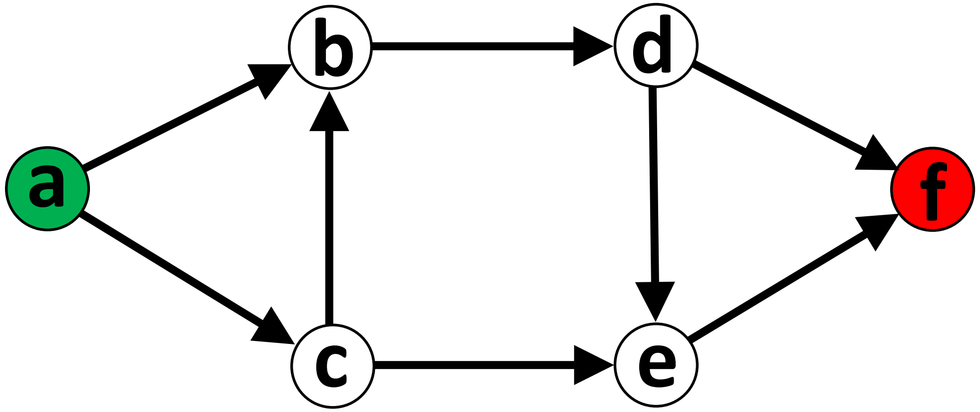
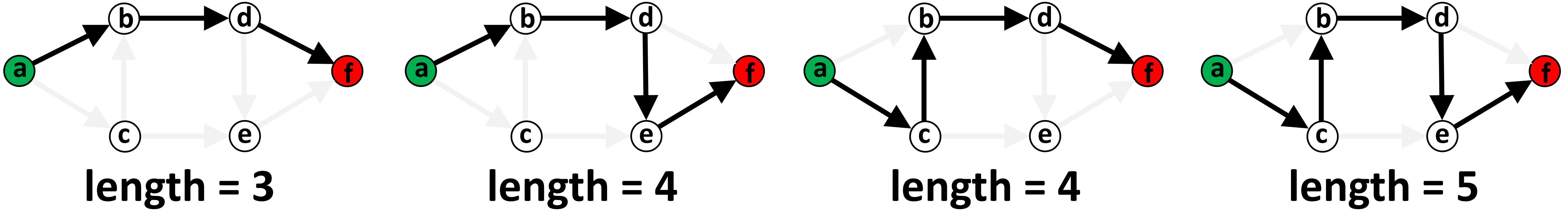
Dynamic Programming
CSCI 532

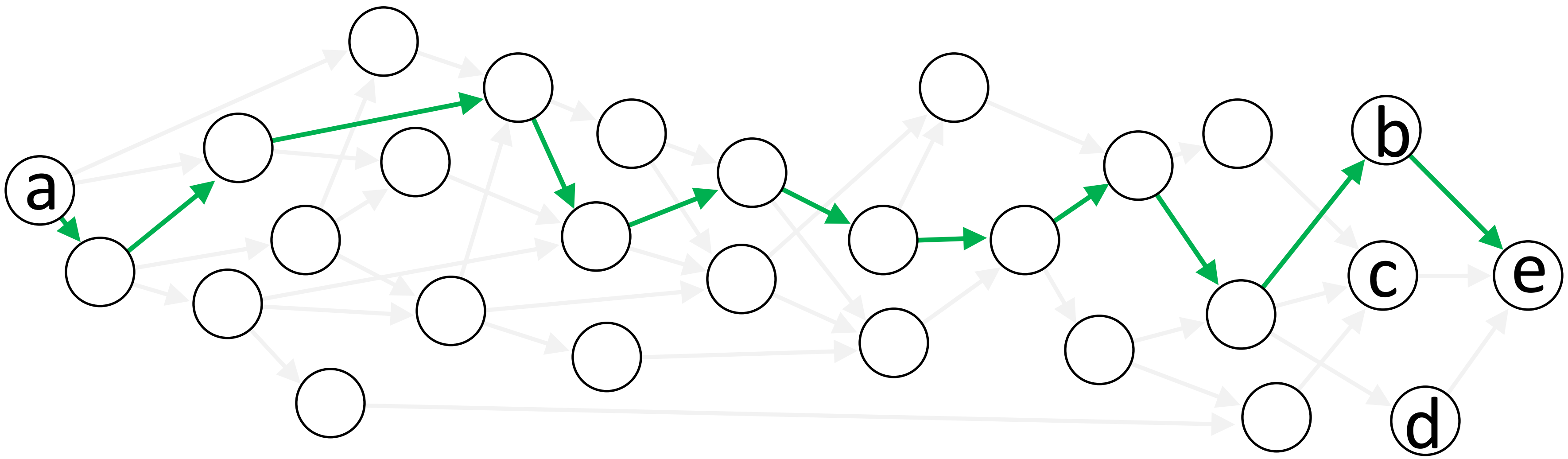
Longest Path in a DAG

Given a DAG, find the longest path between any two vertices in the graph.





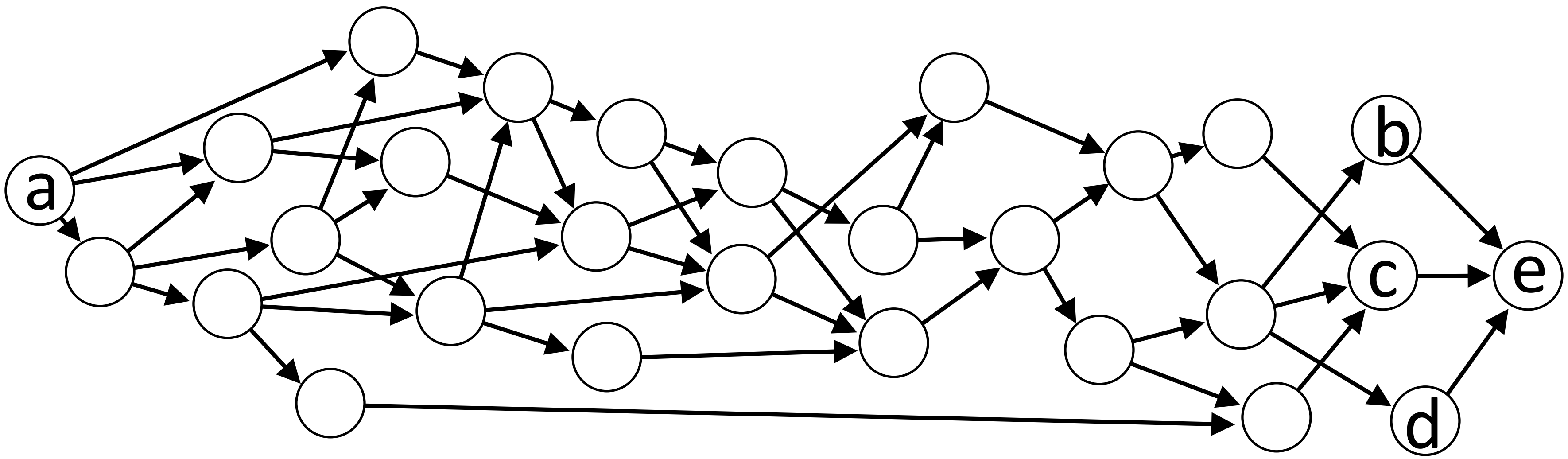
Find the Longest Path in a DAG



Interesting observations?

If the longest path goes from **a** to **e** and passes through **b**, that must be the longest path that ends at **b**. If not, then we could make a longer path.

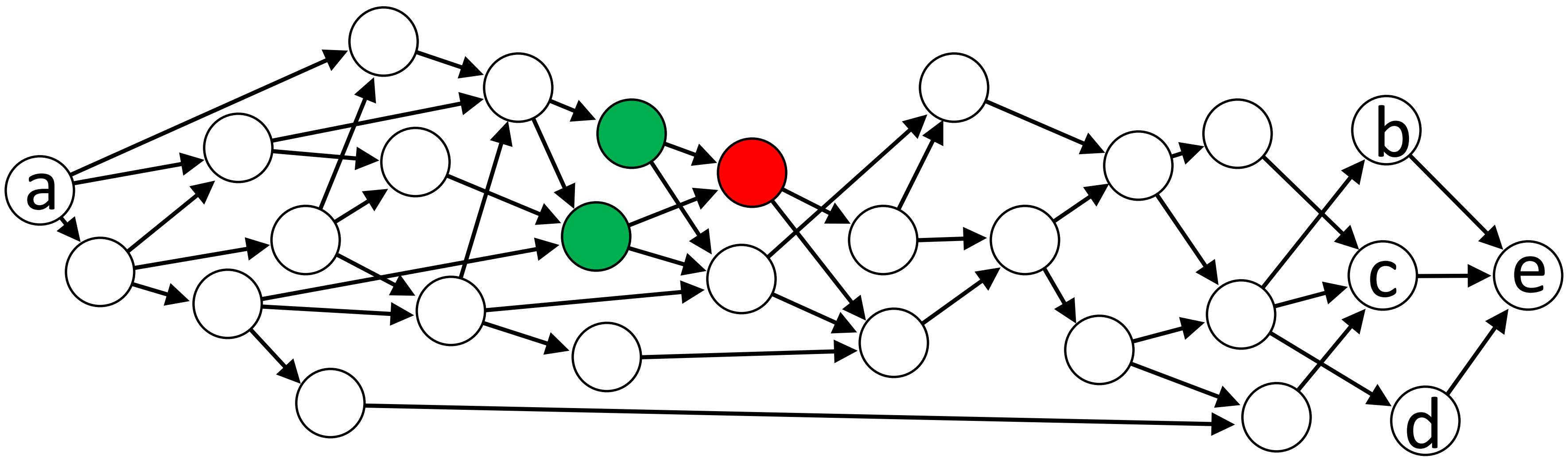
Find the Longest Path in a DAG



Interesting observations?

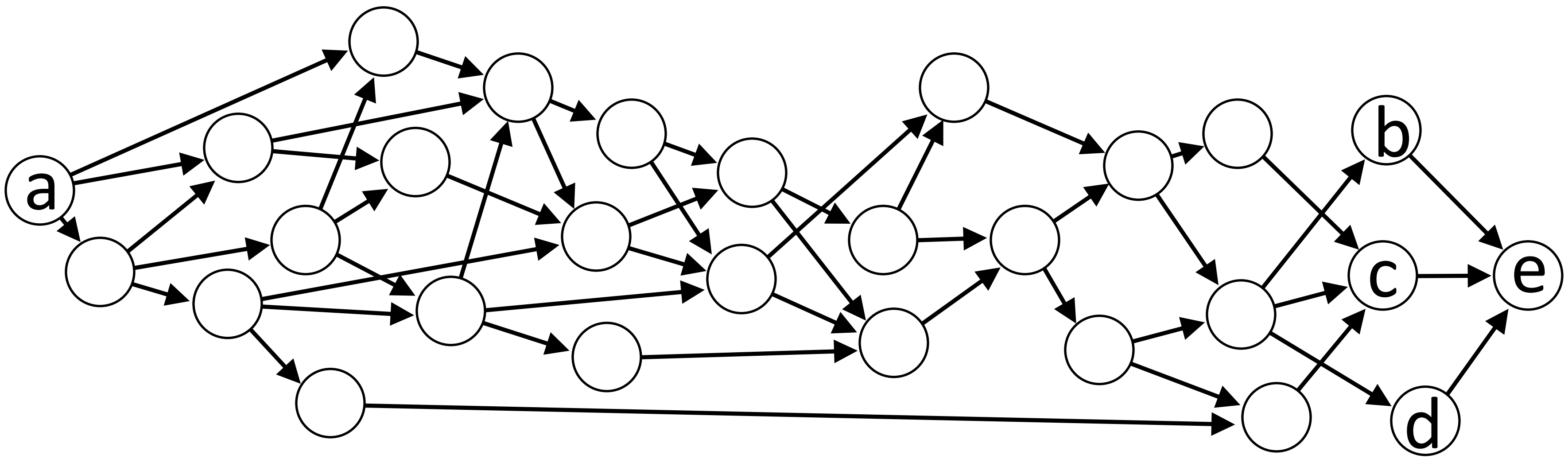
$$\text{The longest path to e} = \max\begin{pmatrix}\text{longest path to b}\\\text{longest path to c}\\\text{longest path to d}\end{pmatrix} + 1$$

Find the Longest Path in a DAG



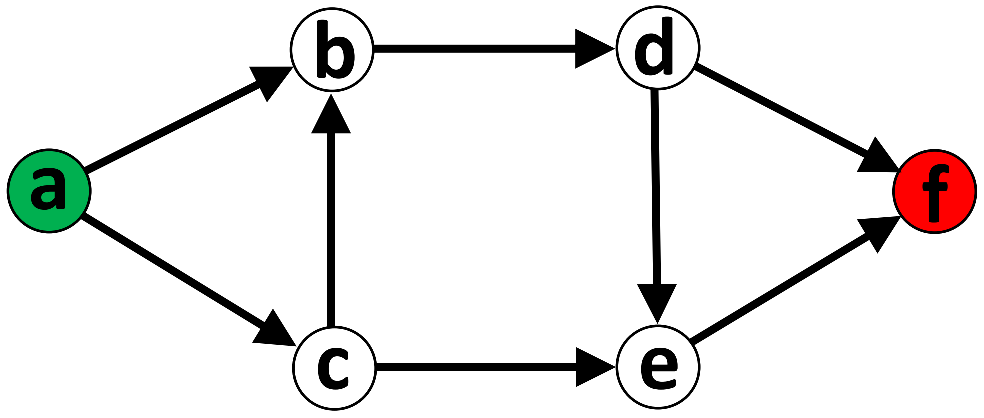
In general: We are ready to calculate the longest path to a vertex if we know the longest path for all incoming neighbors.

Find the Longest Path in a DAG



Topological Ordering of a graph: ordering of its vertices such that for every directed edge (u, v), vertex u comes before vertex v in the ordering.

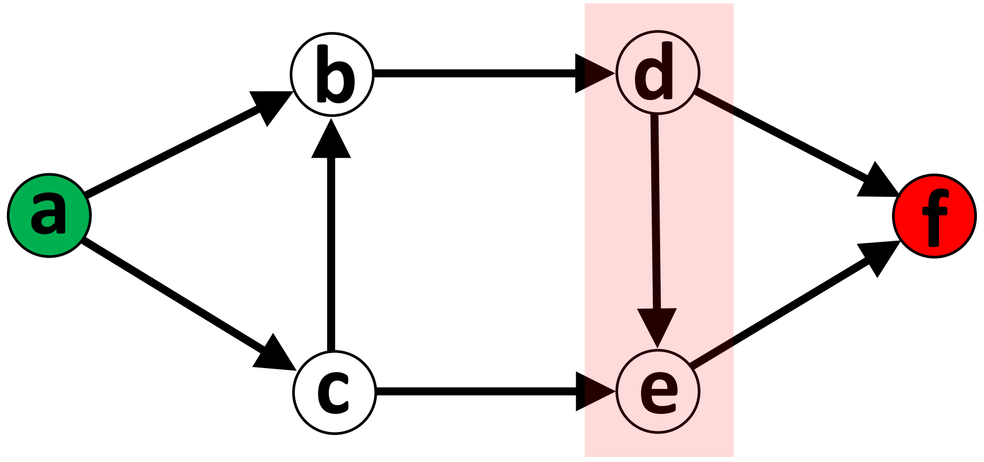
Topological Ordering



Topologically Ordered:

{a, c, b, d, e, f} ✓

Topological Ordering

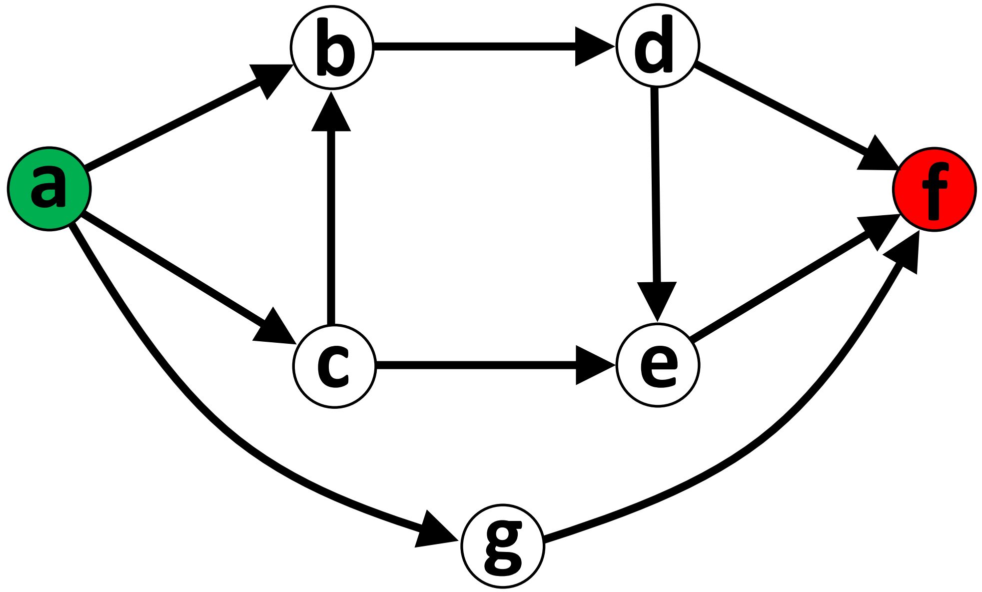


Topologically Ordered:

{a, c, b, d, e, f} ✓

{a, c, b, e, d, f} ✗

Topological Ordering

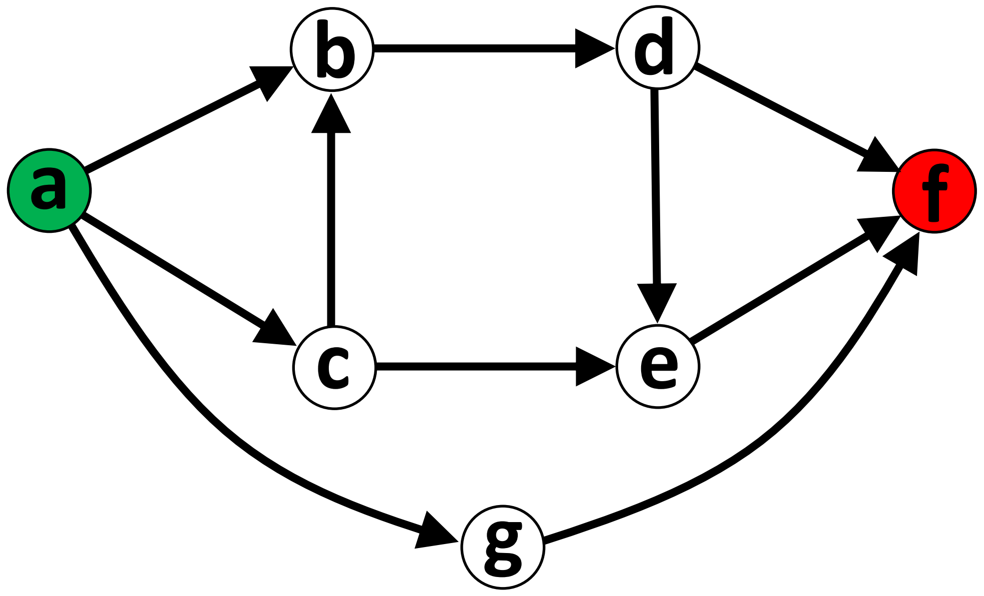


Topologically Ordered:

