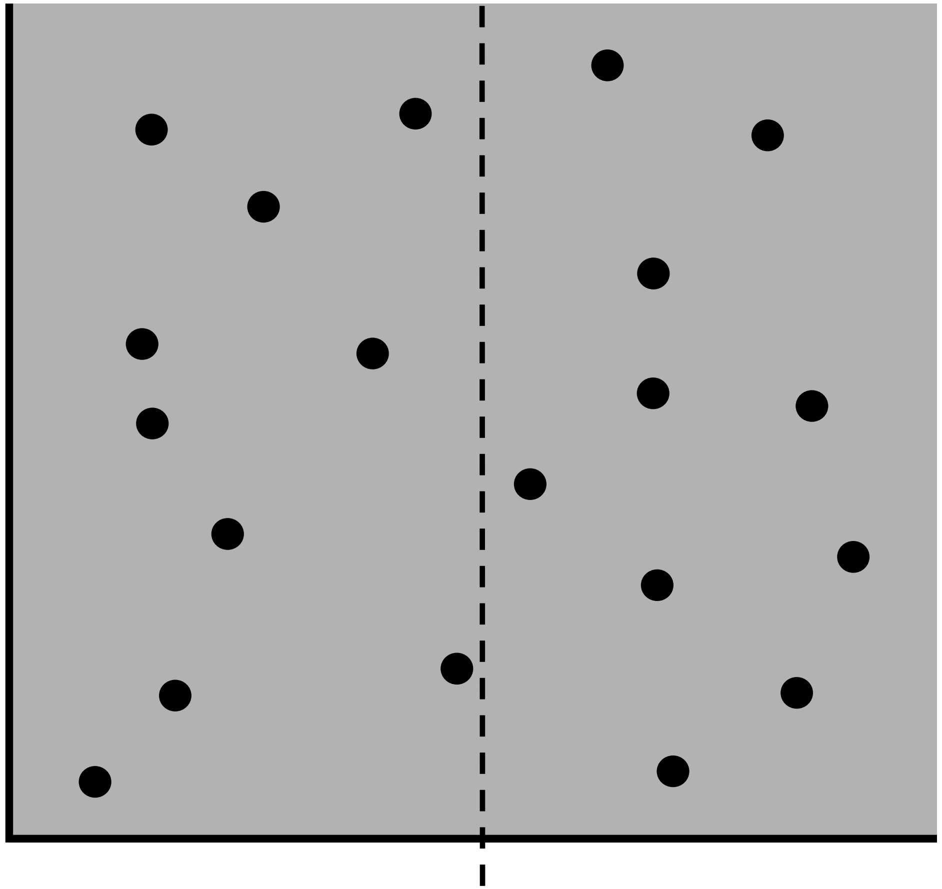
Closest Pair Problem – Divide and Conquer



How can we make the problem smaller and easier?

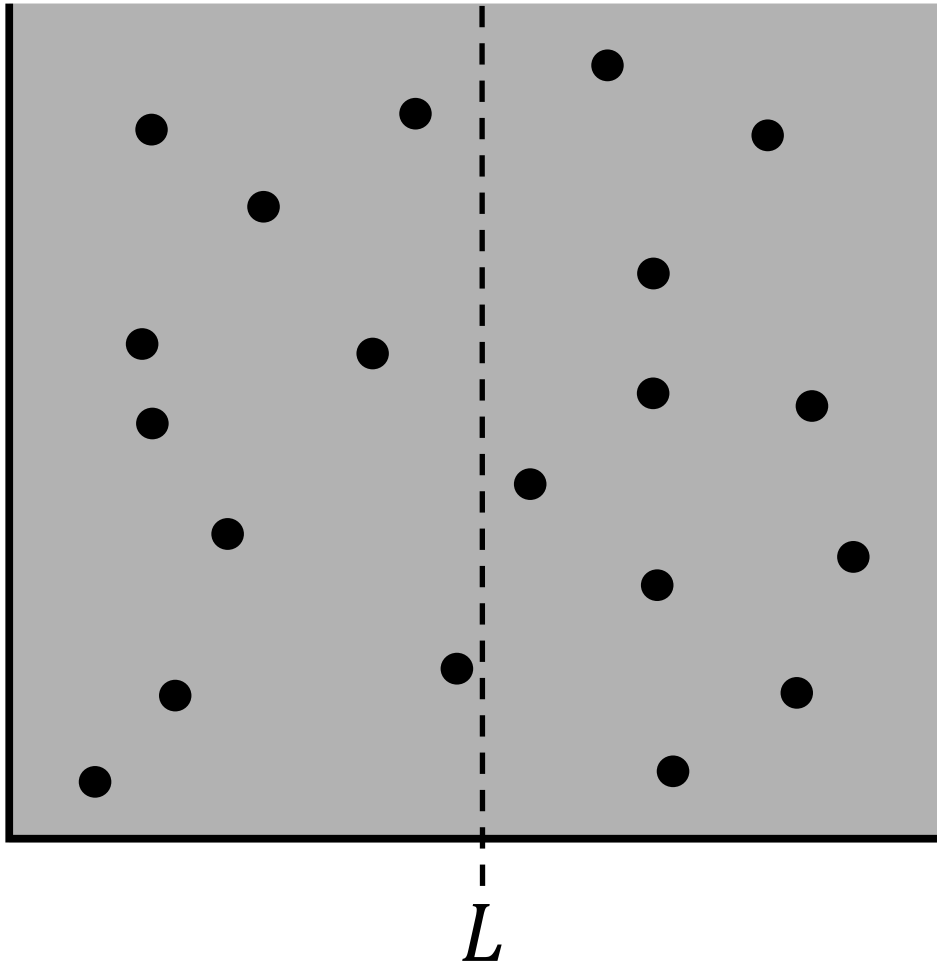
Split it up!

Closest Pair Problem – Divide and Conquer



Divide: How can we draw line so that half of the points are on each side?

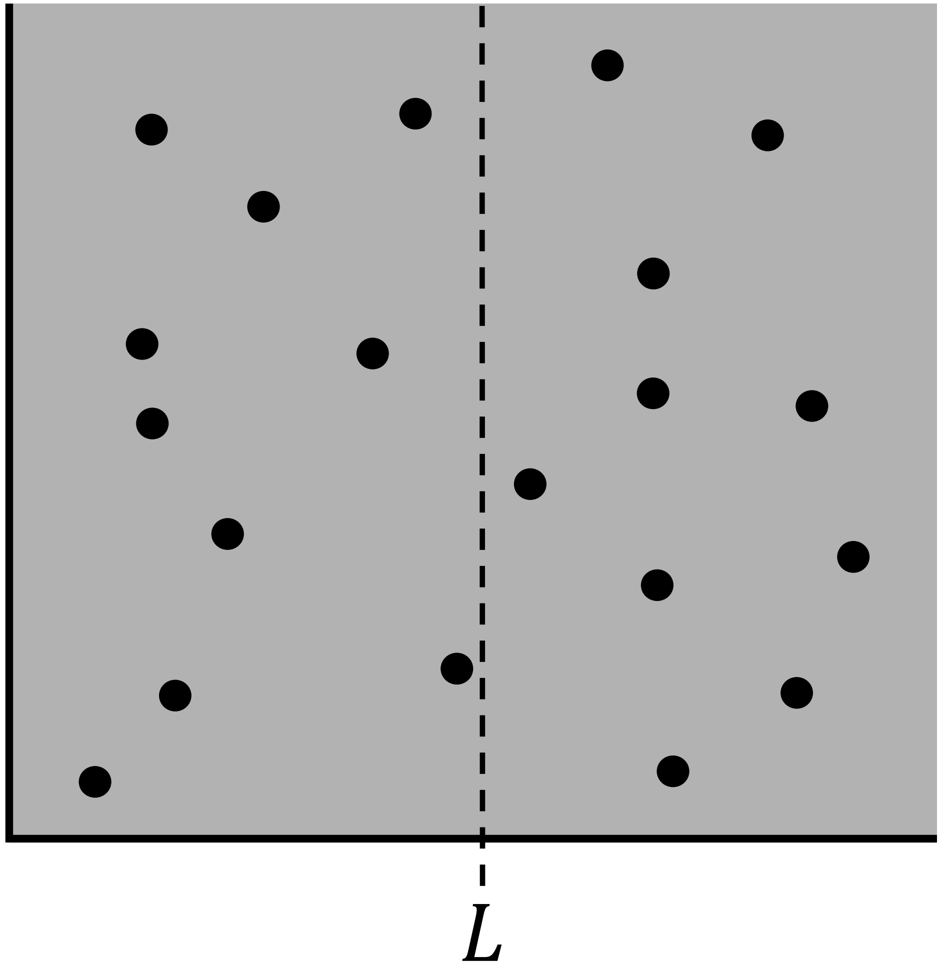
Closest Pair Problem – Divide and Conquer



Divide: How can we draw line so that half of the points are on each side?