{a, c, g, b, d, e, f} ✓

{a, c, b, d, e, g, f} ✓

Topological Ordering

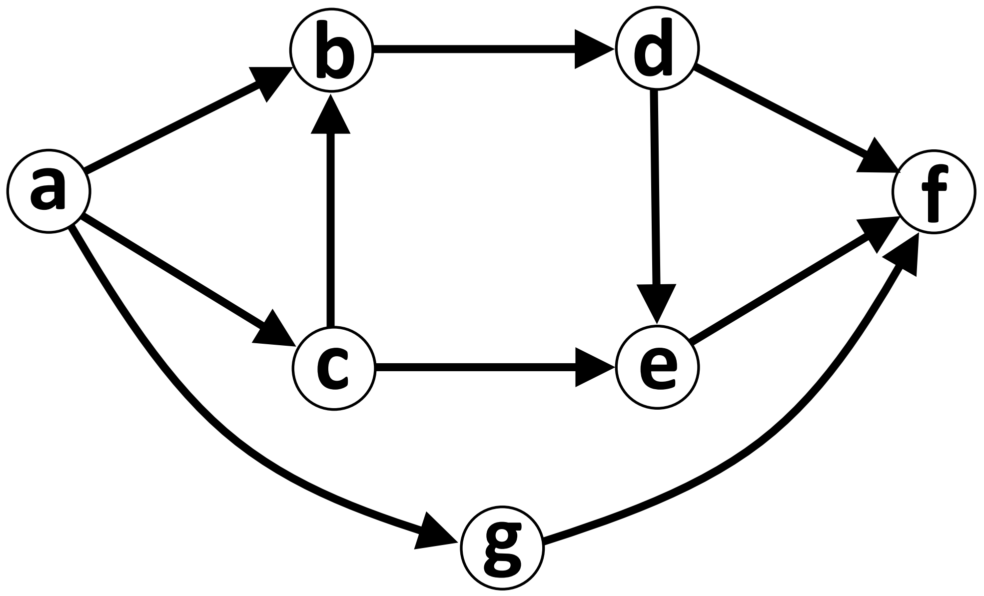


- There are various algorithms to find topological orderings
- Standard running time $= O(|V| + |E|)$.

Find the Longest Path in a DAG

Plan:

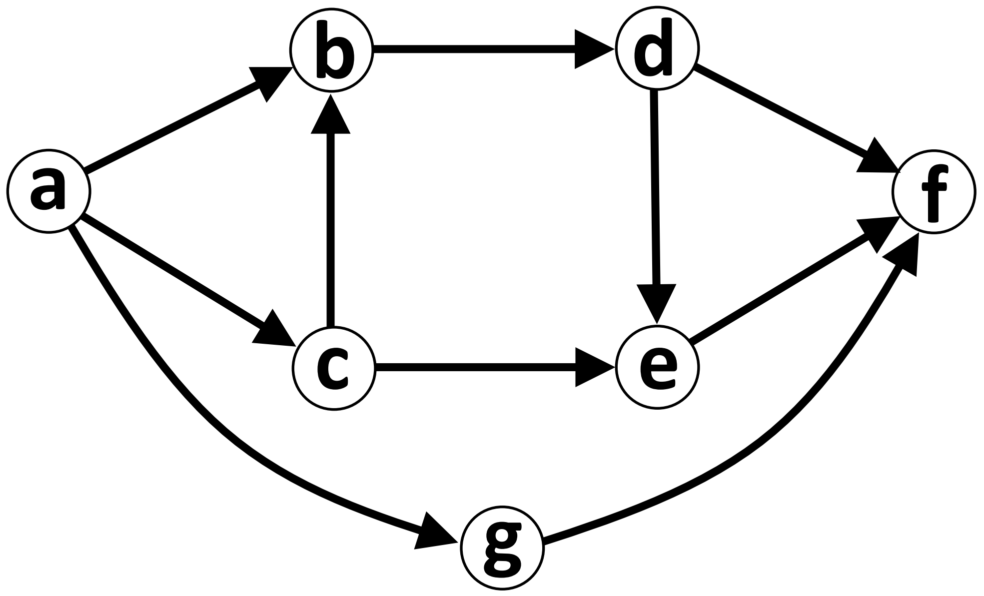
- ??



Find the Longest Path in a DAG

Plan:

- Topologically sort vertices.

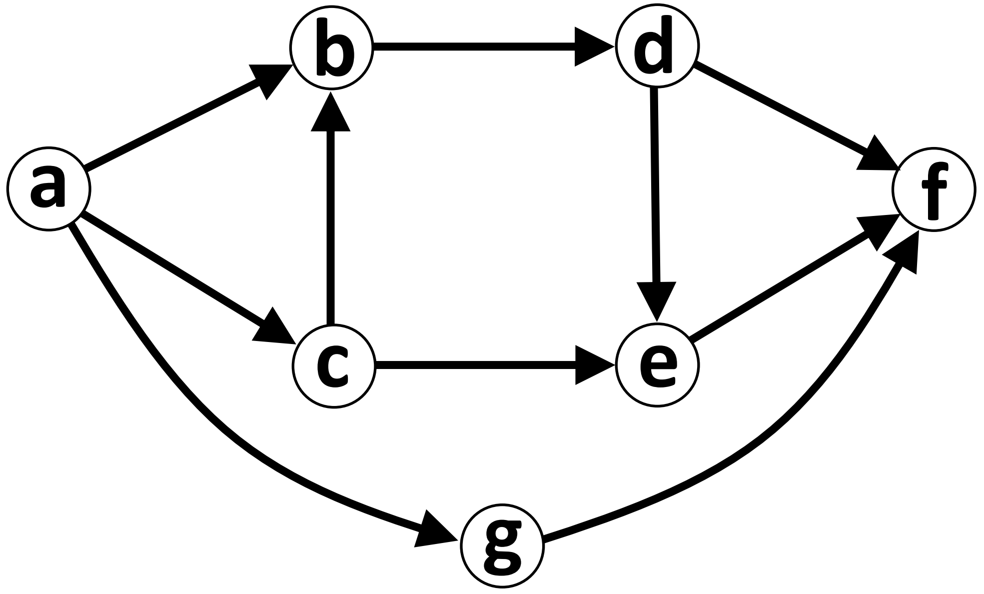


{a, c, g, b, d, e, f}

Find the Longest Path in a DAG

Plan:

- Topologically sort vertices.
- Make array to store length of longest path that ends at each vertex.



{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	0	0	0	0	0	0

Length of longest path that ends at c.

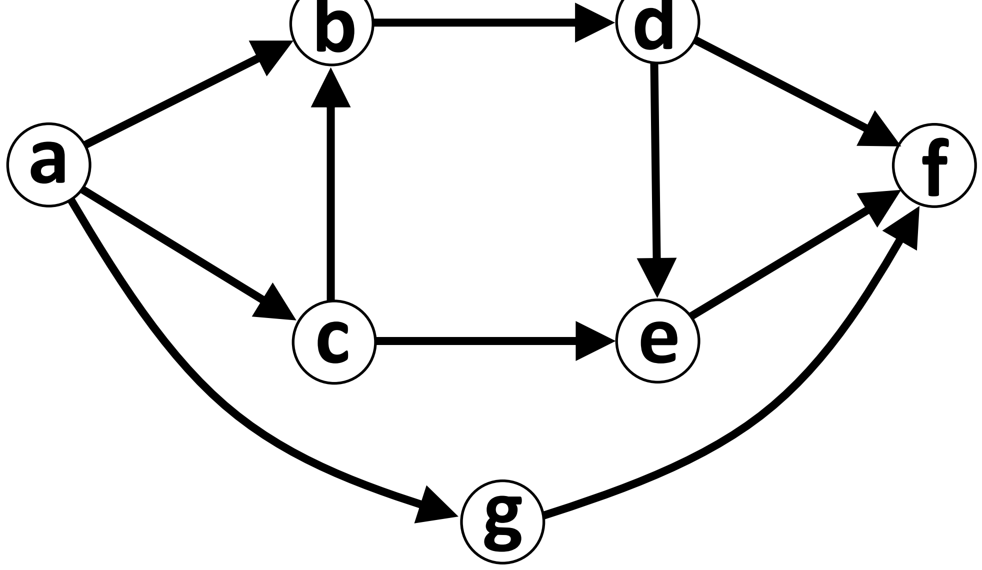
Find the Longest Path in a DAG

Plan:

- Topologically sort vertices.
- Make array to store length of longest path that ends at each vertex.
- For each vertex in order, calculate longest path as:

  $\max_n(\text{longest path to } n) + 1$,
  for all incoming neighbors n.

  (Or 0 if there are no incoming neighbors)



{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	0	0	0	0	0	0

Find the Longest Path in a DAG

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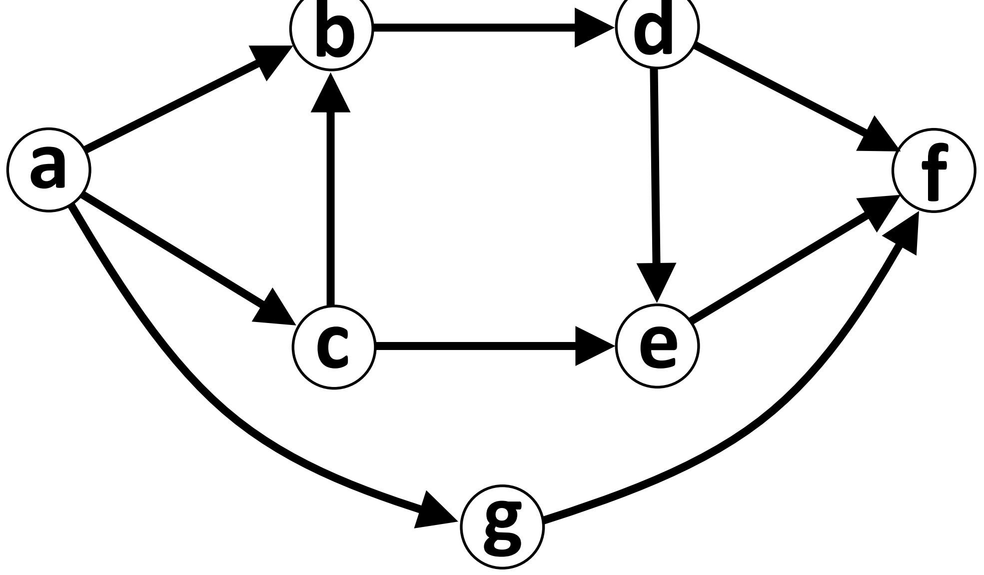

{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	0	0	0	0	0	0

Find the Longest Path in a DAG

Plan:

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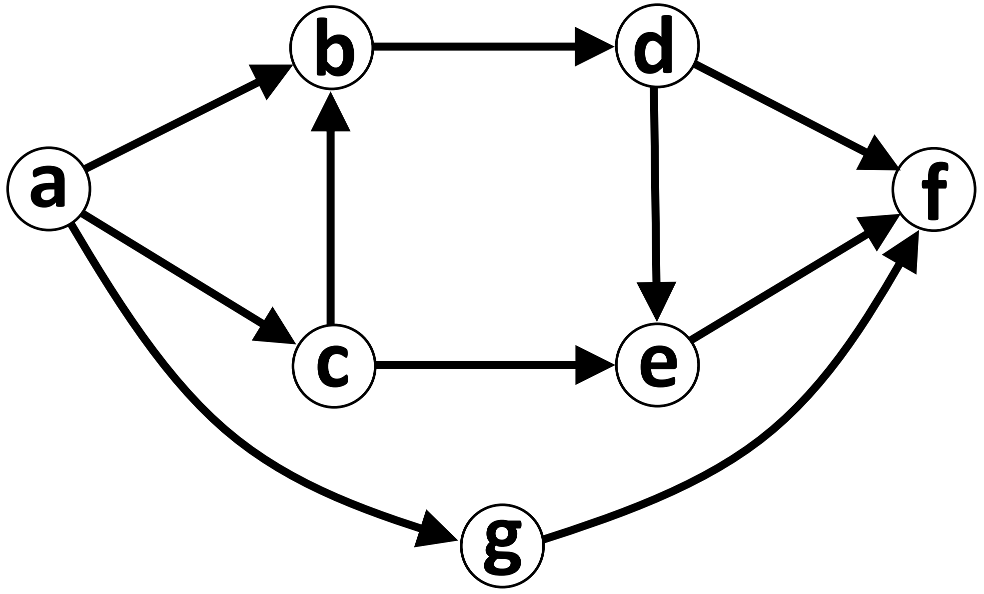

{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	0	0	0	0	0	0

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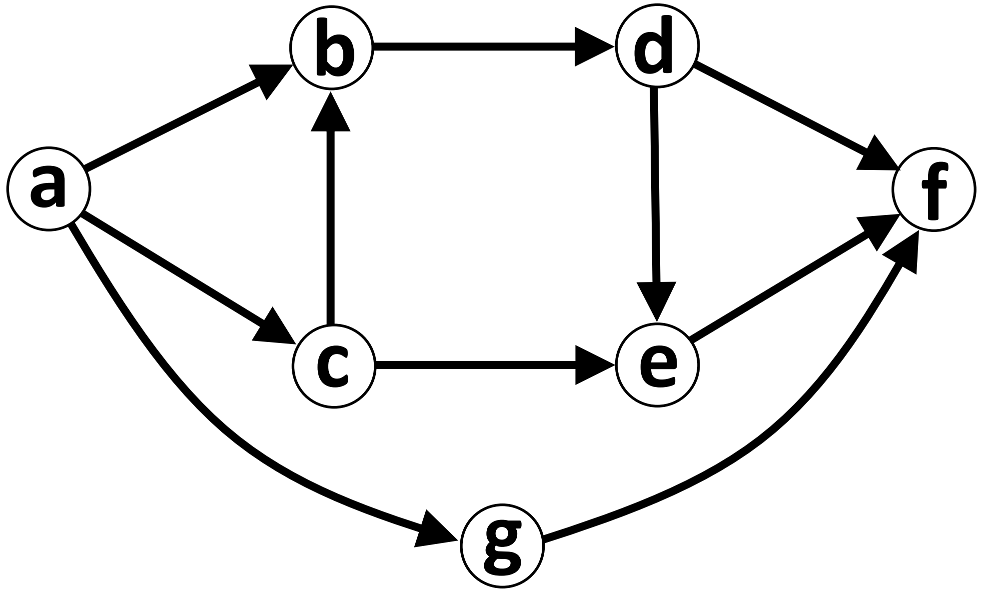

{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	0	1	0	0	0	0

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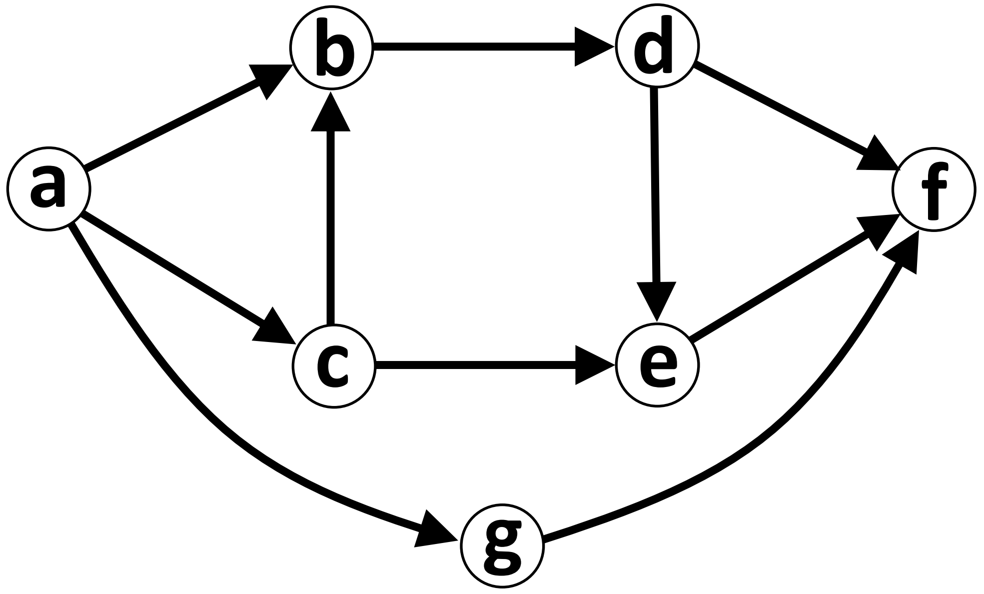

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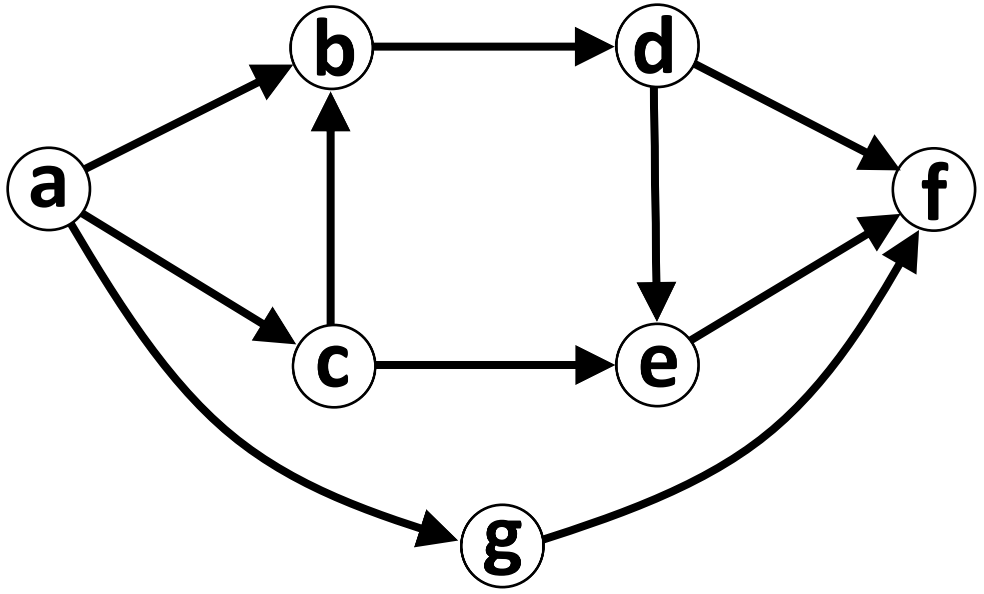