1. Sort by x-coordinate.
2. Put L at median value.

Closest Pair Problem – Divide and Conquer

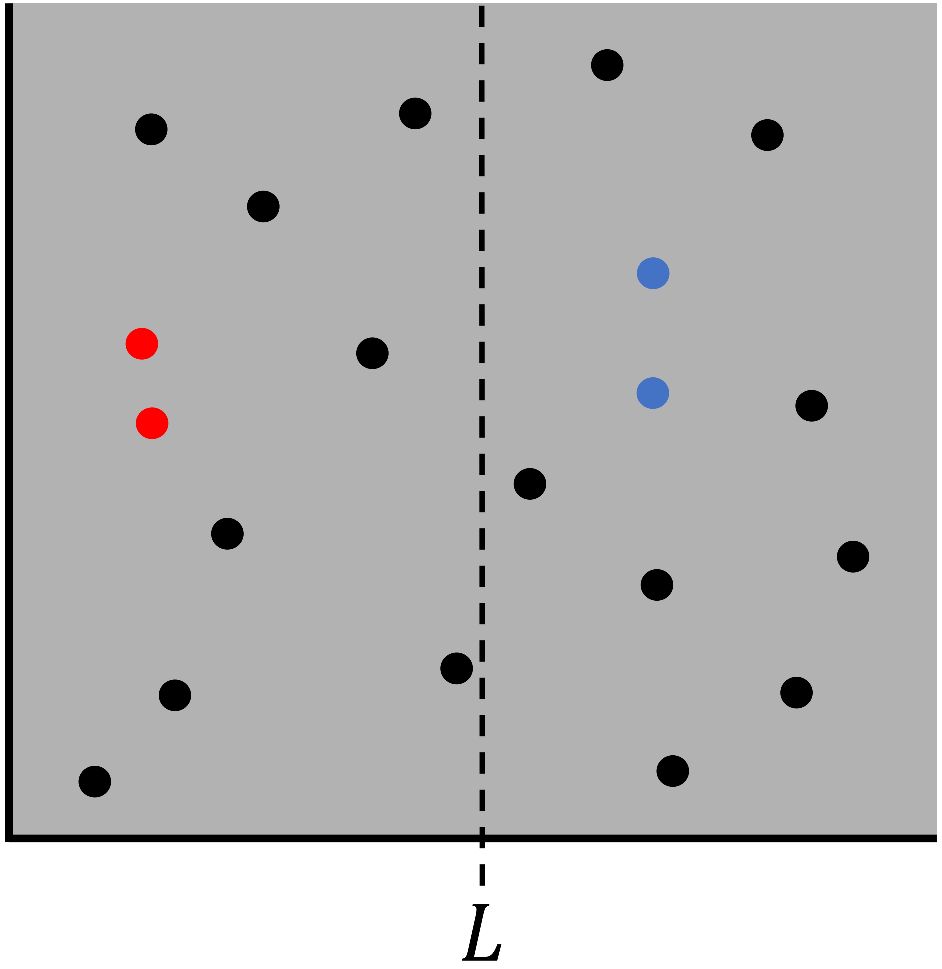


Conquer:

Recursively find closest pairs on each side[1].

[1]Details to follow

Closest Pair Problem – Divide and Conquer



Combine: If we had the closest left pair and the closest right pair, how do we determine actual closest?

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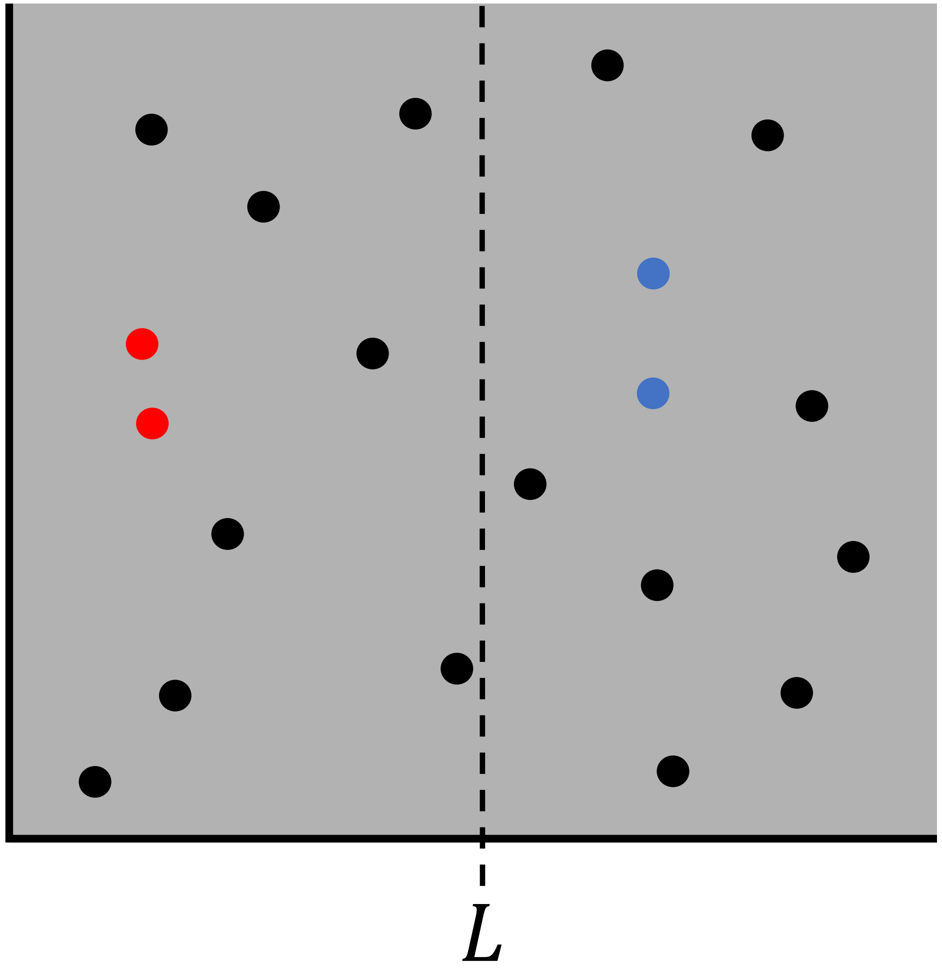

Combine: If we had the closest left pair and the closest right pair, how do we determine actual closest?