{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	0	1	0	0	0	1

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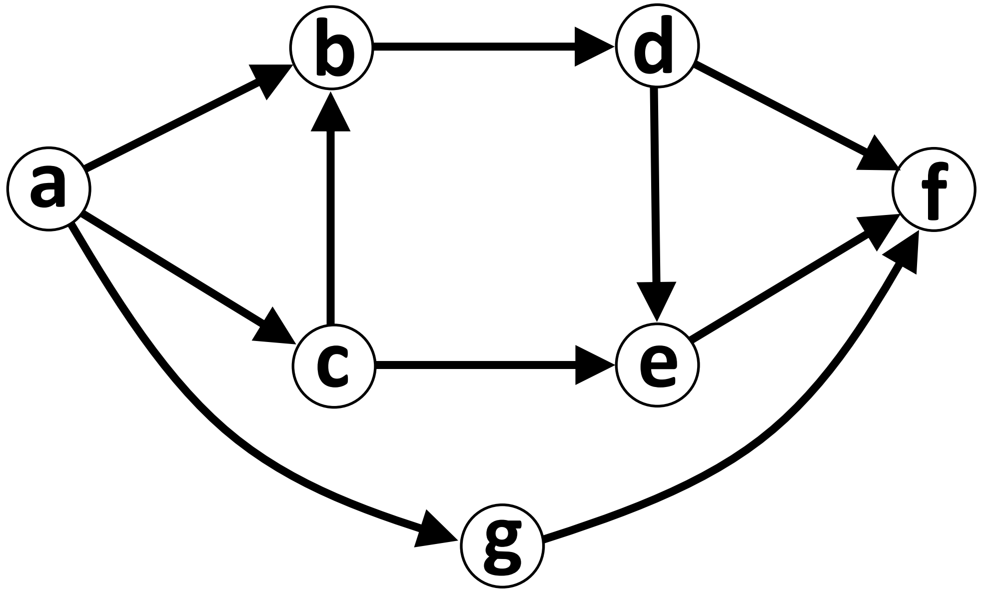

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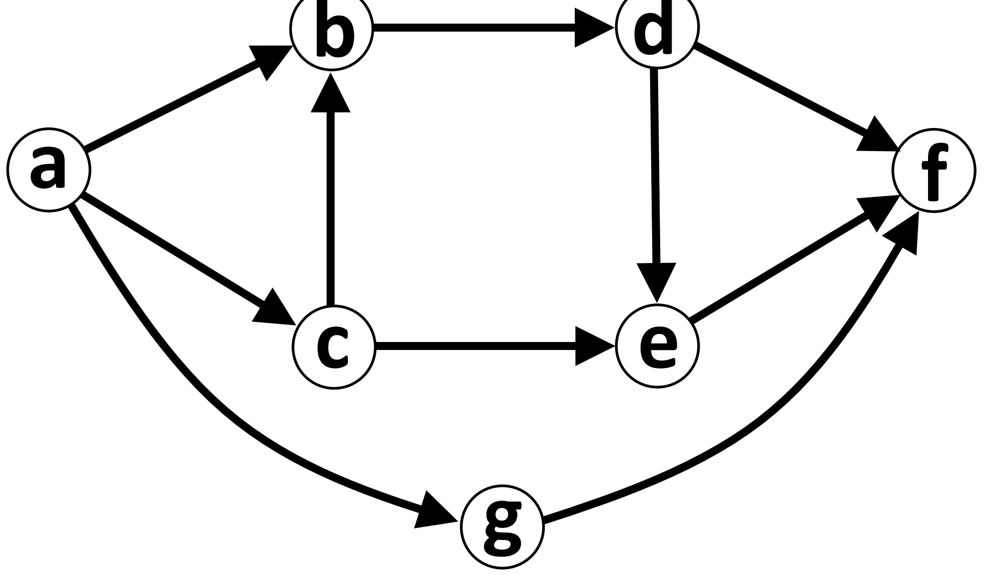

{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	2	1	0	0	0	1

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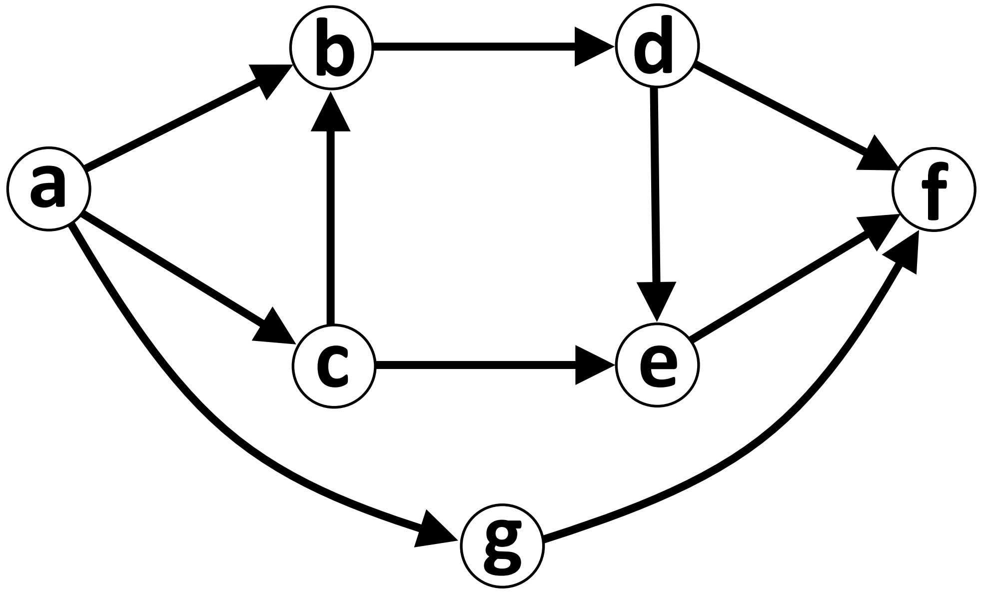

{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	2	1	3	4	5	1

Find the Longest Path in a DAG

Plan:

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- Make array to store length of longest path that ends at each vertex.
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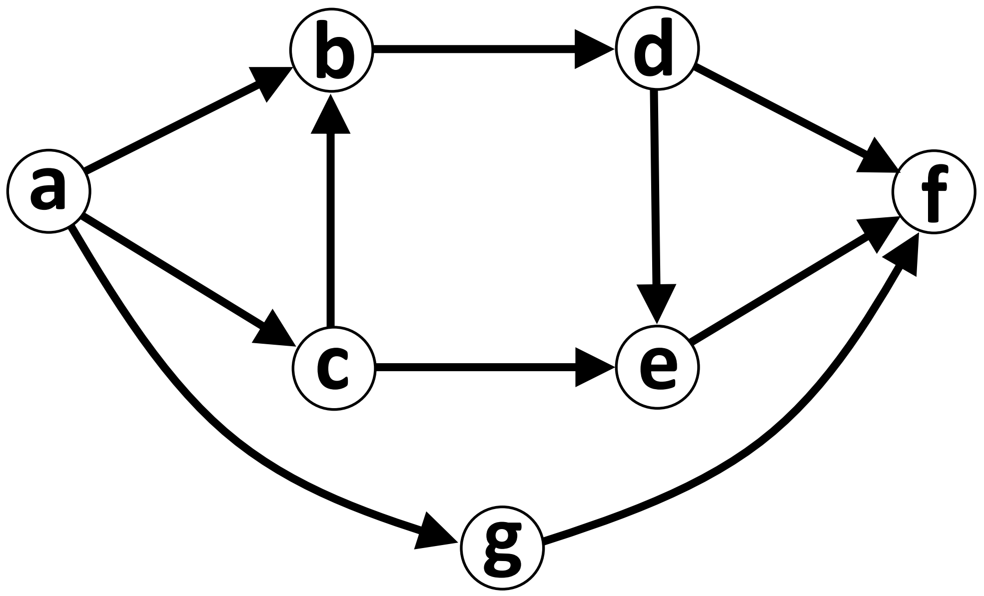
  for all incoming neighbors n.
- Largest value in array = Longest path.



{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	2	1	3	4	5	1

Find the Longest Path in a DAG

```
longest_path(G=(V,E)):
    pathLengths = [0,...,0]
    Let Vsort be topologically sort vertices
    for each vertex v in Vsort:
        for each incoming neighbor n of v:
            if pathLengths[n] + 1 > pathLengths[v]:
                pathLengths[v] = pathLengths[n] + 1
    return maxValue(pathLengths)
```

Running time: ?

Find the Longest Path in a DAG

```
longest_path(G=(V,E)):
    pathLengths = [0,...,0]
    Let V_sort be topologically sort vertices
    for each vertex v in V_sort:
        for each incoming neighbor n of v:
            if pathLengths[n] + 1 > pathLengths[v]:
                pathLengths[v] = pathLengths[n] + 1
    return maxValue(pathLengths)
```

Running time: $O(\text{Topological Sort} + |V|^2) \in O(|V|^2)$

Find the Longest Path in a DAG

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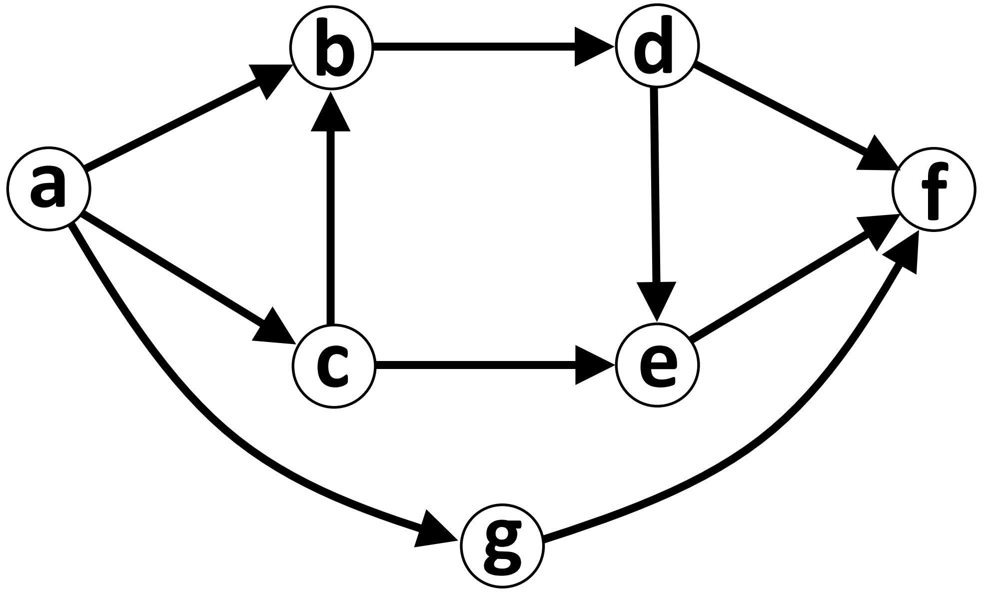


{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	2	1	3	4	5	1

Find the Longest Path in a DAG

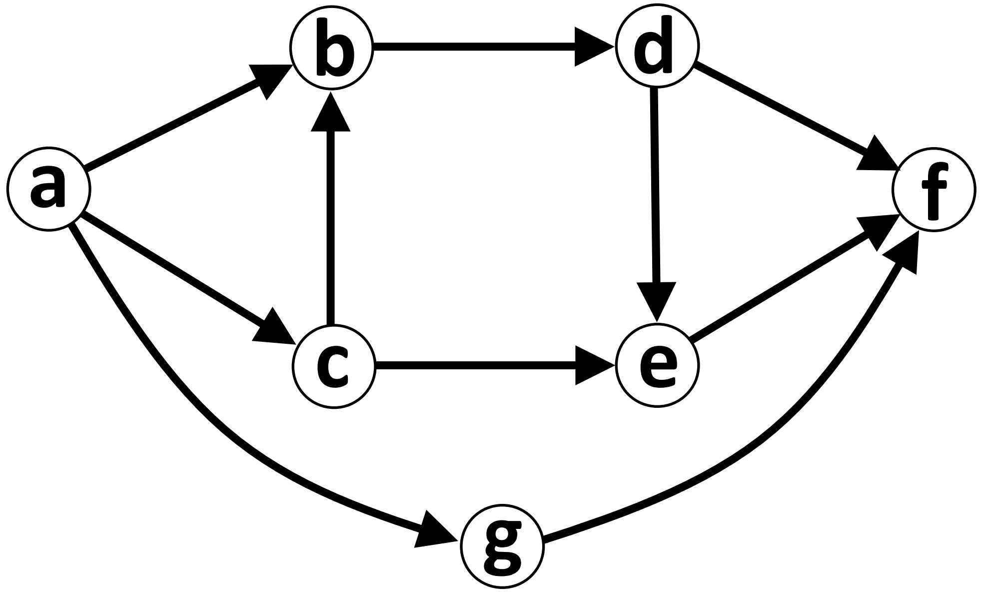
Plan:

- Topologically sort vertices.
- Make array to store le ongest path that ends at e tex.
- For each verte der, calculate longest pa

  ma gest path to n) $+ 1$,

  ncoming neighbors n.
- Larg st value in array = Longest path.

So, what's the path??



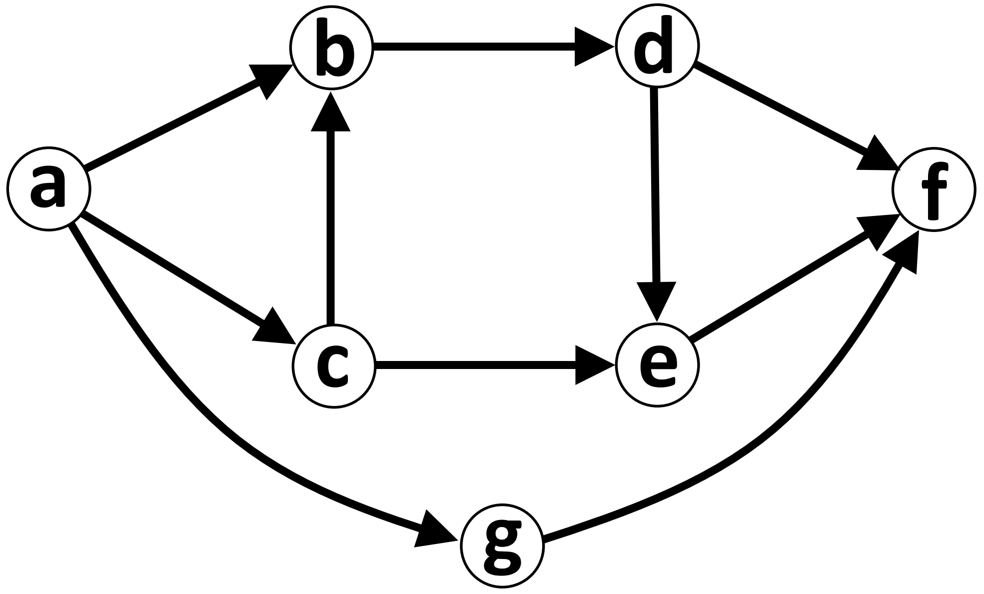
{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	2	1	3	4	5	1

Find the Longest Path in a DAG

Plan:

- Make second array that tracks where longest path came from.



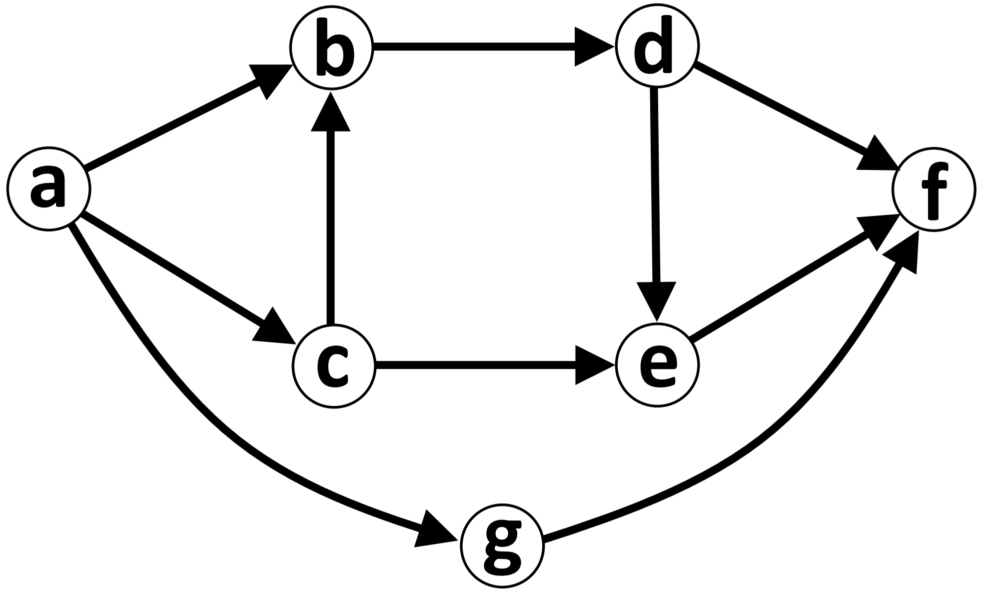
{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	2	1	3	4	5	1
-	-	-	-	-	-	-

Find the Longest Path in a DAG

Plan:

- Make second array that tracks where longest path came from.
- When neighbor with longest path is determined, save that neighbor.