1. Return minimum of: d_{left}, d_{right}.

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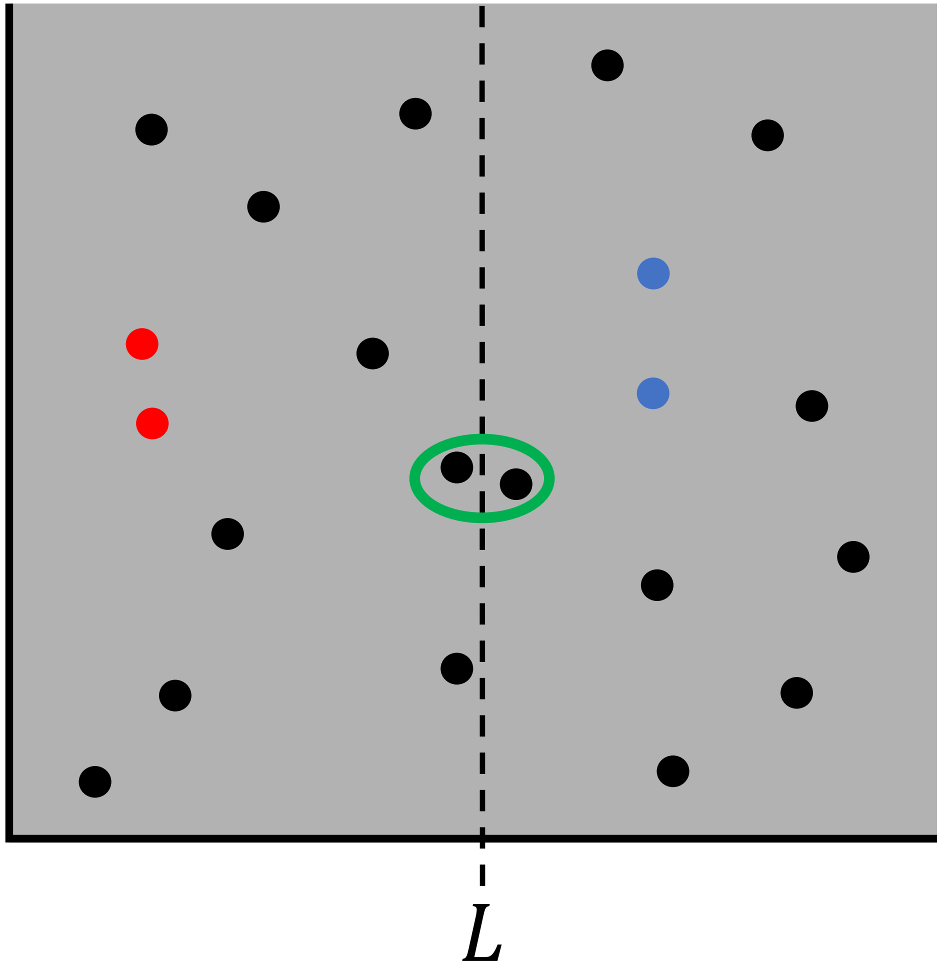

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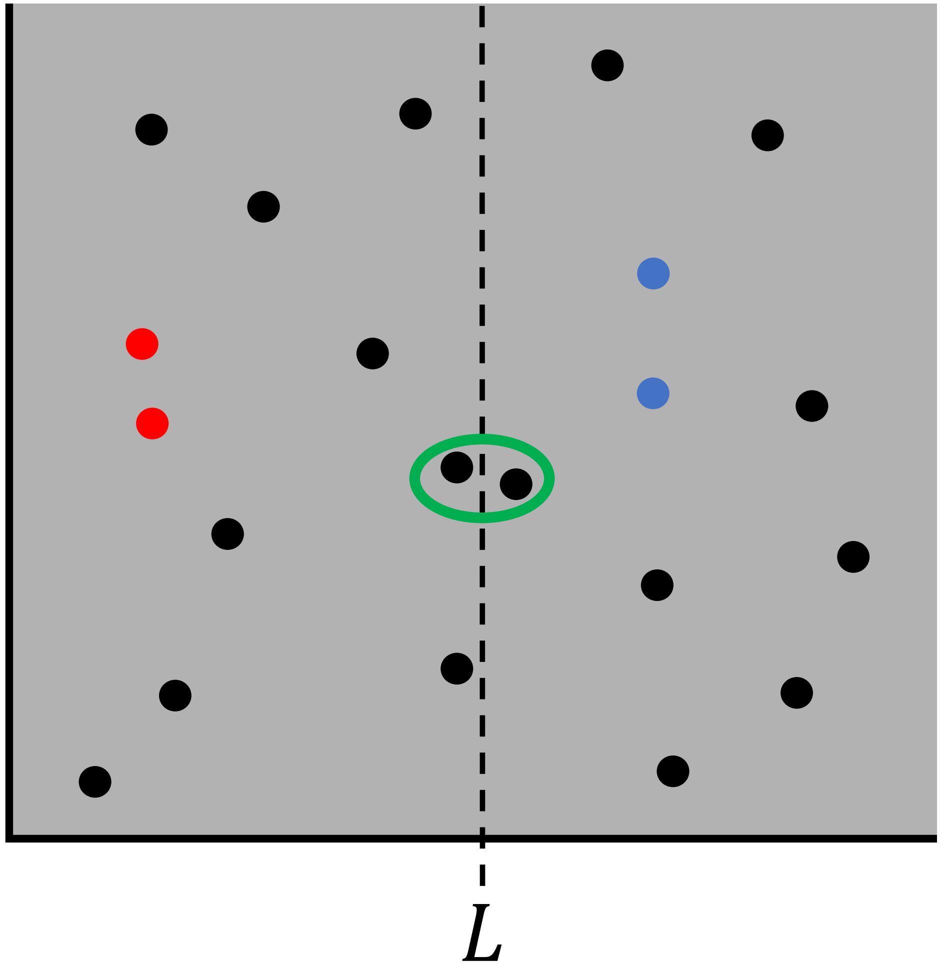
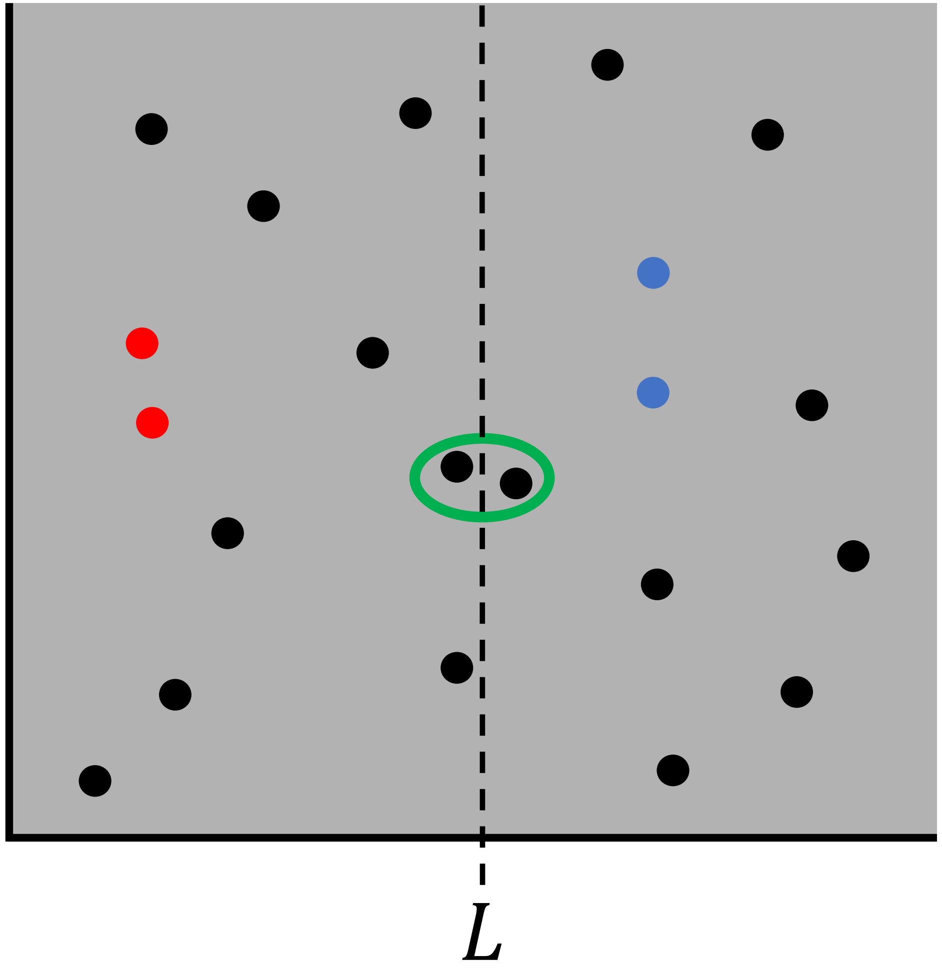


Combine: If we had the closest left pair and the closest right pair, how do we determine actual closest?

1. Return minimum of: d_{left}, d_{right}, $d_{\text{min\_straddle}}$.

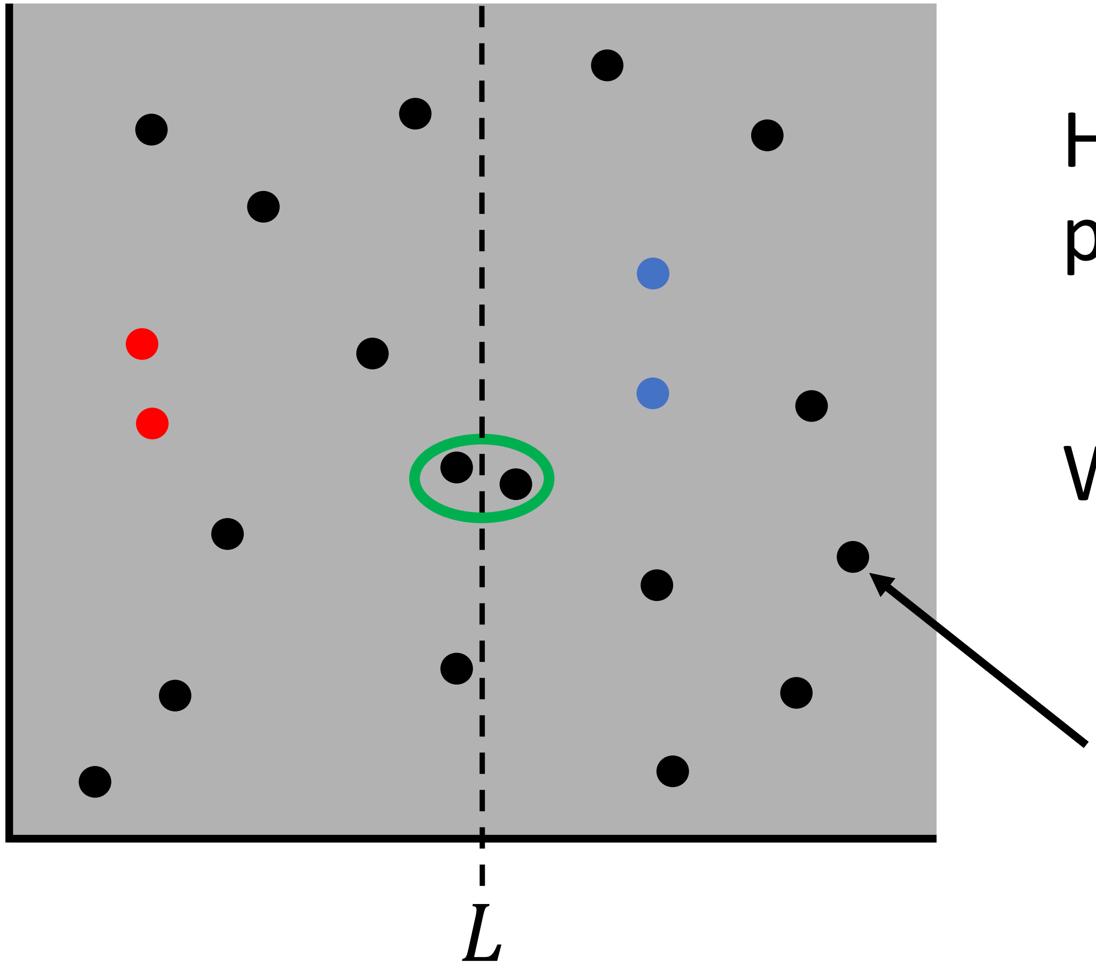
Closest Pair Problem – Divide and Conquer



How should we search for "straddle points"?

We know $\delta = \min(d_{\text{left}}, d_{\text{right}})$.

Closest Pair Problem – Divide and Conquer

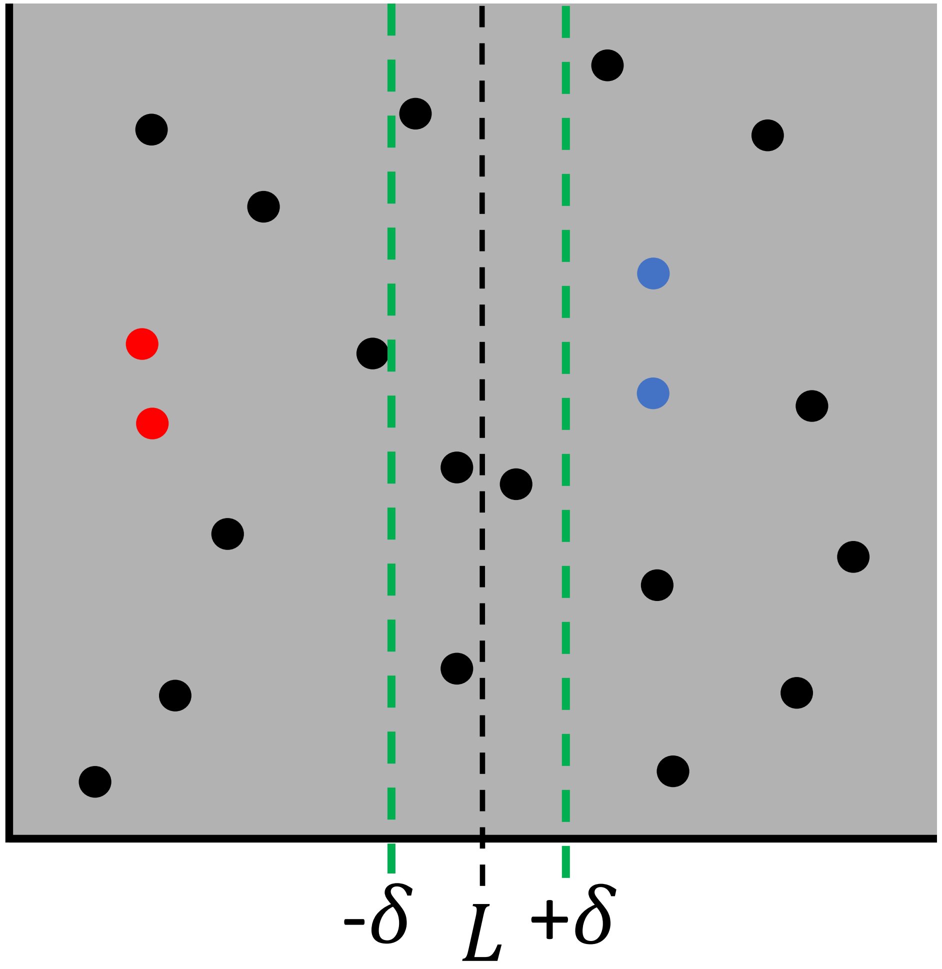


How should we search for "straddle points"?

We know $\delta = \min(d_{\text{left}}, d_{\text{right}})$.

Do we need to consider this point when looking for straddle points?

Closest Pair Problem – Divide and Conquer



Rule: We only need to hunt for straddle points at most δ away from L.

Reason: Points outside $L \pm \delta$ cannot reach the other side in less than δ.

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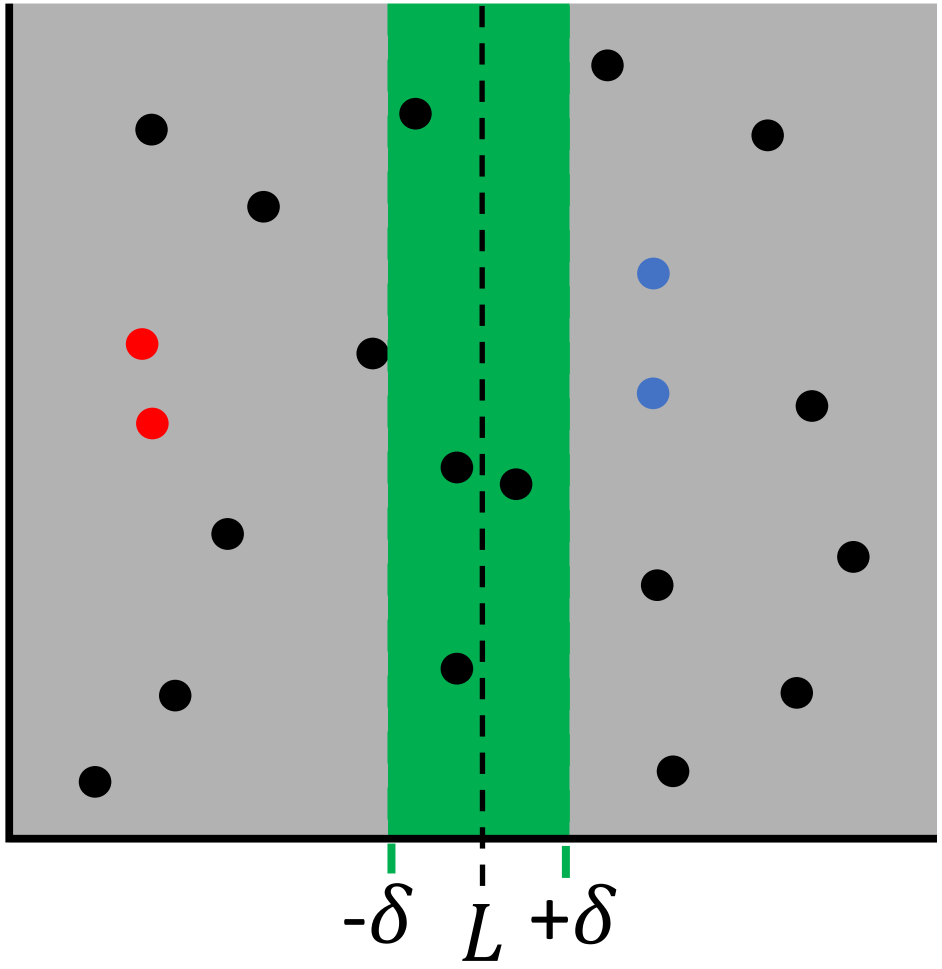