{a, c, g, b, d, e, f}

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0	2	1	3	4	5	1
-	-	-	-	-	-	-

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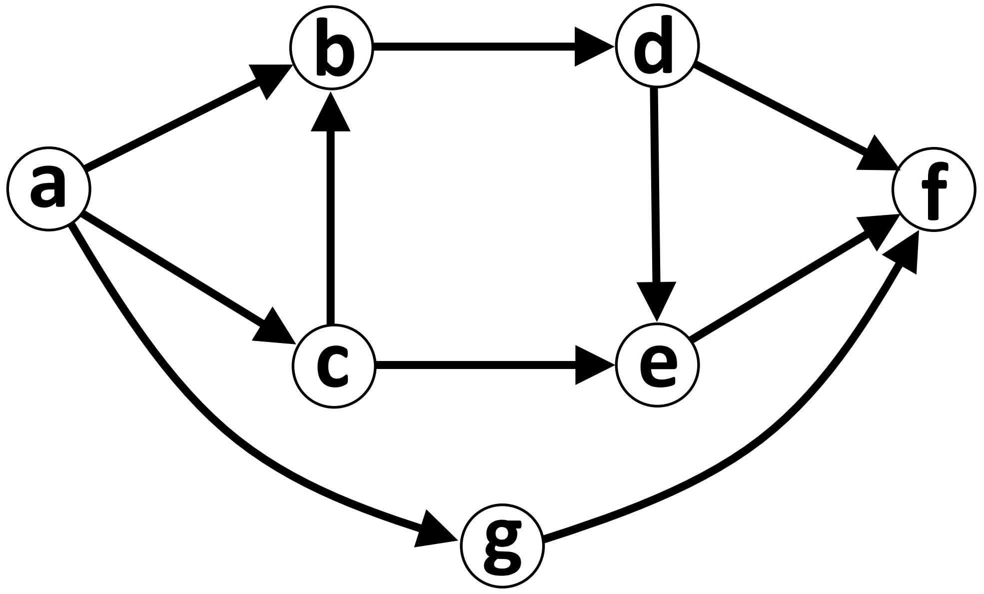

{a, c, g, b, d, e, f}

For each vertex in order, calculate longest path as:

$\max_n(\text{longest path to } n) + 1$,

for all incoming neighbors n.

a	b	c	d	e	f	g
0	0	0	0	0	0	0
-	-	-	-	-	-	-

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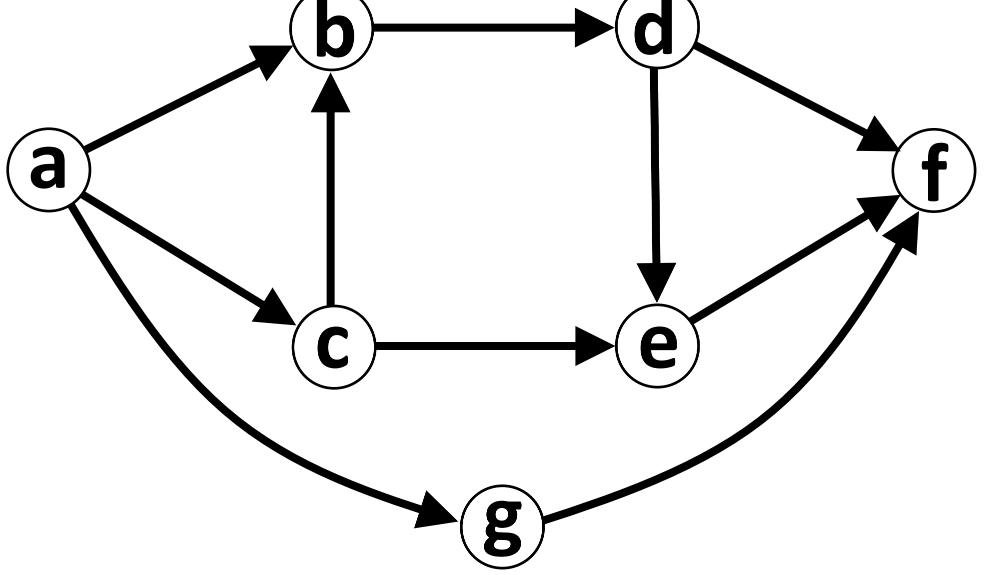

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a	b	c	d	e	f	g
0	0	0	0	0	0	0
-	-	-	-	-	-	-

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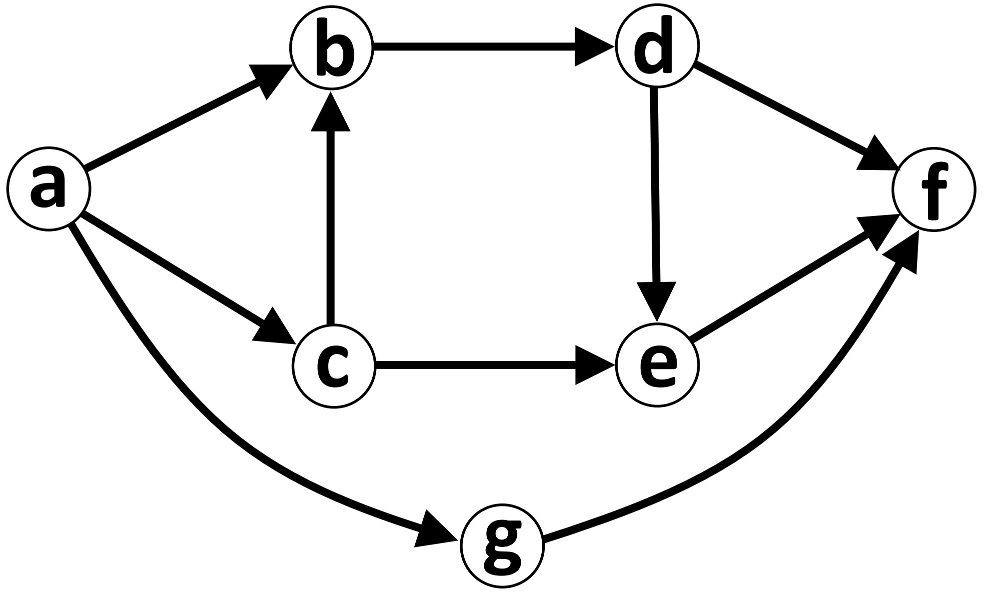

{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	0	0	0	0	0	0
-	-	-	-	-	-	-

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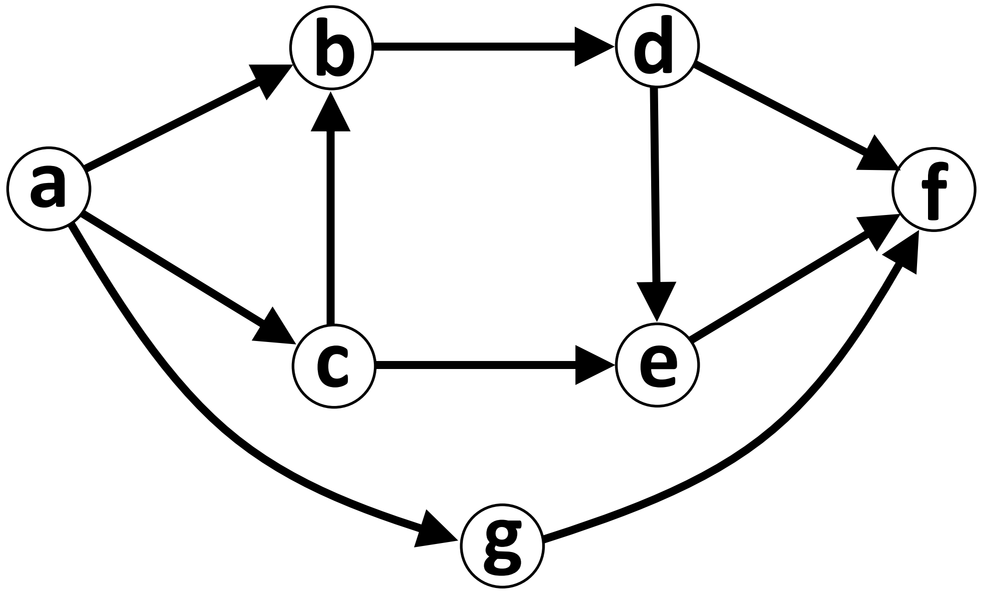

{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	0	1	0	0	0	0
-	-	-	-	-	-	-

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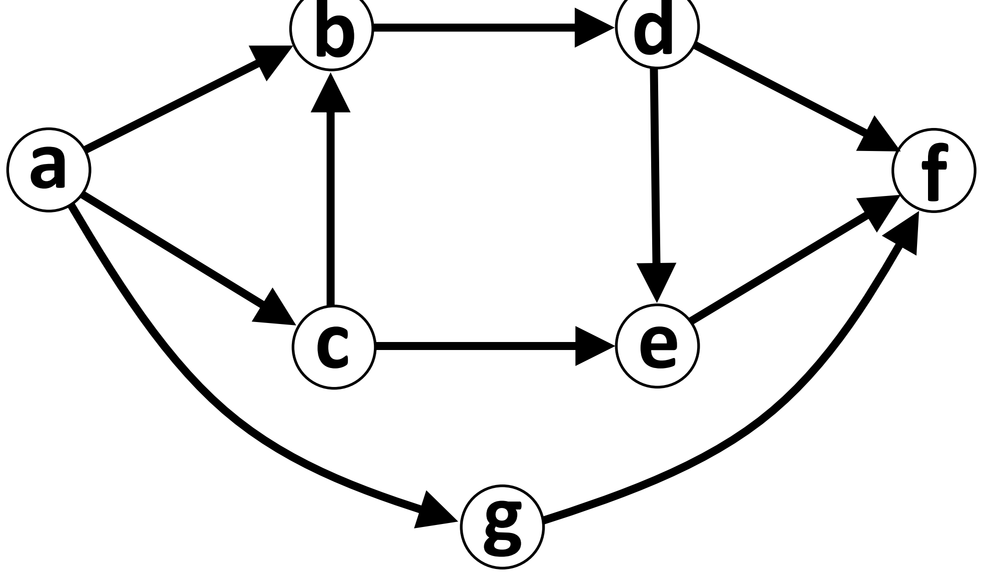

{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	0	1	0	0	0	0
-	-	a	-	-	-	-

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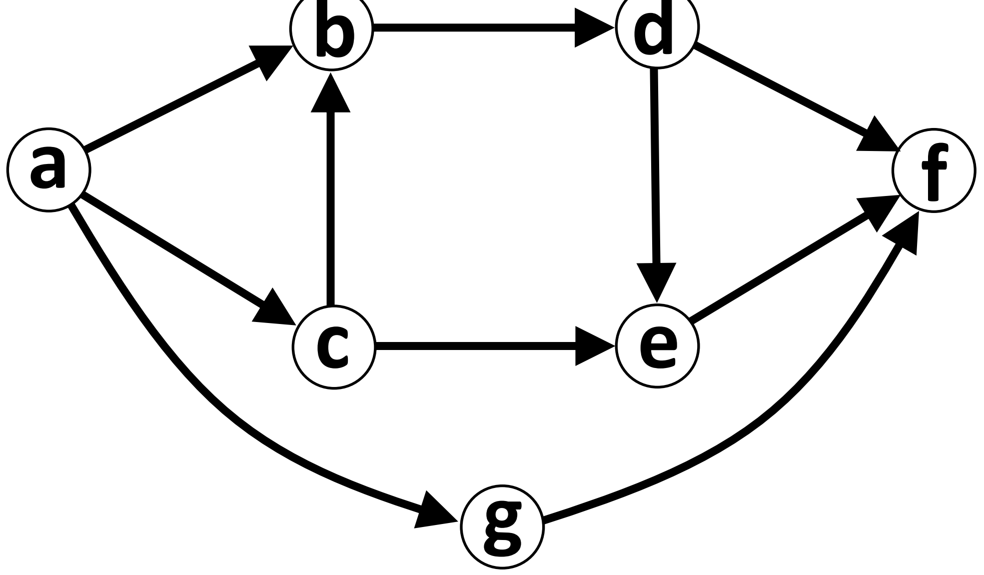

{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	0	1	0	0	0	1
-	-	a	-	-	-	a

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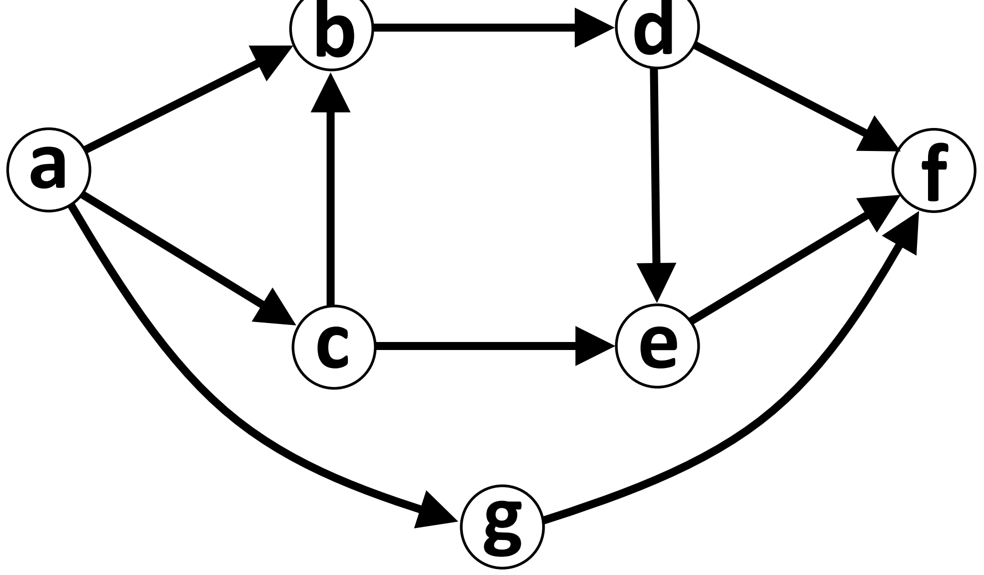

{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	2	1	0	0	0	1
-	c	a	-	-	-	a

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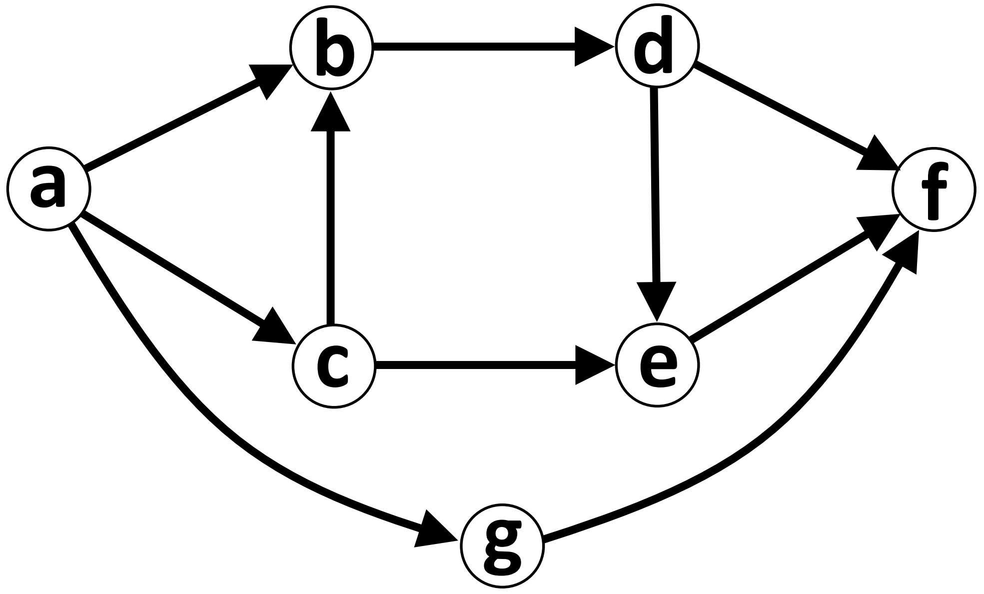


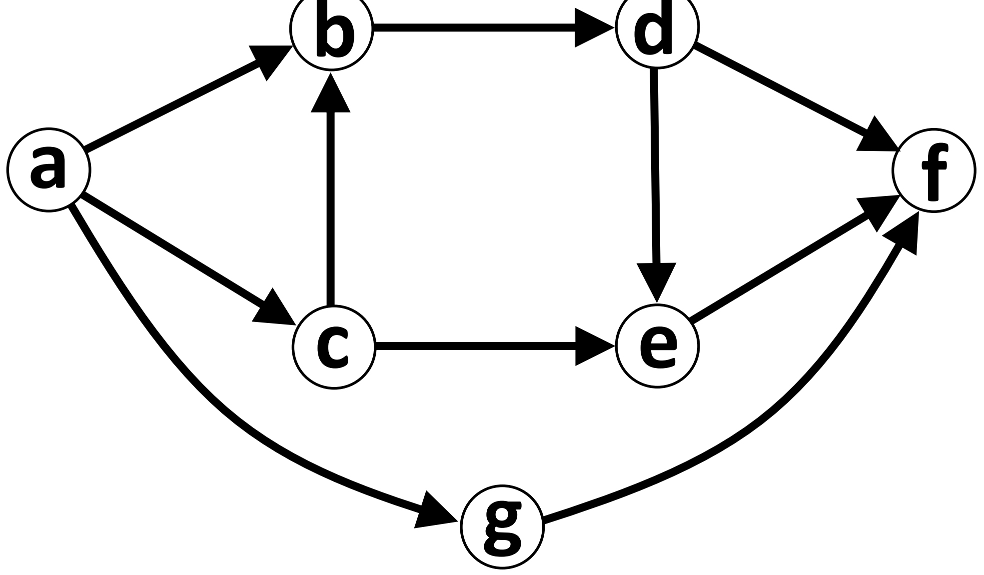
{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	2	1	3	4	5	1
-	c	a	b	d	e	a

Find the Longest Path in a DAG

Plan:

- Make second array that tracks where longest path came from.
- When neighbor with longest path is determined, save that neighbor.
- Backtrack through array to construct path.