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Let $\boldsymbol{S}$ be the set of straddle points.

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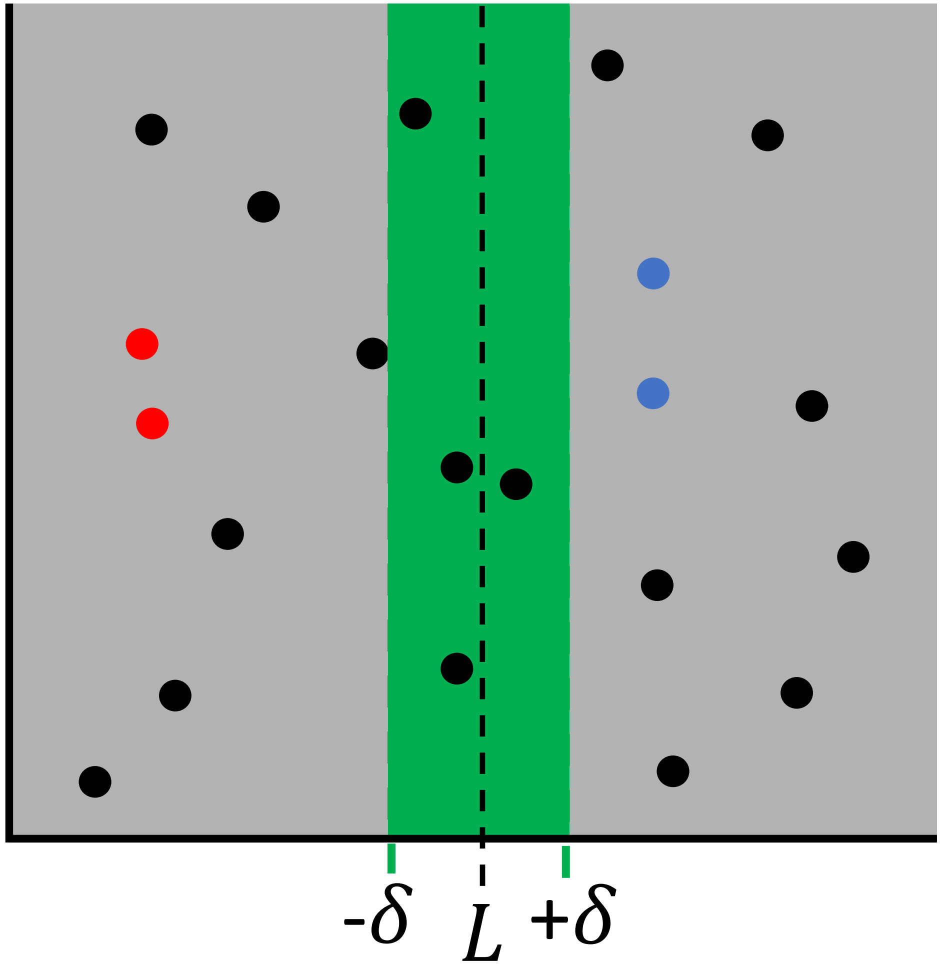
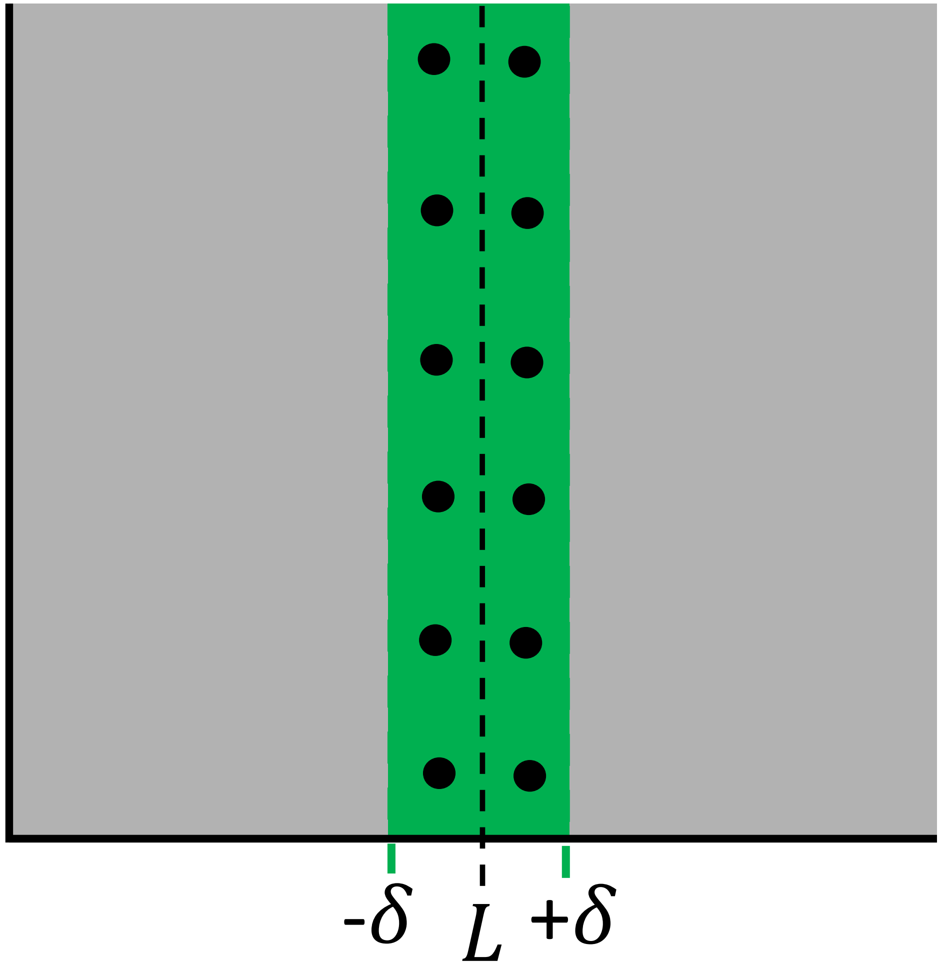


Can we just compare all left straddle points to all right straddle points?

Closest Pair Problem – Divide and Conquer



Can we just compare all left straddle points to all right straddle points?

So, we need to reduce the number of straddle points we have to consider.

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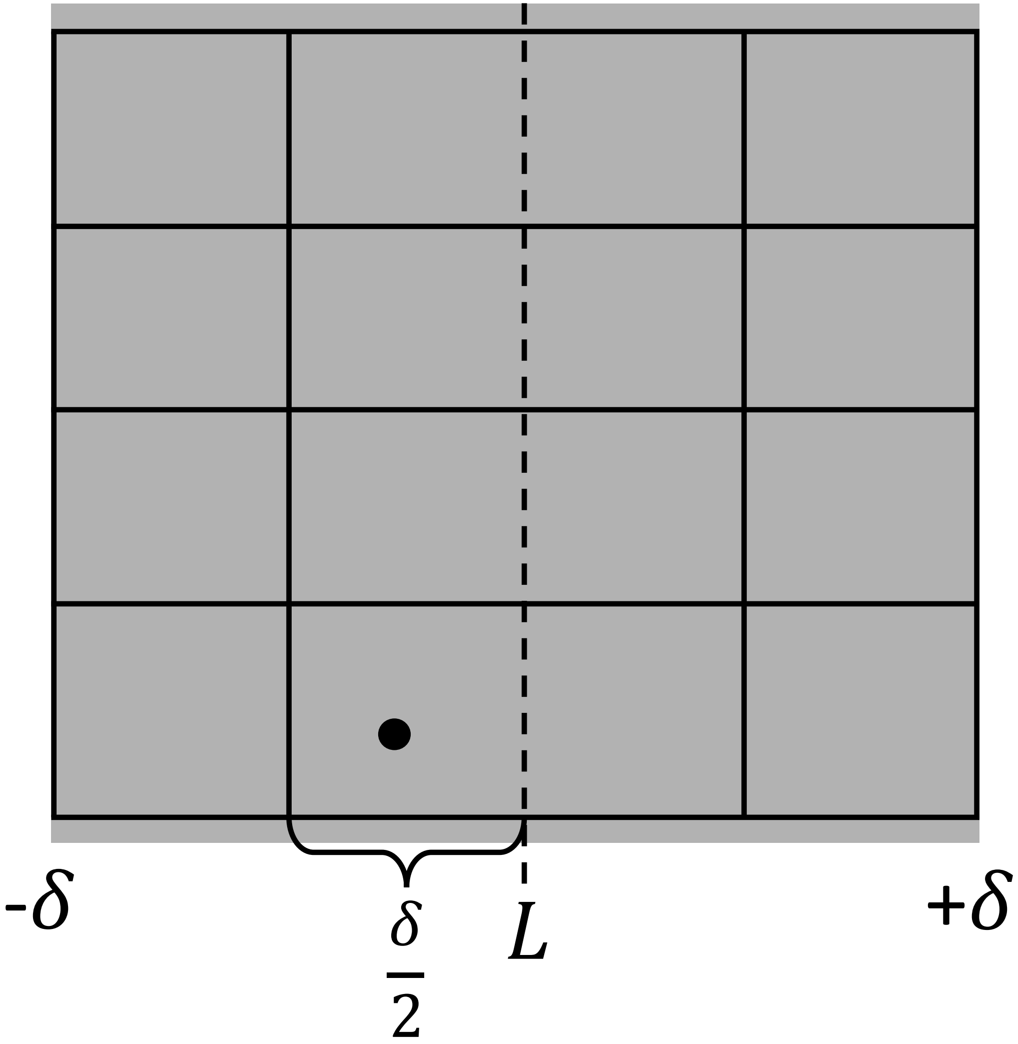

Divide S into $\frac{\delta}{2} \times \frac{\delta}{2}$ boxes.

Can we focus our search to certain boxes?

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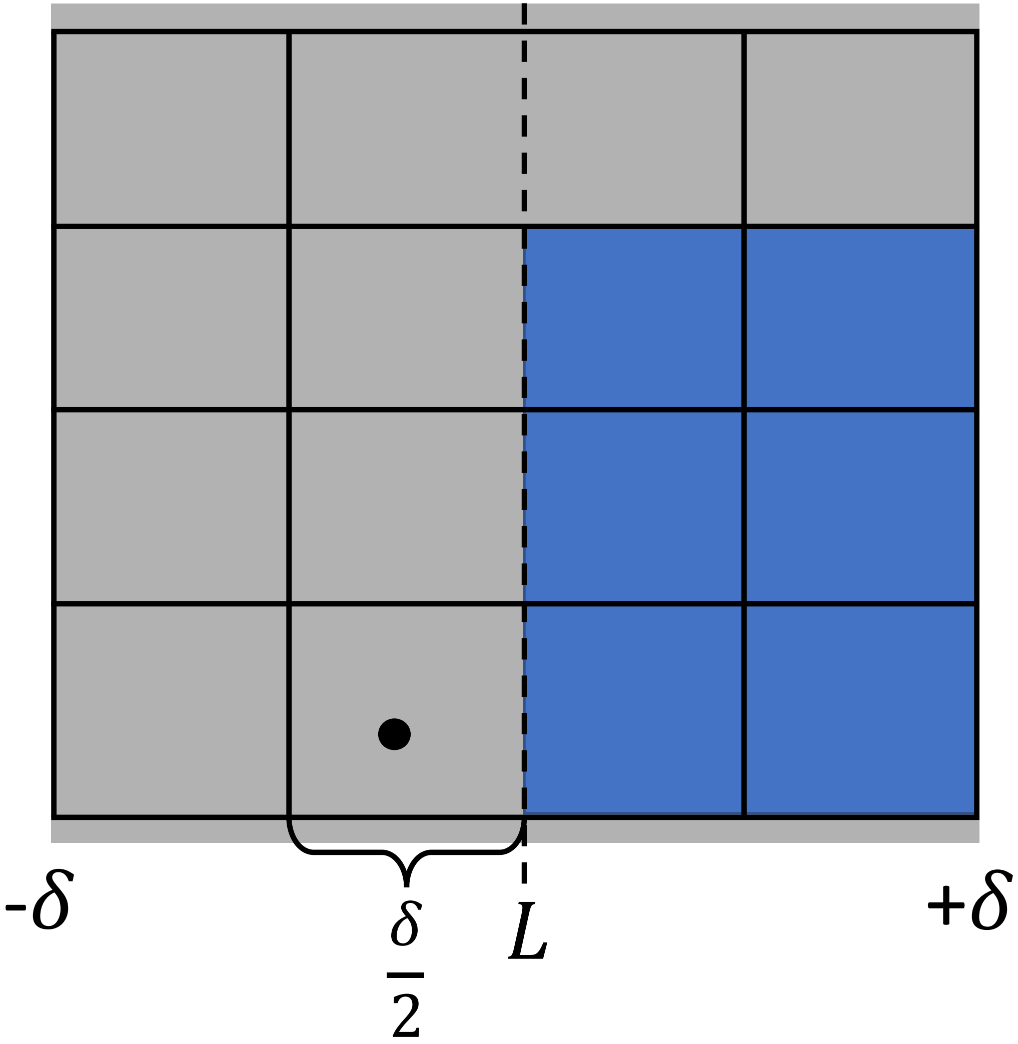

Divide S into $\frac{\delta}{2} \times \frac{\delta}{2}$ boxes.

Can we focus our search to certain boxes?

Yes – we only care about points within δ.