{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	2	1	3	4	5	1
-	c	a	b	d	e	a

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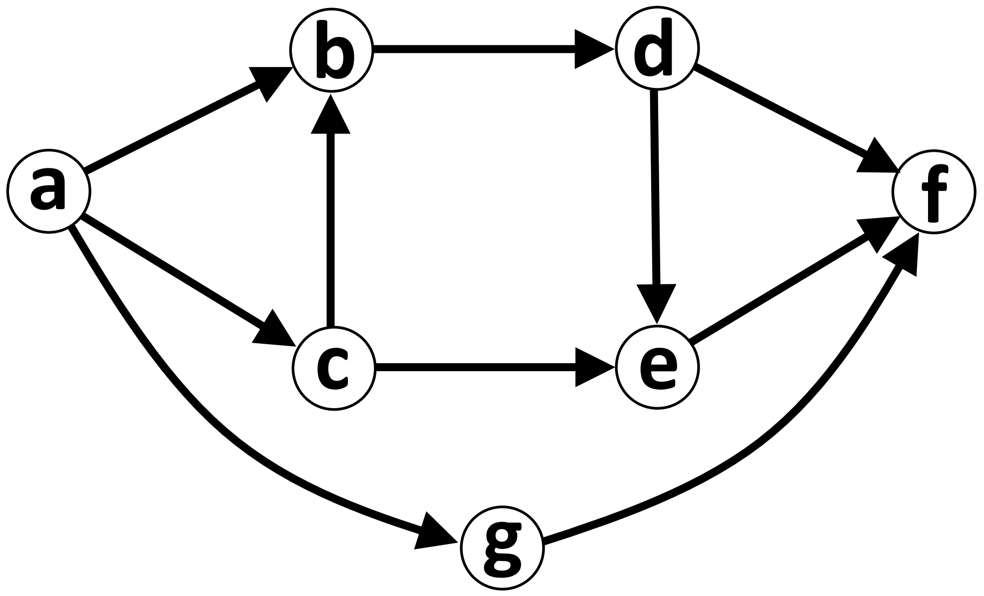

{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	2	1	3	4	5	1
-	c	a	b	d	e	a

path: f

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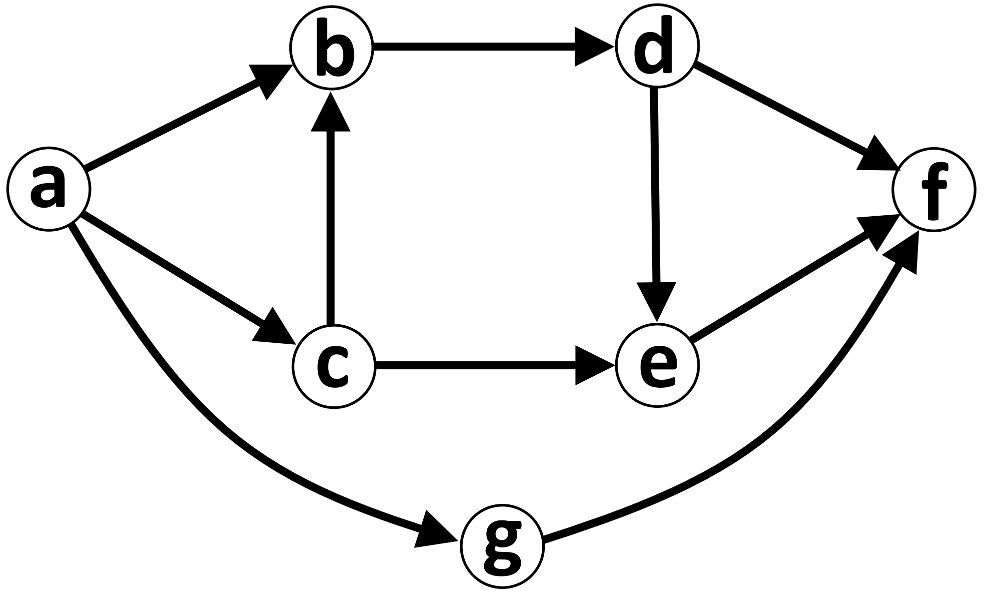


{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	2	1	3	4	5	1
-	c	a	b	d	e	a

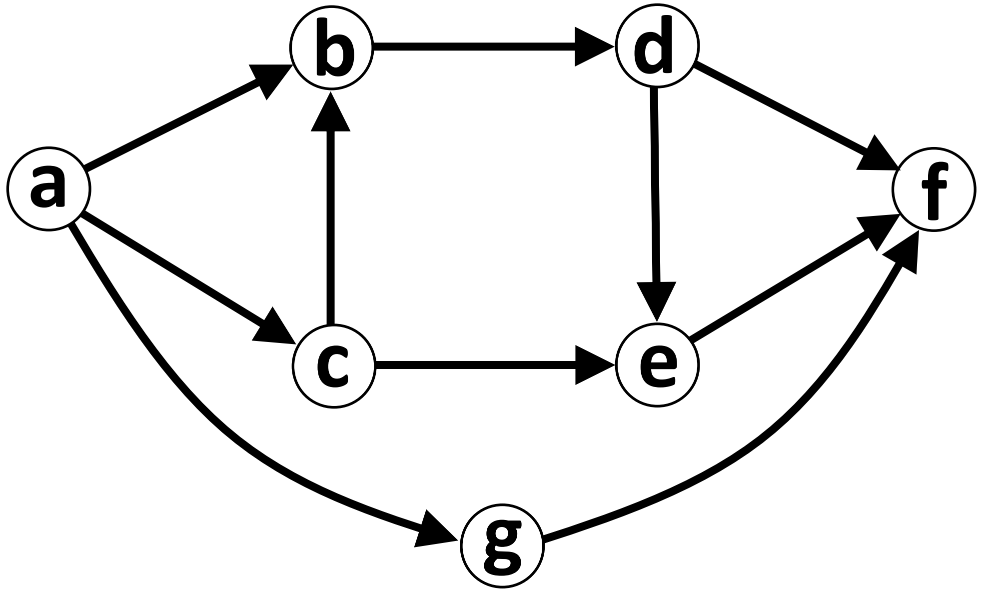
path: f <- e

Find the Longest Path in a DAG

Plan:

- Make second array that tracks where longest path came from.
- When neighbor with longest path is determined, save that neighbor.
- Backtrack through array to construct path.

path: f <- e <- d



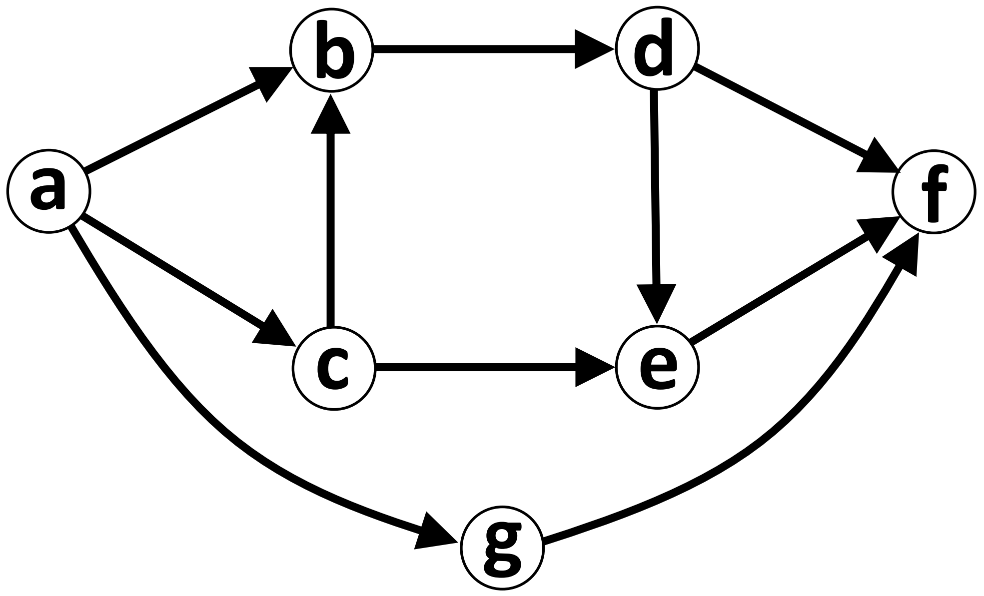
{a, c, g, b, d, e, f}

a	b	c	d	e	f	g
0	2	1	3	4	5	1
-	c	a	b	d	e	a

Find the Longest Path in a DAG

Plan:

- Make second array that tracks where longest path came from.
- When neighbor with longest path is determined, save that neighbor.
- Backtrack through array to construct path.



{a, c, g, b, d, e, f}

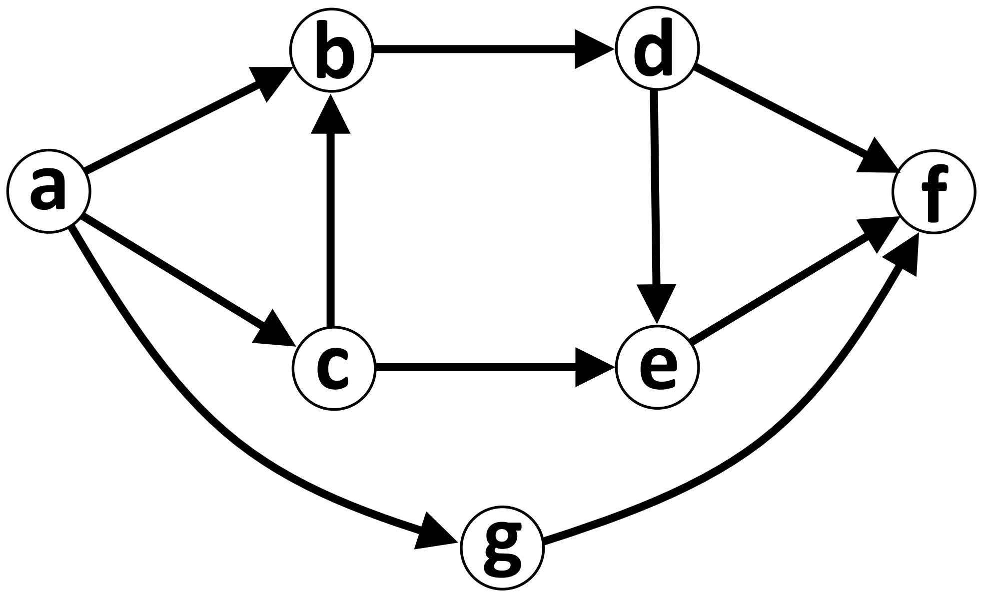
a	b	c	d	e	f	g
0	2	1	3	4	5	1
-	c	a	b	d	e	a

path: f <- e <- d <- b

Find the Longest Path in a DAG

Plan:

- Make second array that tracks where longest path came from.
- When neighbor with longest path is determined, save that neighbor.
- Backtrack through array to construct path.



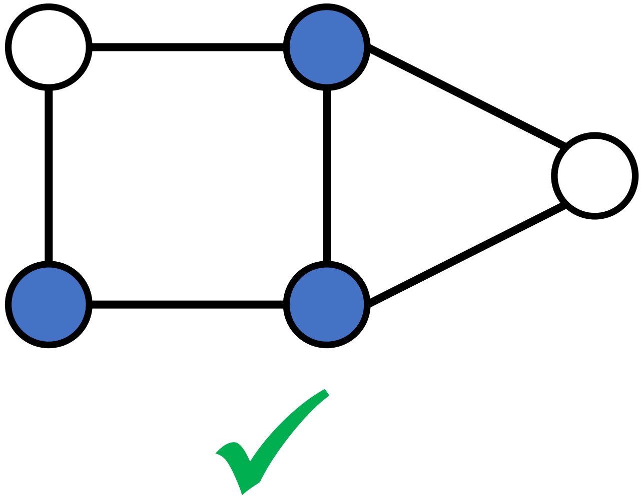
{a, c, g, b, d, e, f}

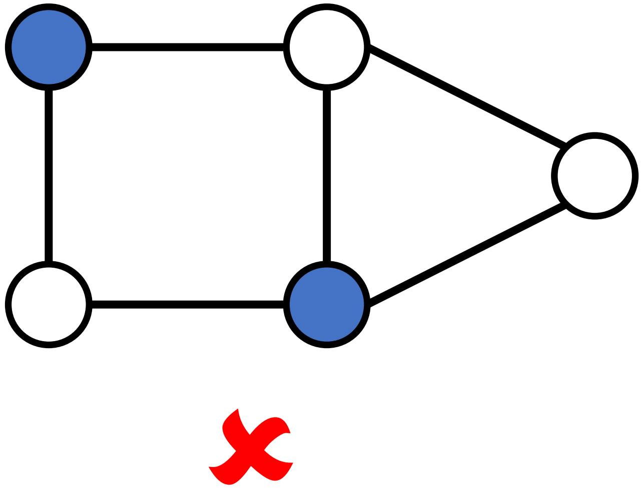
a	b	c	d	e	f	g
0	2	1	3	4	5	1
-	c	a	b	d	e	a

path: f <- e <- d <- b <- c <- a

Vertex Cover

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.



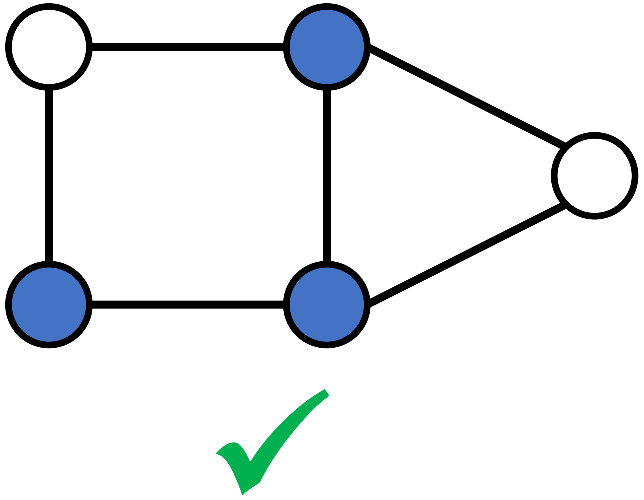


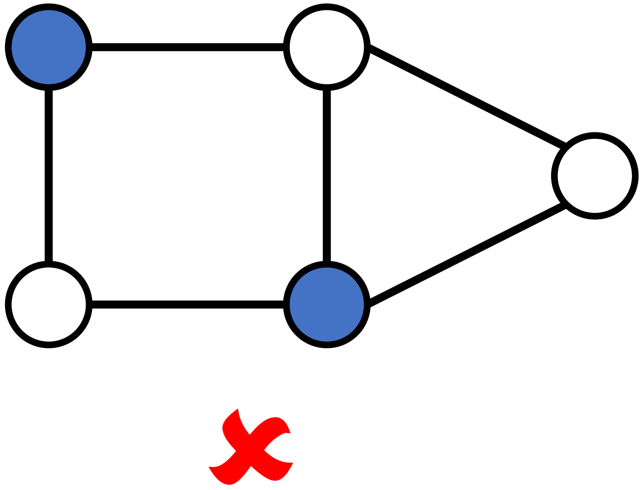
Vertex Cover

tree

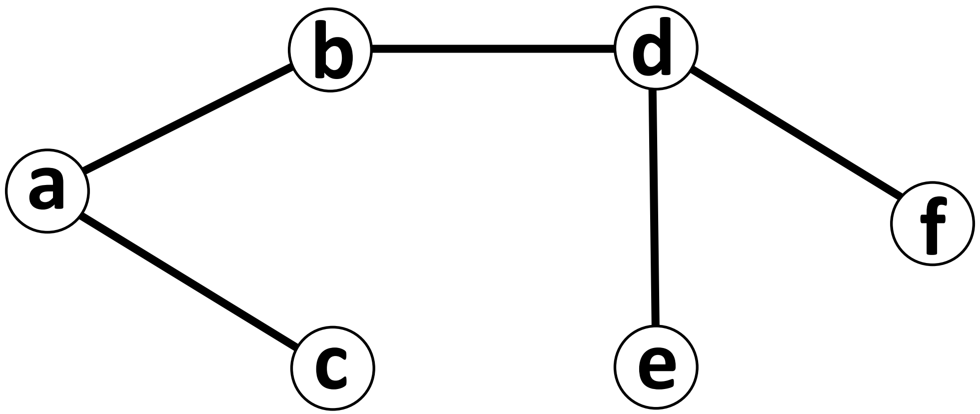
Vertex Cover: Given ~~graph~~, find the smallest subset of vertices such that every edge in the ~~graph~~ has at least one vertex in the subset.

tree





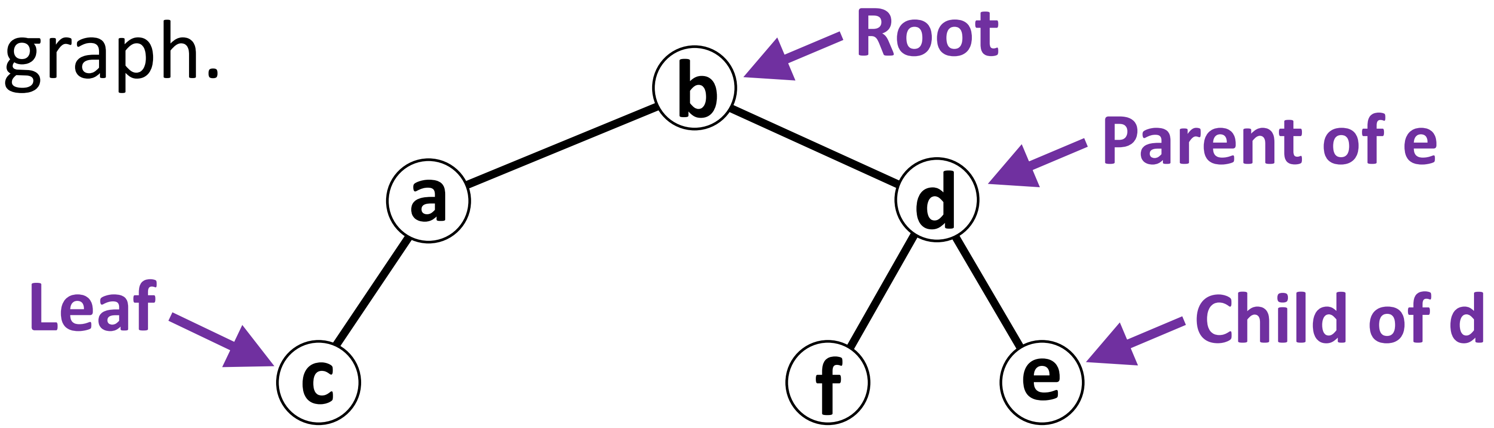
Special Graphs



Vertices (or Nodes)
Edges

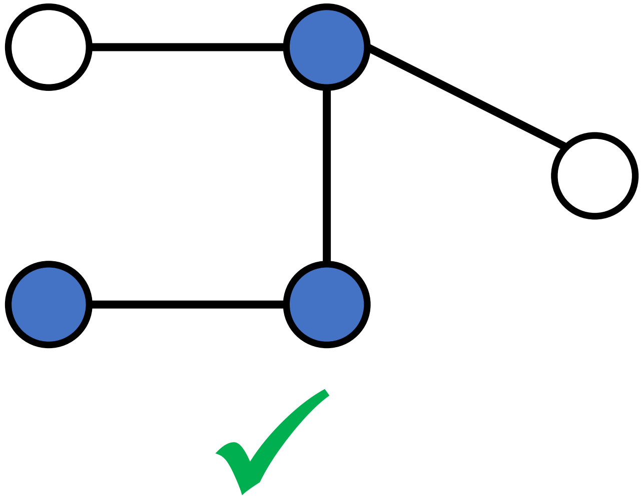
$$\boldsymbol{G = (V, E)}$$

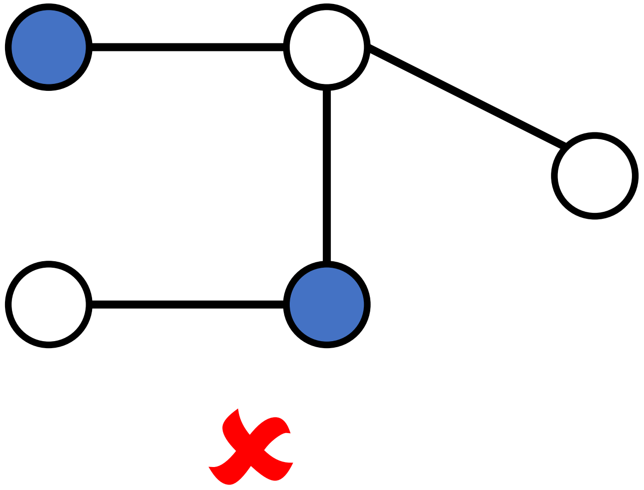
- Connected Graph = Graph that has a path between every vertex pair.
- Acyclic Graph = Graph with no cycles.
- Directed Acyclic Graph (DAG) = Directed graph with no cycles.
- Tree = Connected acyclic graph.