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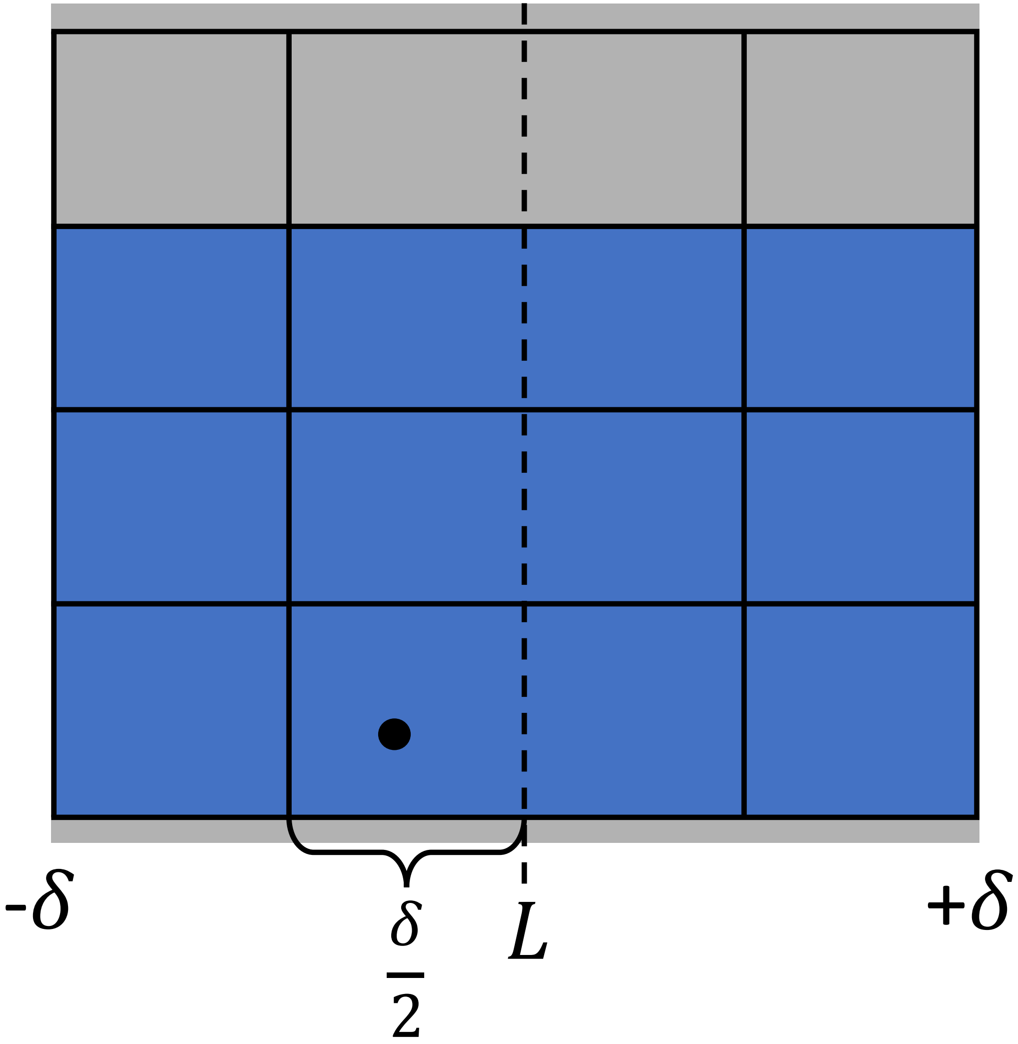

Divide S into $\frac{\delta}{2} \times \frac{\delta}{2}$ boxes.

Can we focus our search to certain boxes?

> Yes – we only care about points within δ.

This still gives us possibly lots of points to look at.

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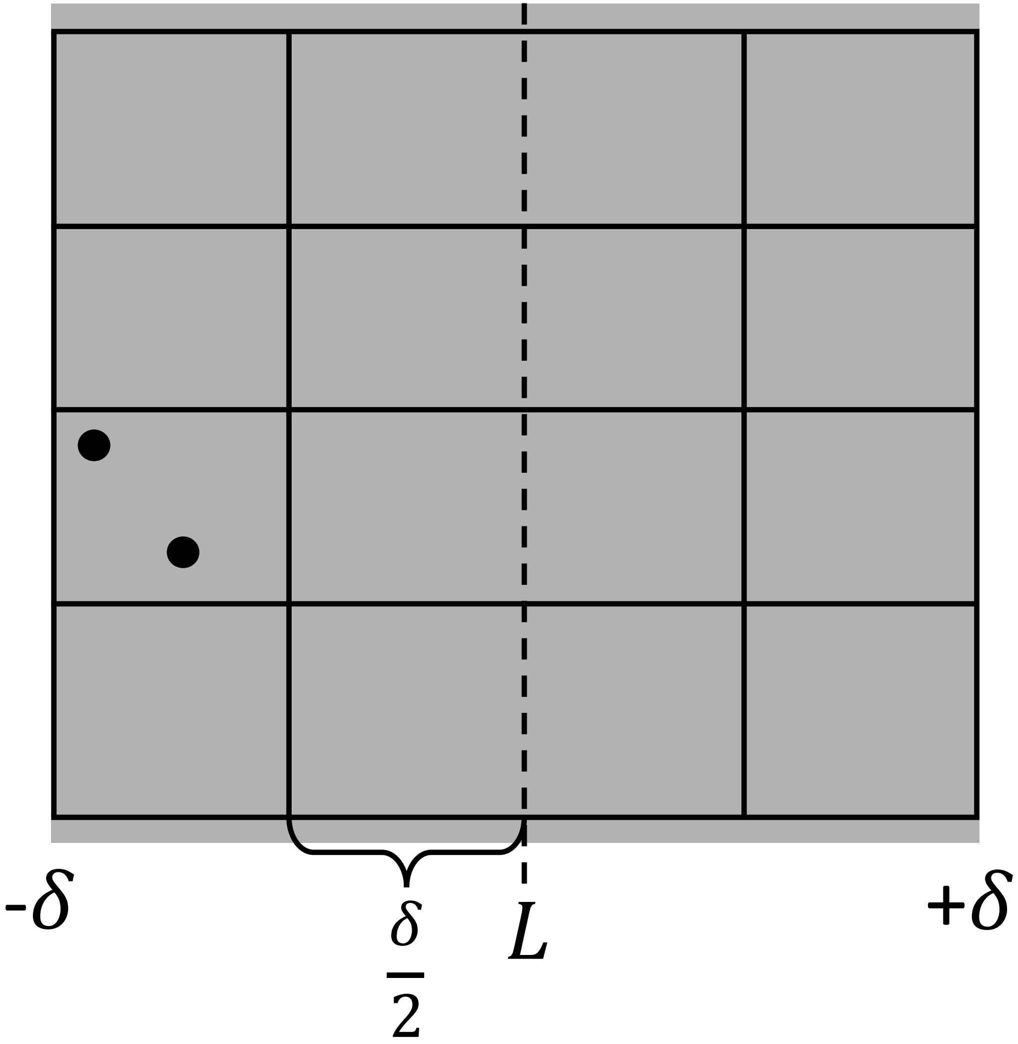

Can we have multiple points in one box?

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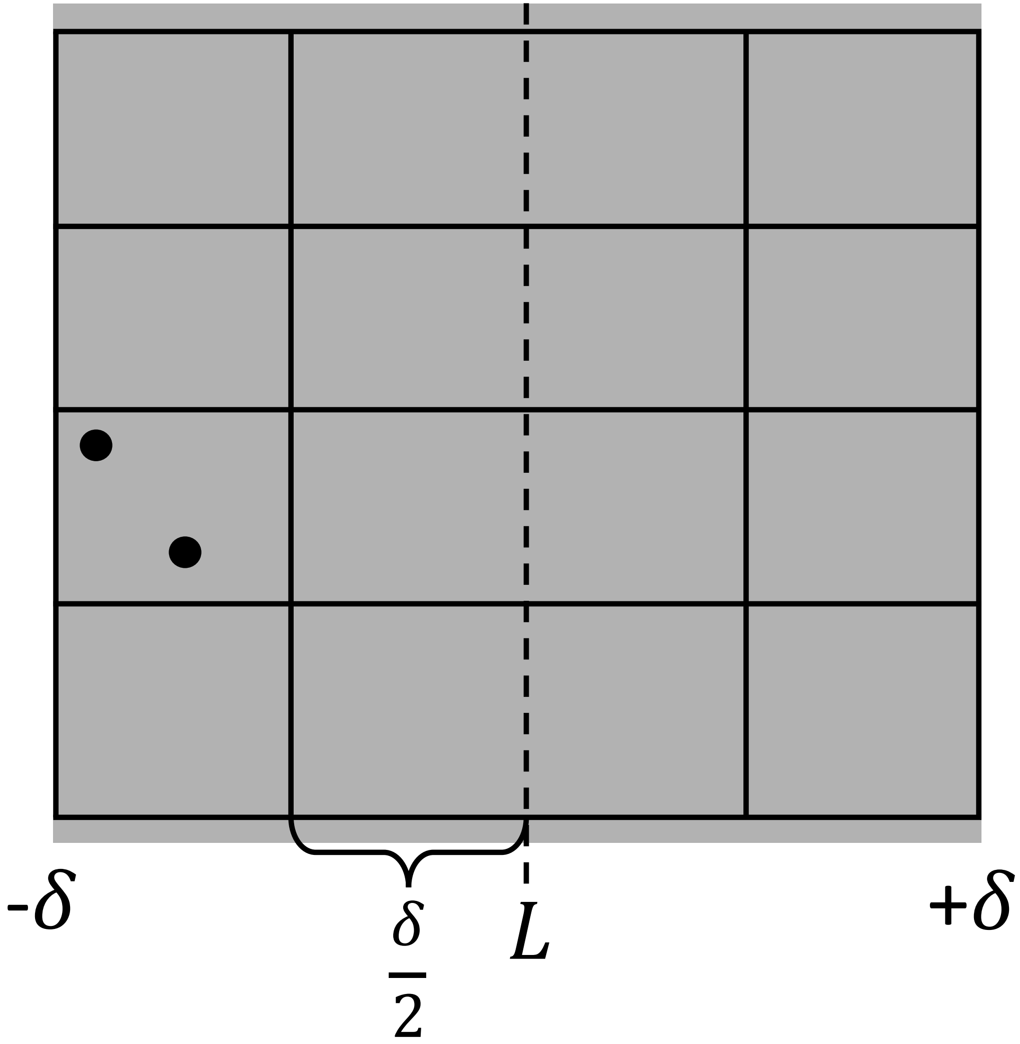

Can we have multiple points in one box?

No. δ is the smallest distance on either side of L.

⇒ at most one point per box.

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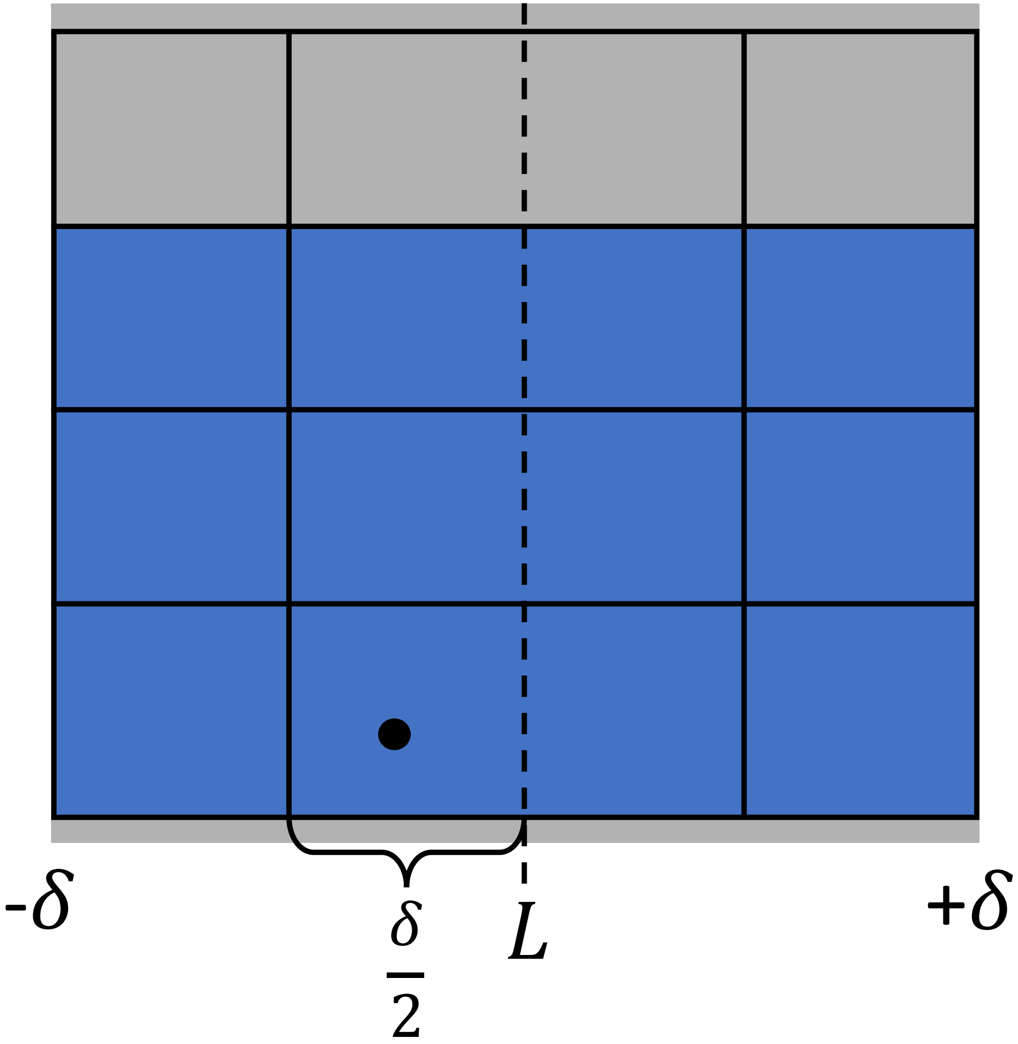

Only care about certain boxes
+ At most one point per box

Fixed number of points to check

1. Sort straddle points by y coordinate.
2. Only possible "δ-busting" points are the 11 points after our point being considered.

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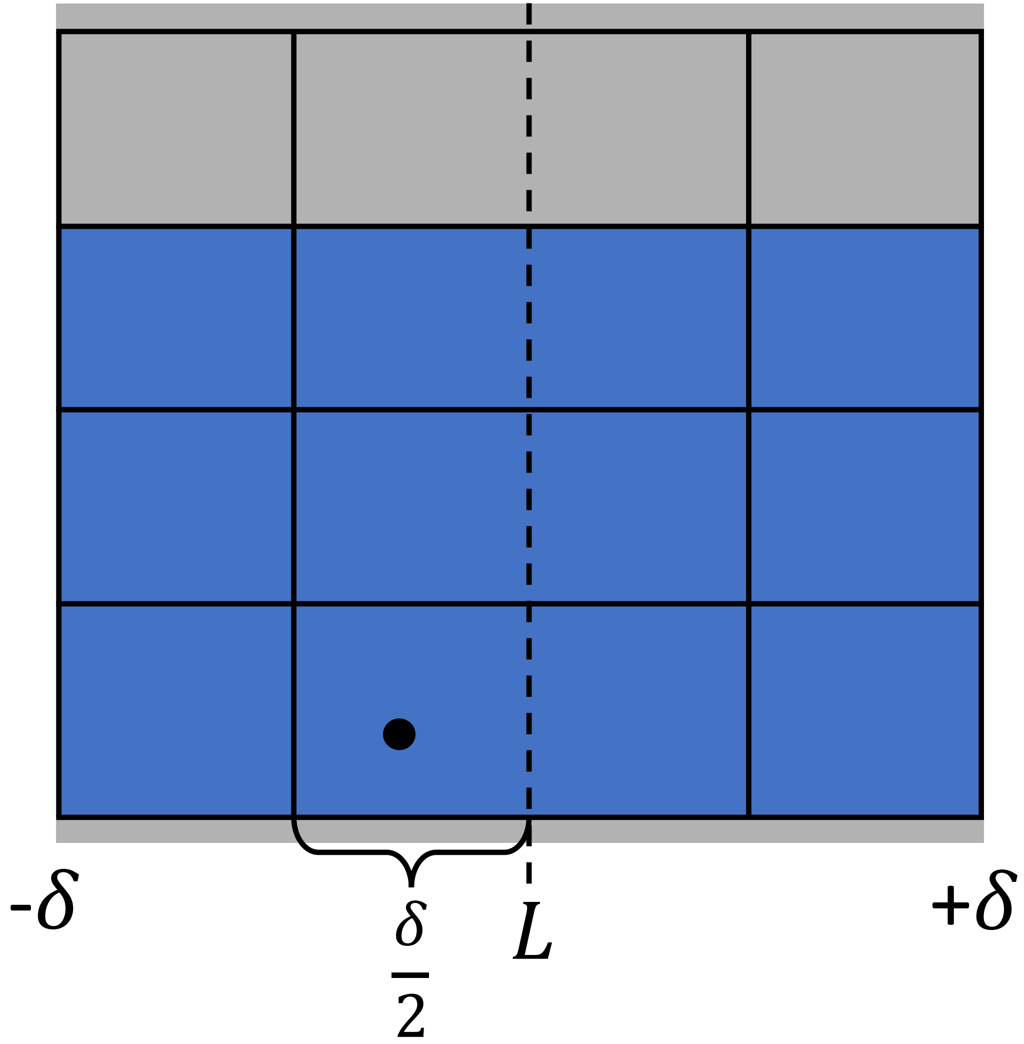

Only care about certain boxes

\+ At most one point per box

Fixed number of points to check

Straddle point hunting:

$$O(n^2) \longrightarrow O(n \log n)$$