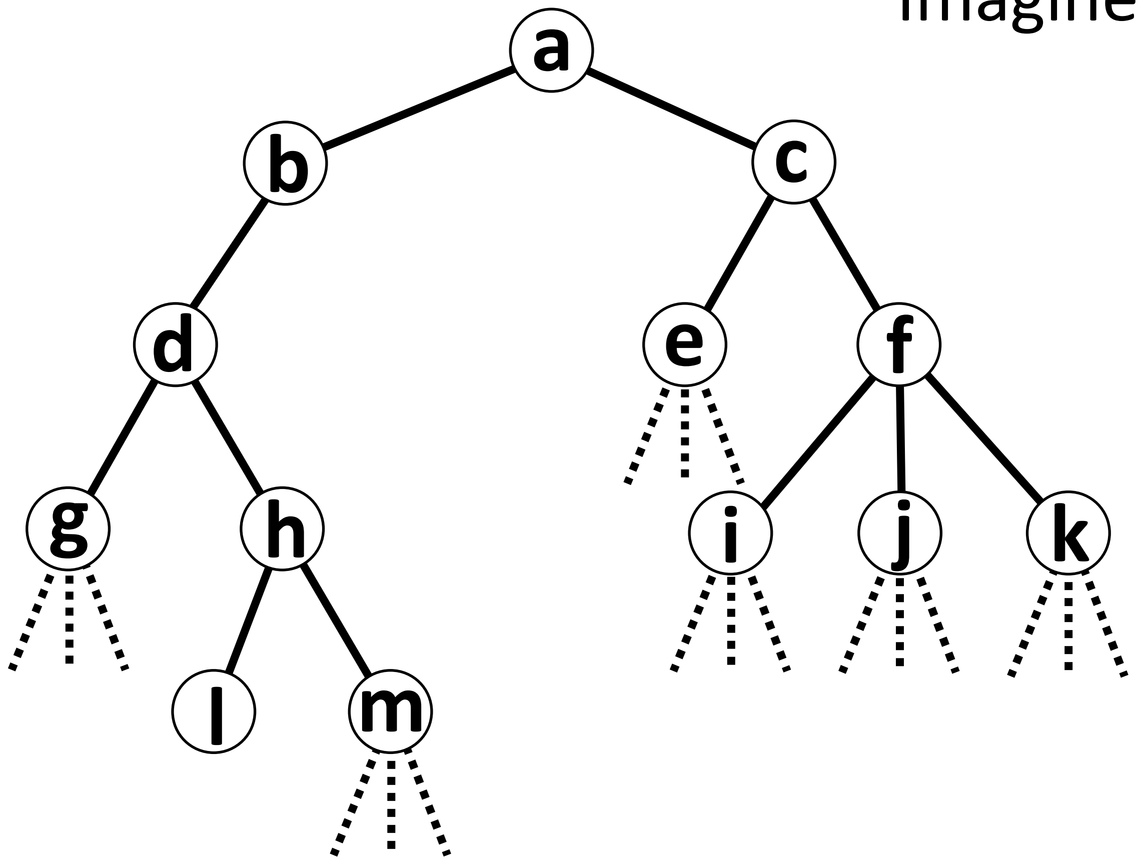
Vertex Cover

Vertex Cover: Given ~~graph~~ **tree**, find the smallest subset of vertices such that every edge in the ~~graph~~ **tree** has at least one vertex in the subset.



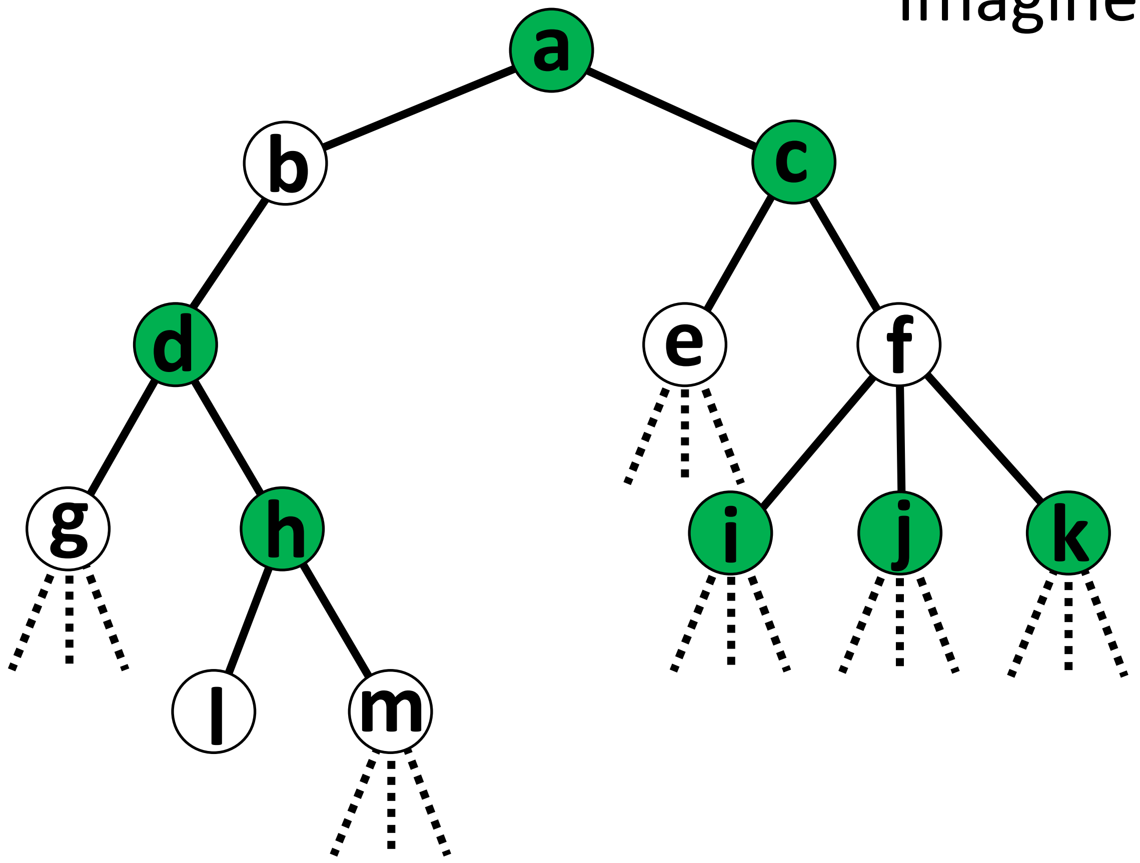


Vertex Cover in Trees



Imagine the minimum vertex cover.

Vertex Cover in Trees



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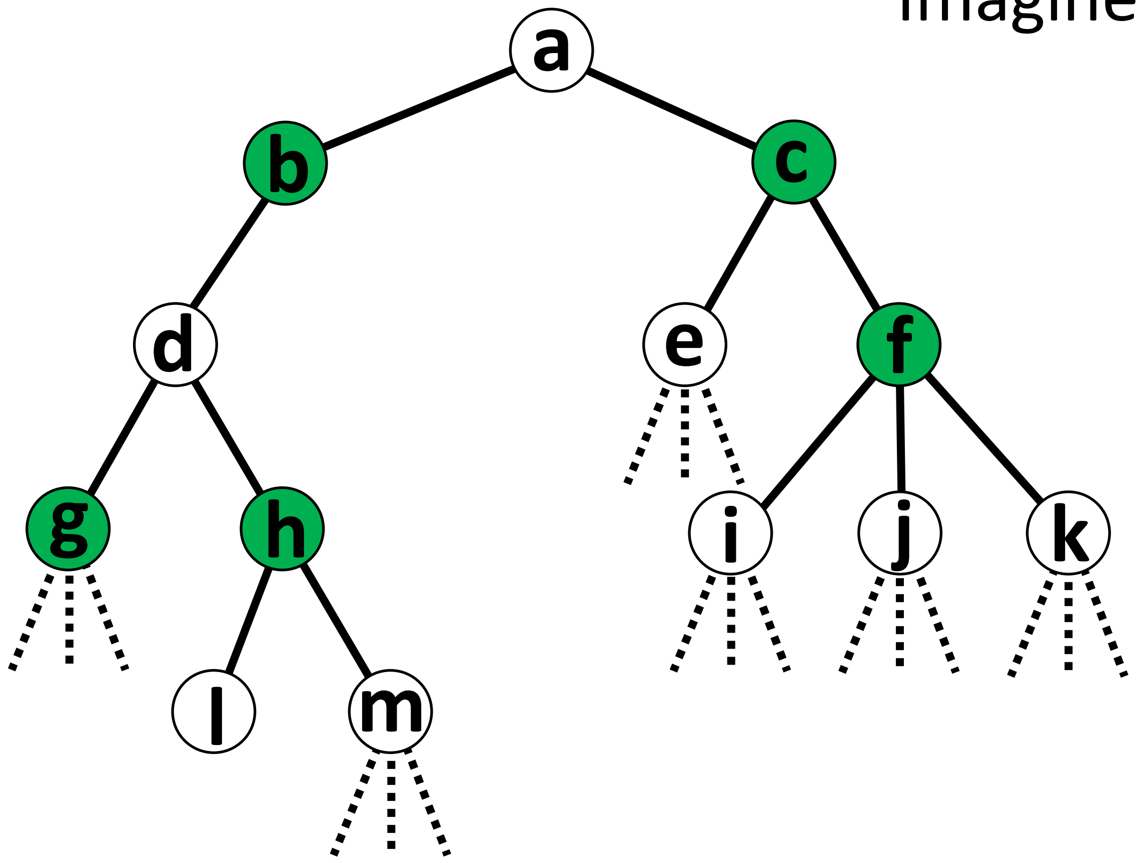
Vertex Cover in Trees



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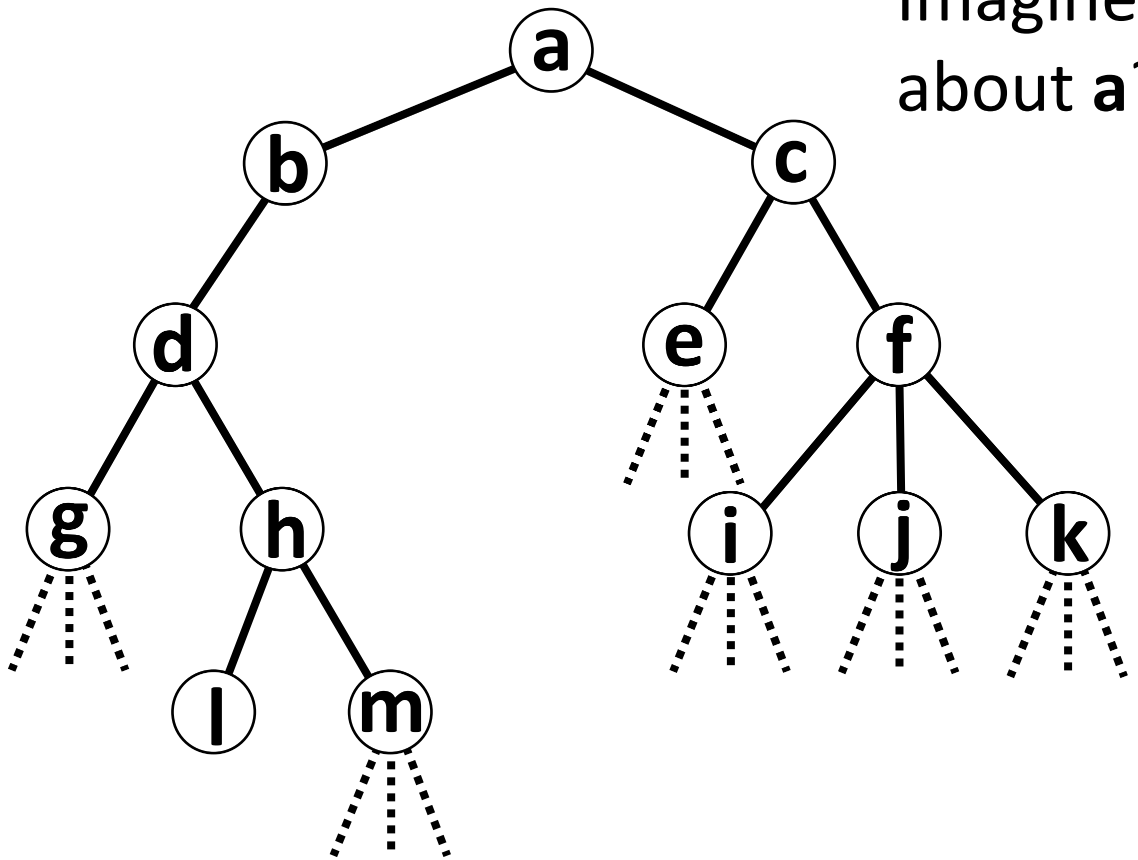
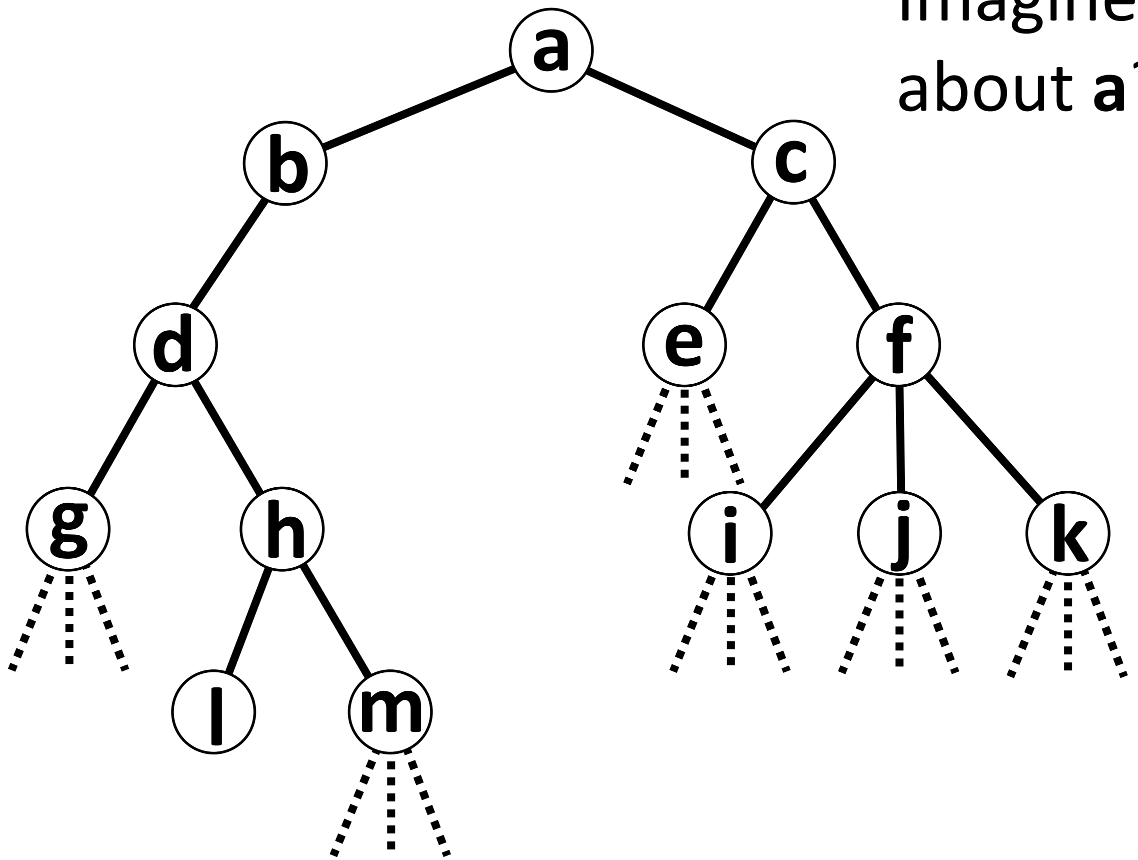
Vertex Cover in Trees



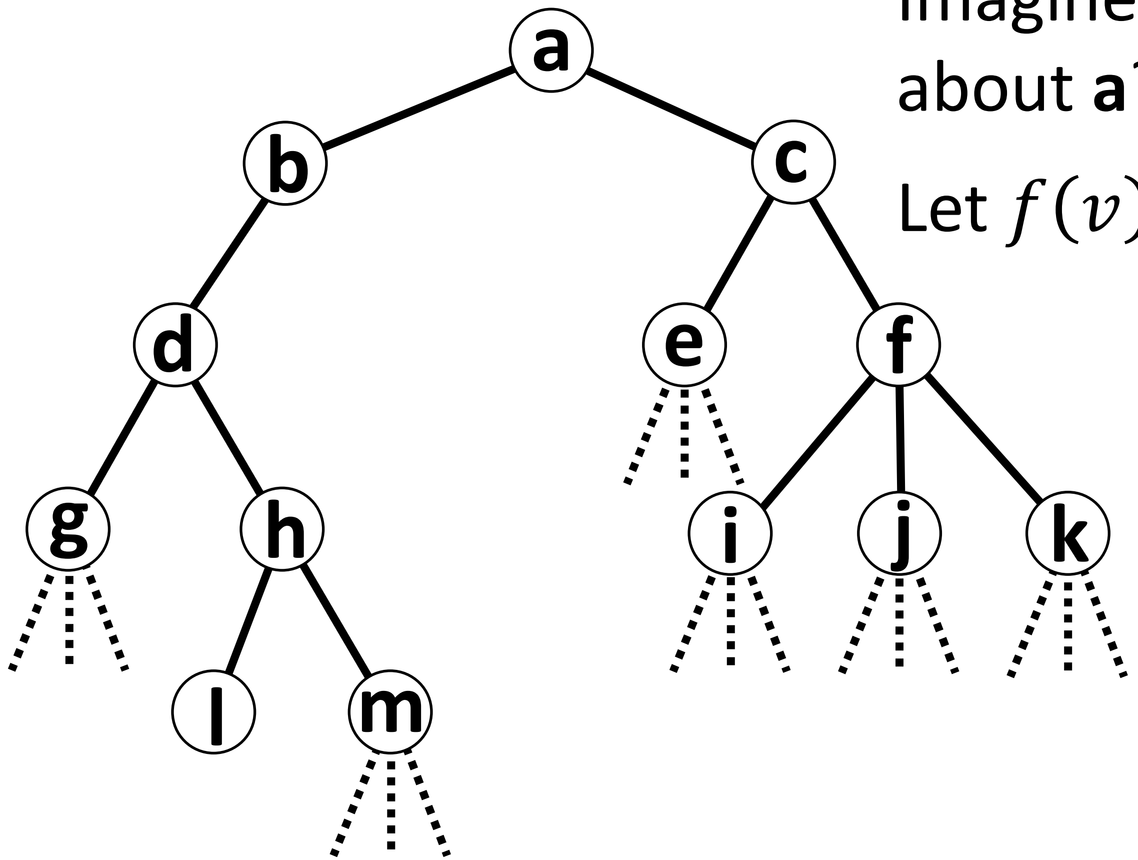
Imagine the minimum vertex cover. What can we say about **a**?

Vertex Cover in Trees



Imagine the minimum vertex cover. What can we say about **a**? **a** is in a minimum vertex cover, or it's not.

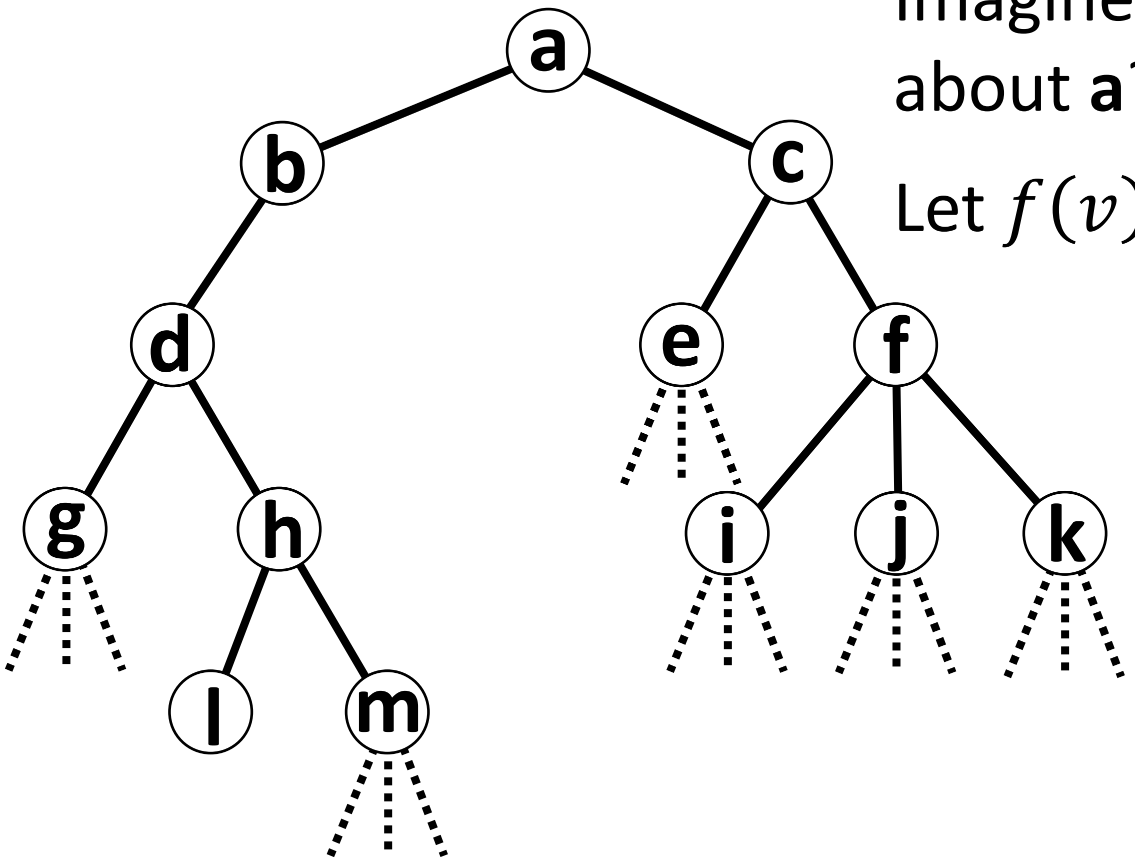
Vertex Cover in Trees



Imagine the minimum vertex cover. What can we say about **a**? **a** is in a minimum vertex cover, or it's not.

Let $f(v) =$ Size of minimum vertex cover rooted at v.

Vertex Cover in Trees



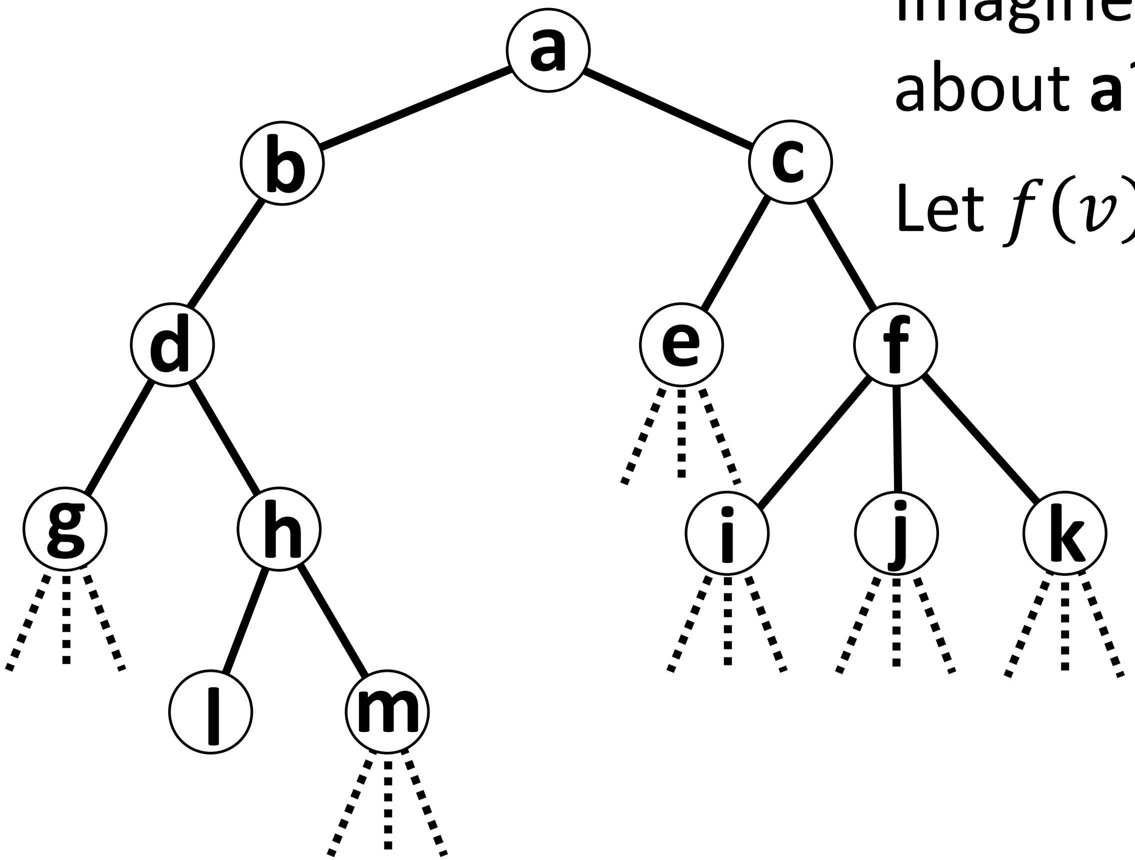
Imagine the minimum vertex cover. What can we say about **a**? **a** is in a minimum vertex cover, or it's not.

Let $f(v) =$ Size of minimum vertex cover rooted at v.

If a <u>**is**</u> in a minimum VC

If a <u>**is not**</u> in a minimum VC

Vertex Cover in Trees



Imagine the minimum vertex cover. What can we say about **a**? **a** is in a minimum vertex cover, or it's not.

Let $f(v)$ = Size of minimum vertex cover rooted at v.

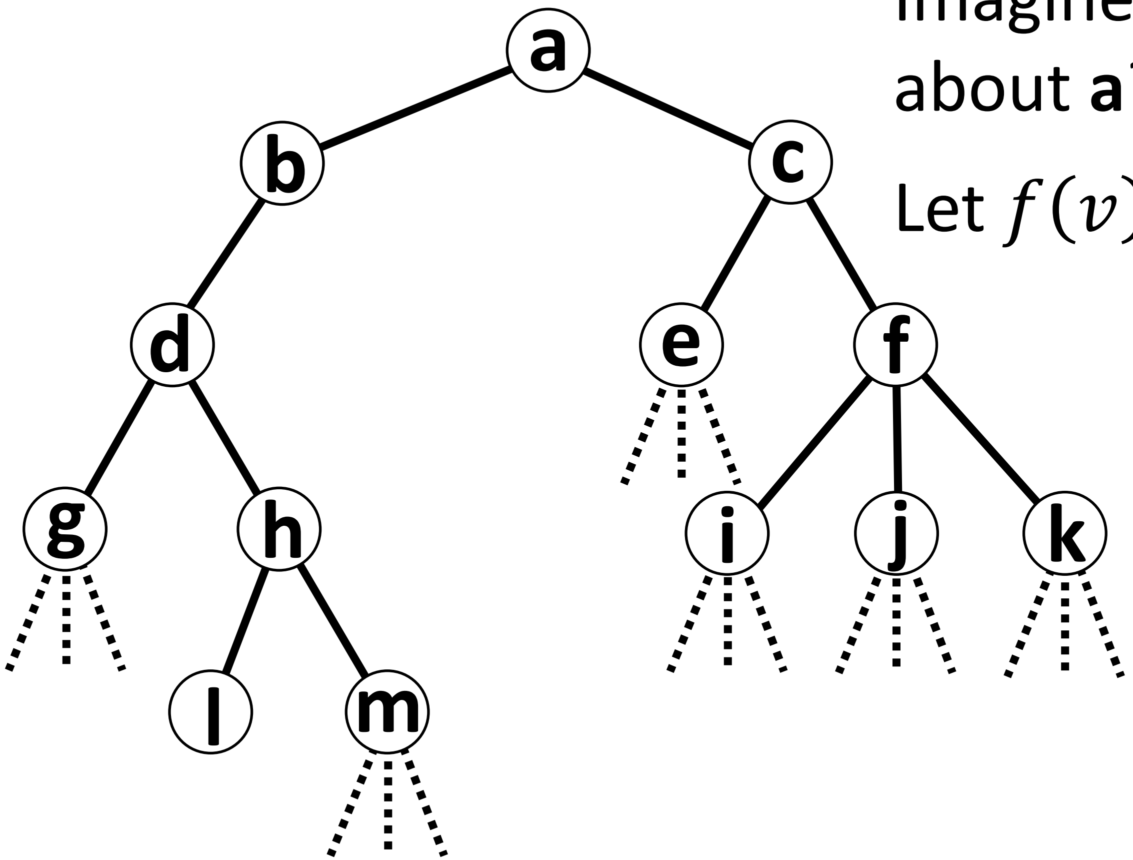
If a **<u>is</u>** in a minimum VC

$f(a) = ??$

If a **<u>is not</u>** in a minimum VC

$f(a) = ??$

Vertex Cover in Trees



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If a <u>**is**</u> in a minimum VC

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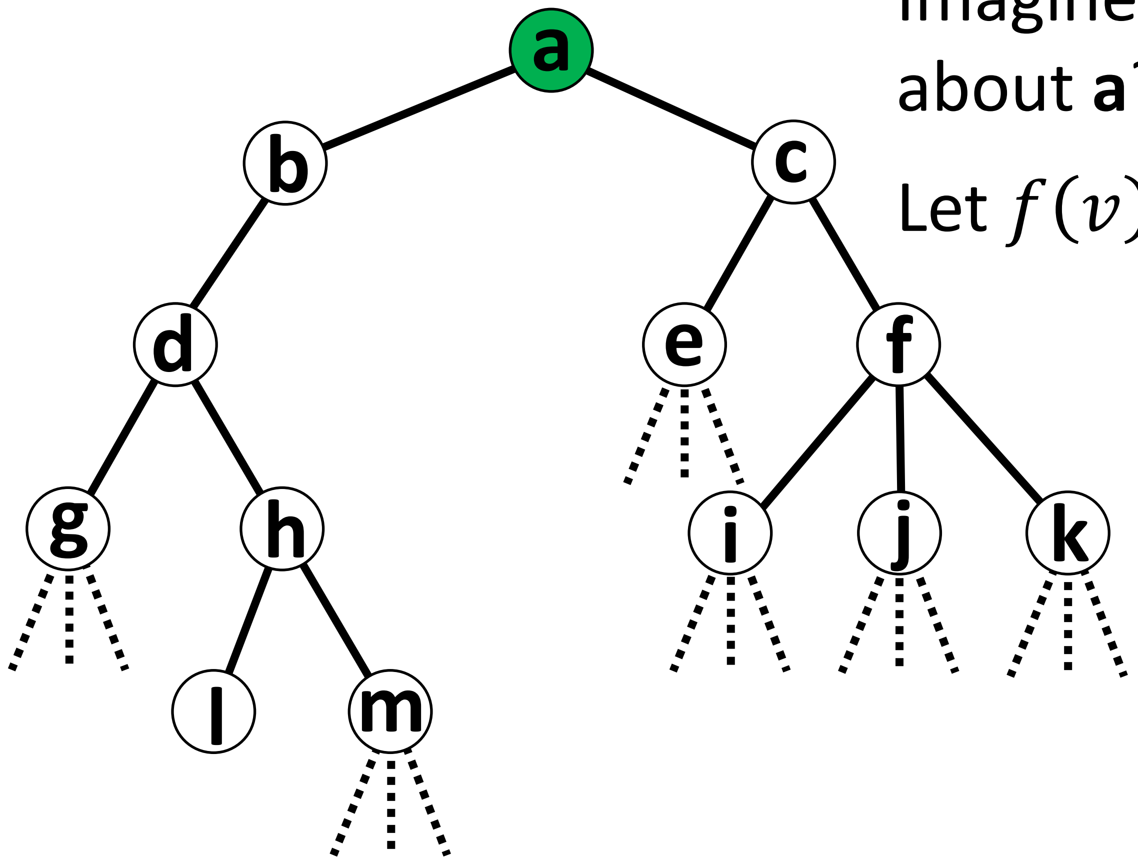
Vertex Cover in Trees



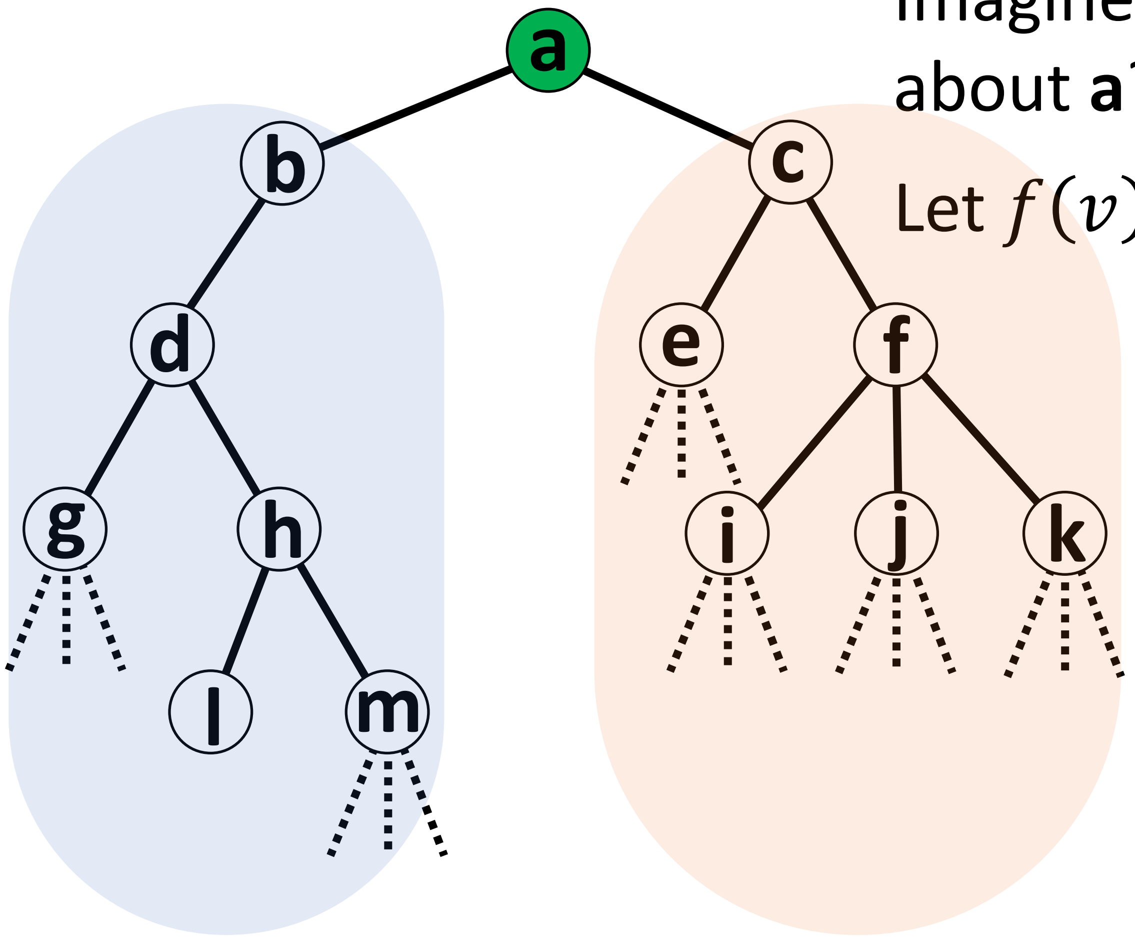
Imagine the minimum vertex cover. What can we say about **a**? **a** is in a minimum vertex cover, or it's not.

Let $f(v)$ = Size of minimum vertex cover rooted at v.

If a <u>**is**</u> in a minimum VC

$f(a) = 1 + ??$

Vertex Cover in Trees



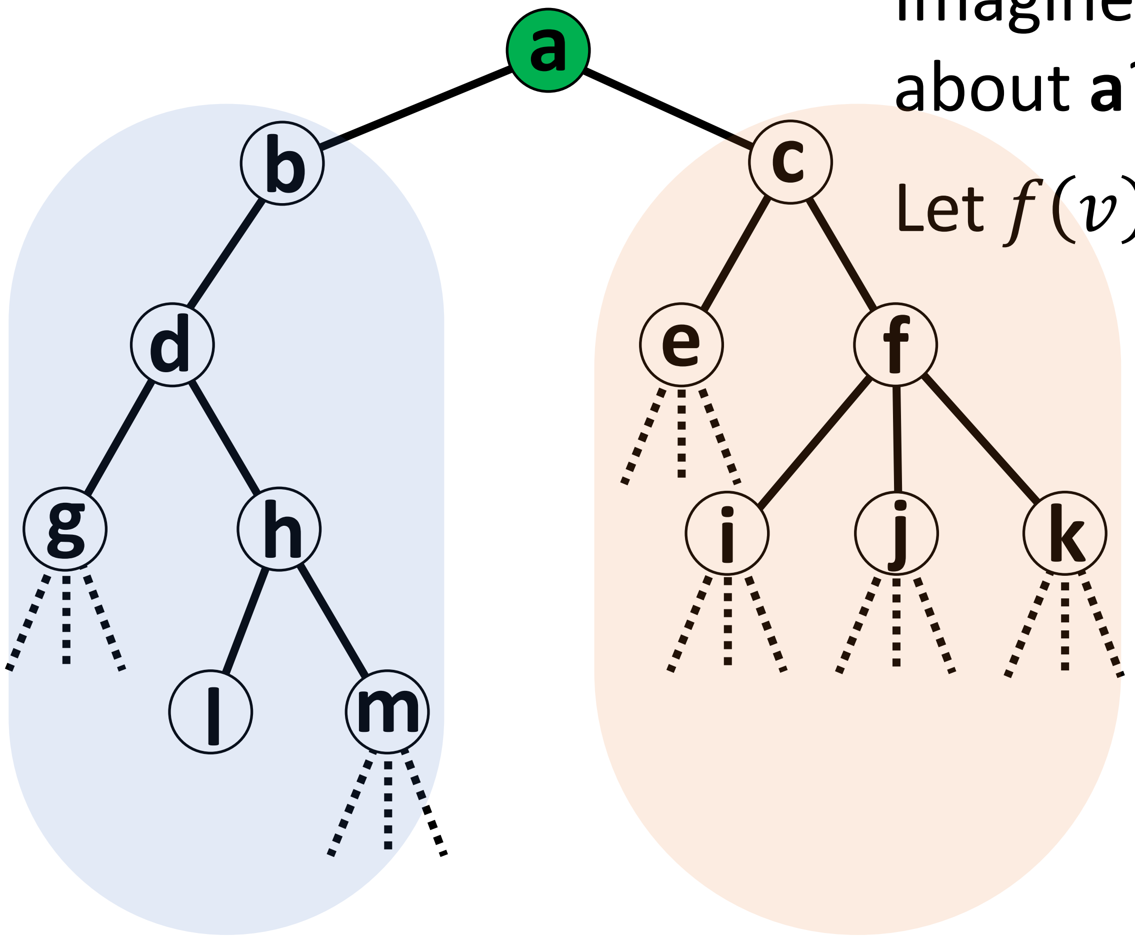
Imagine the minimum vertex cover. What can we say about **a**? **a** is in a minimum vertex cover, or it's not.

Let $f(v) =$ Size of minimum vertex cover rooted at v.

If a **is** in a minimum VC

$f(a) = 1 + ??$

Vertex Cover in Trees



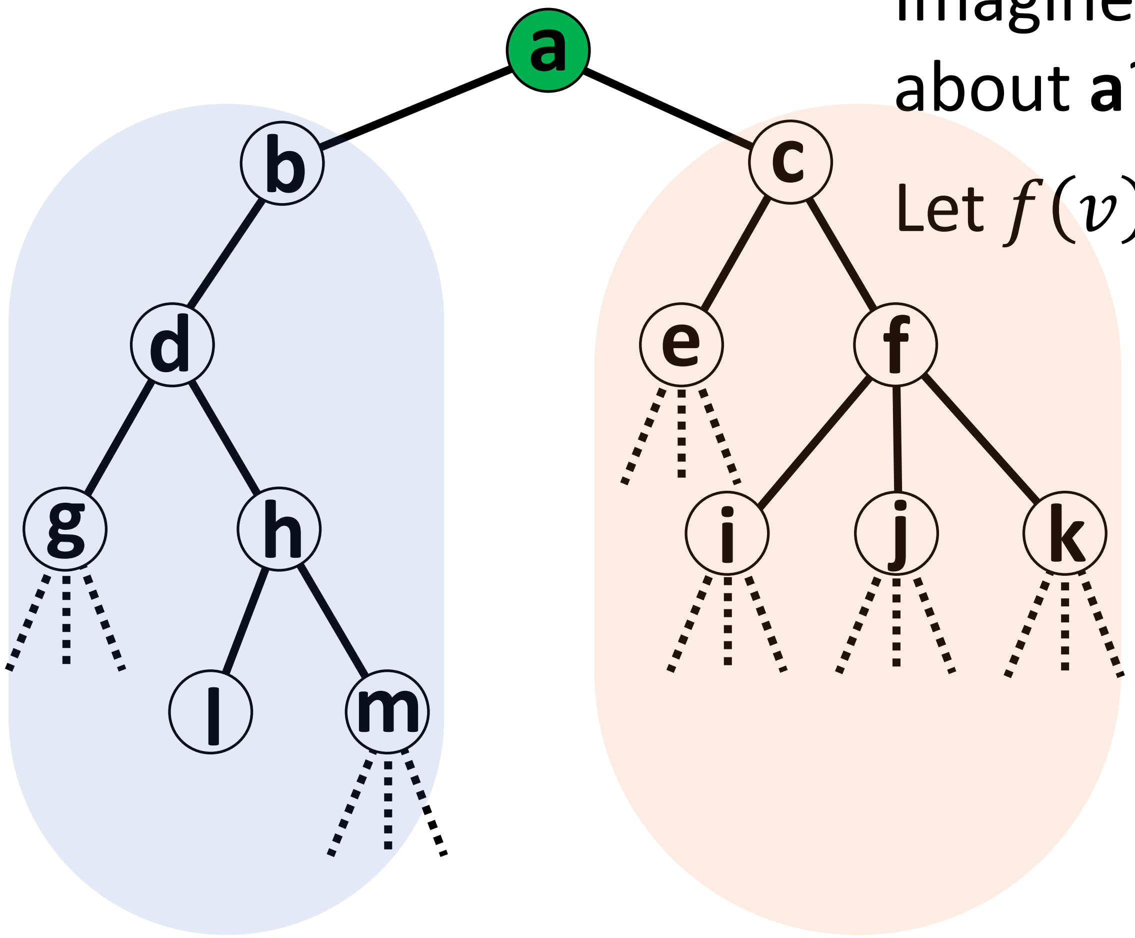
Imagine the minimum vertex cover. What can we say about **a**? **a** is in a minimum vertex cover, or it's not.

Let $f(v)$ = Size of minimum vertex cover rooted at v.

If a **<u>is</u>** in a minimum VC

$f(a) = 1 + f(b) + f(c)$

Vertex Cover in Trees



Imagine the minimum vertex cover. What can we say about **a**? **a** is in a minimum vertex cover, or it's not.

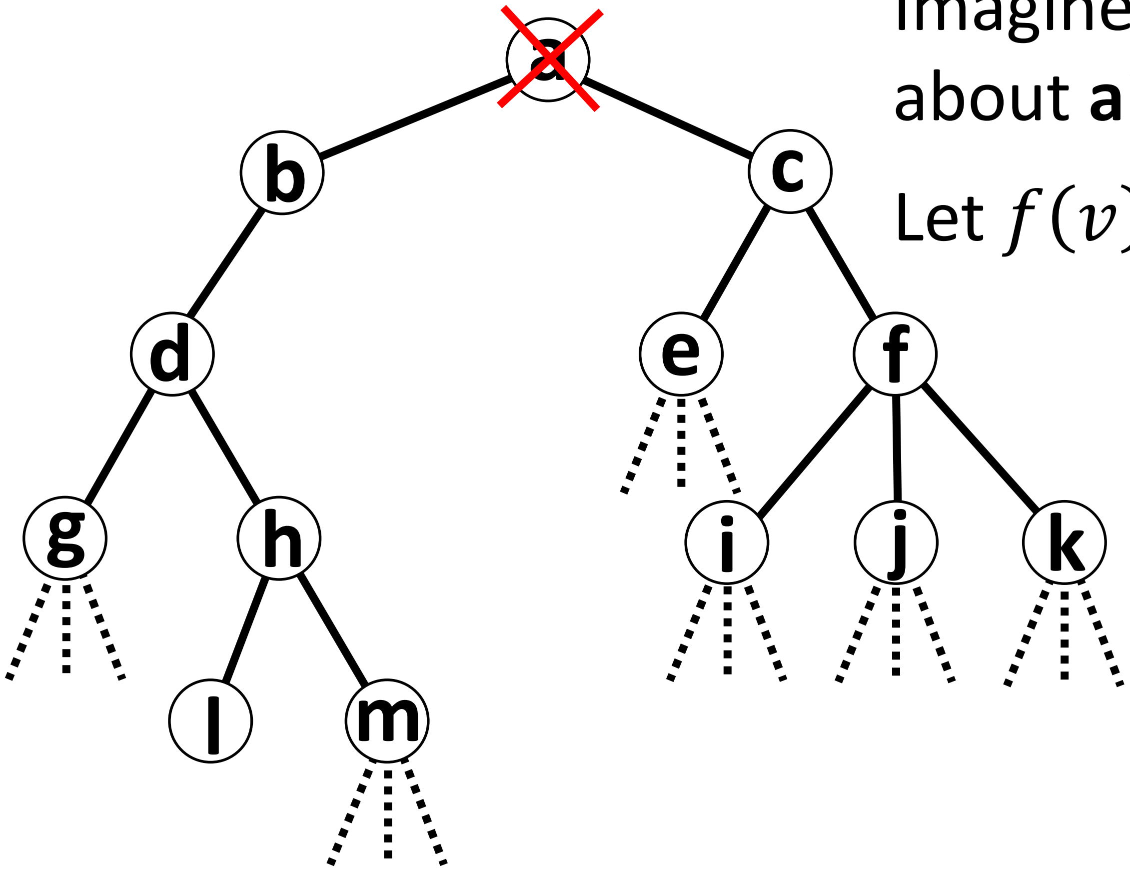
Let $f(v)$ = Size of minimum vertex cover rooted at v.

If a **is** in a minimum VC

$f(a) = 1 + f(b) + f(c)$

*"If there was a smaller VC rooted at **b**, it would give us a smaller VC rooted at **a**."*

Vertex Cover in Trees



Imagine the minimum vertex cover. What can we say about **a**? **a** is in a minimum vertex cover, or it's not.

Let $f(v)$ = Size of minimum vertex cover rooted at v.

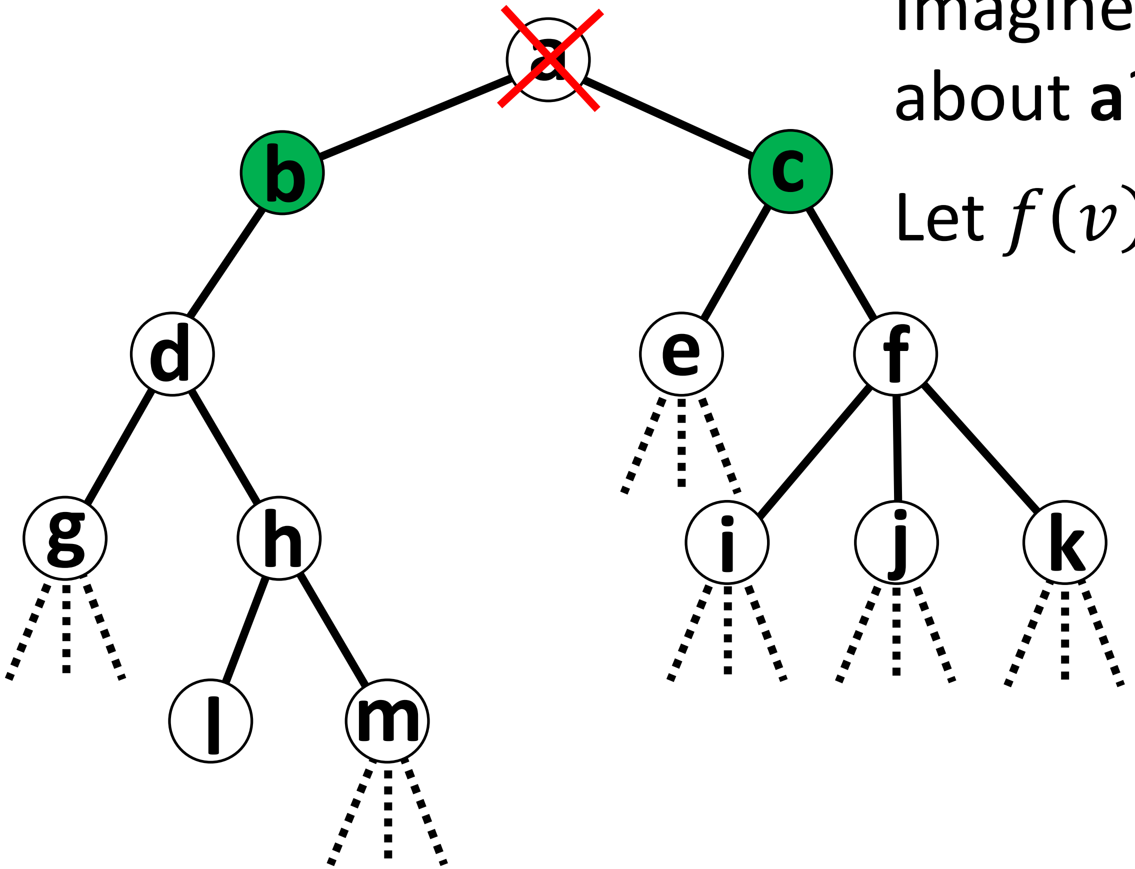
If a **is** in a minimum VC

$$f(a) = 1 + f(b) + f(c)$$

If a **is not** in a minimum VC

$$f(a) = ??$$

Vertex Cover in Trees



Imagine the minimum vertex cover. What can we say about **a**? **a** is in a minimum vertex cover, or it's not.

Let $f(v)$ = Size of minimum vertex cover rooted at v.

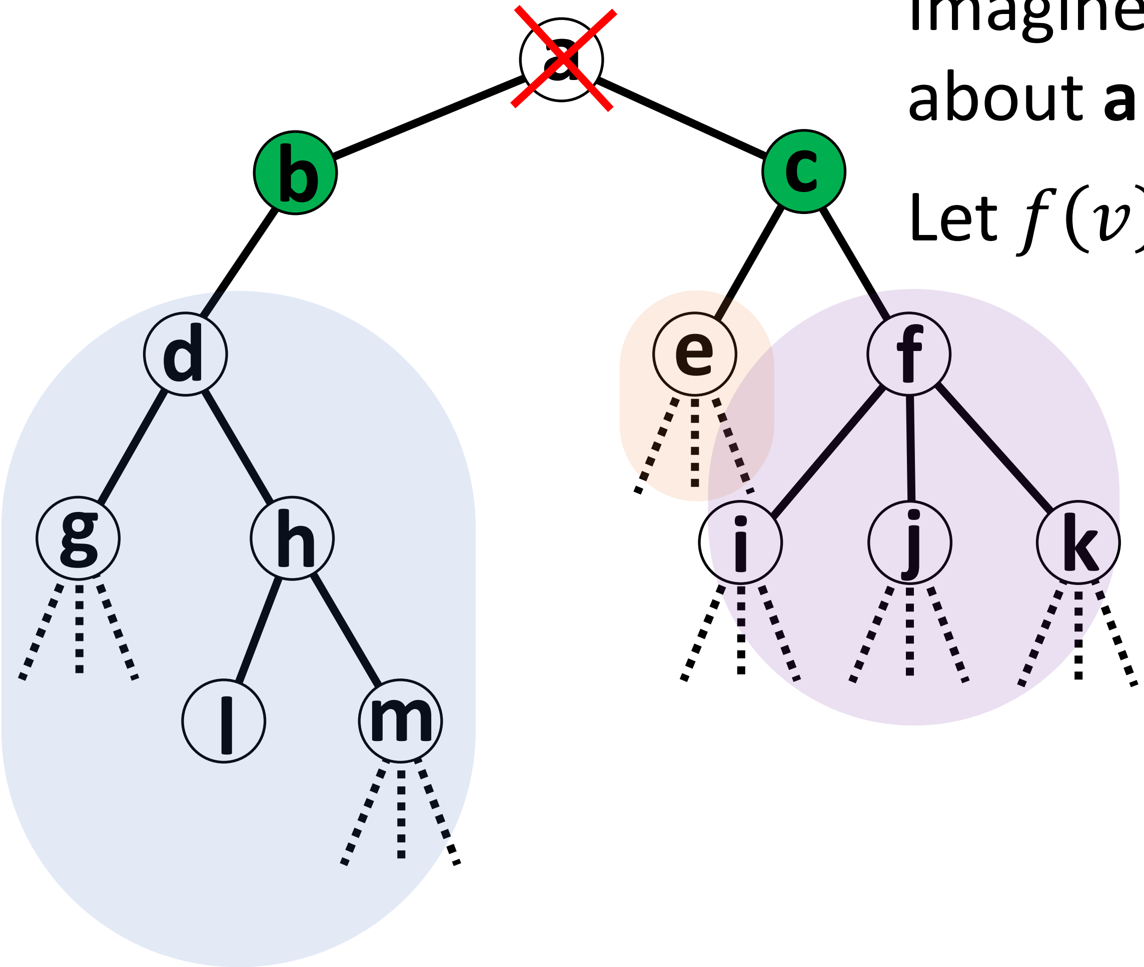
If a **<u>is</u>** in a minimum VC

$f(a) = 1 + f(b) + f(c)$

If a **<u>is not</u>** in a minimum VC

$f(a) = 2 + ??$

Vertex Cover in Trees



Imagine the minimum vertex cover. What can we say about **a**? **a** is in a minimum vertex cover, or it's not.

Let $f(v)$ = Size of minimum vertex cover rooted at v.

If a **<u>is</u>** in a minimum VC

$f(a) = 1 + f(b) + f(c)$

If a **<u>is not</u>** in a minimum VC

$f(a) = 2 + f(d) + f(e) + f(f)$