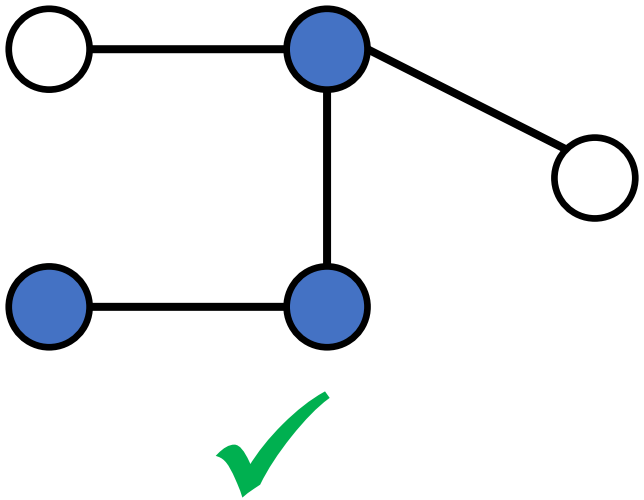
Dynamic Programming
CSCI 532

Vertex Cover

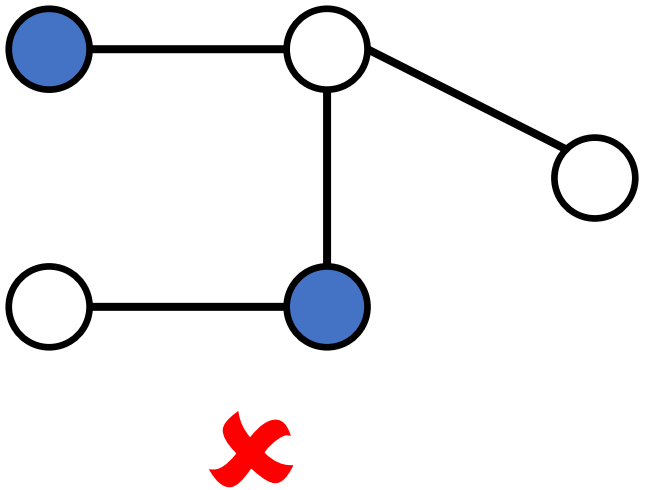
tree

Vertex Cover: Given ~~graph~~, find the smallest subset of vertices such that every edge in the ~~graph~~ has at least one vertex in the subset.

tree

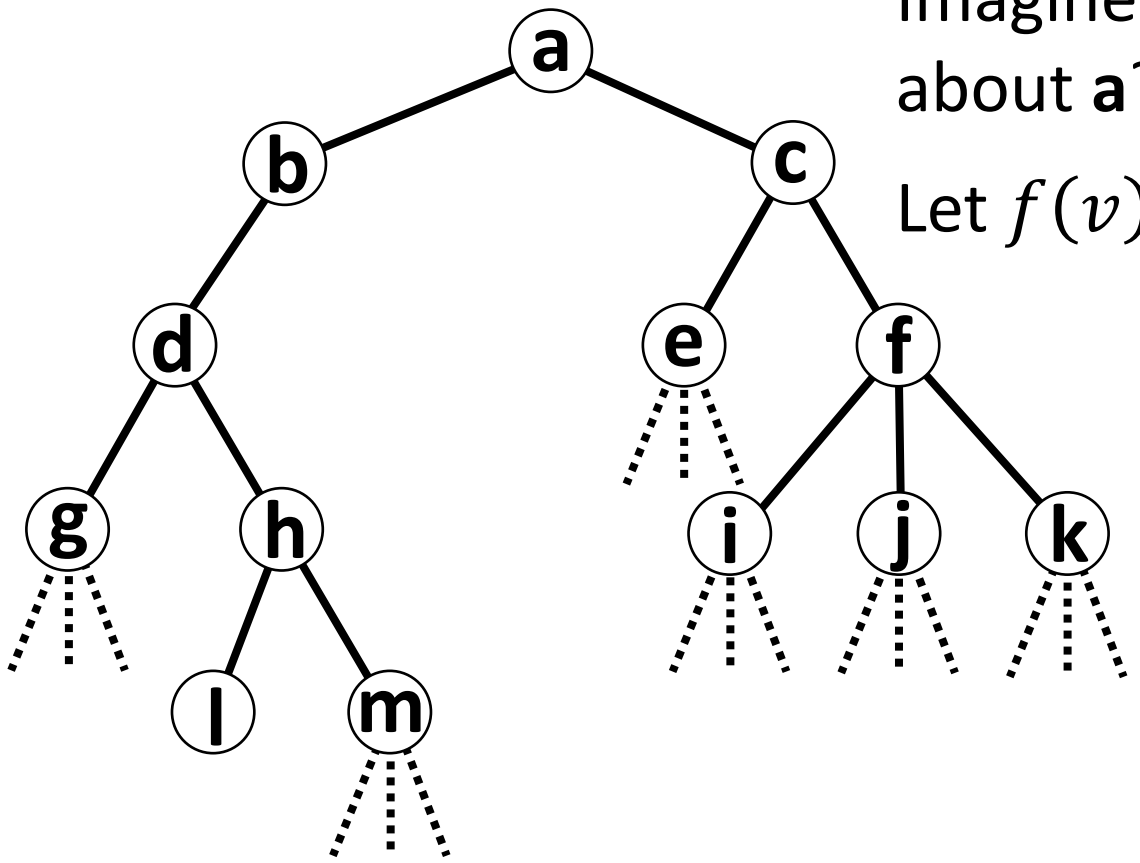


✓



✗

Vertex Cover in Trees



Imagine the minimum vertex cover. What can we say about **a**? **a** is in a minimum vertex cover, or it's not.

Let $f(v) =$ Size of minimum vertex cover rooted at v.

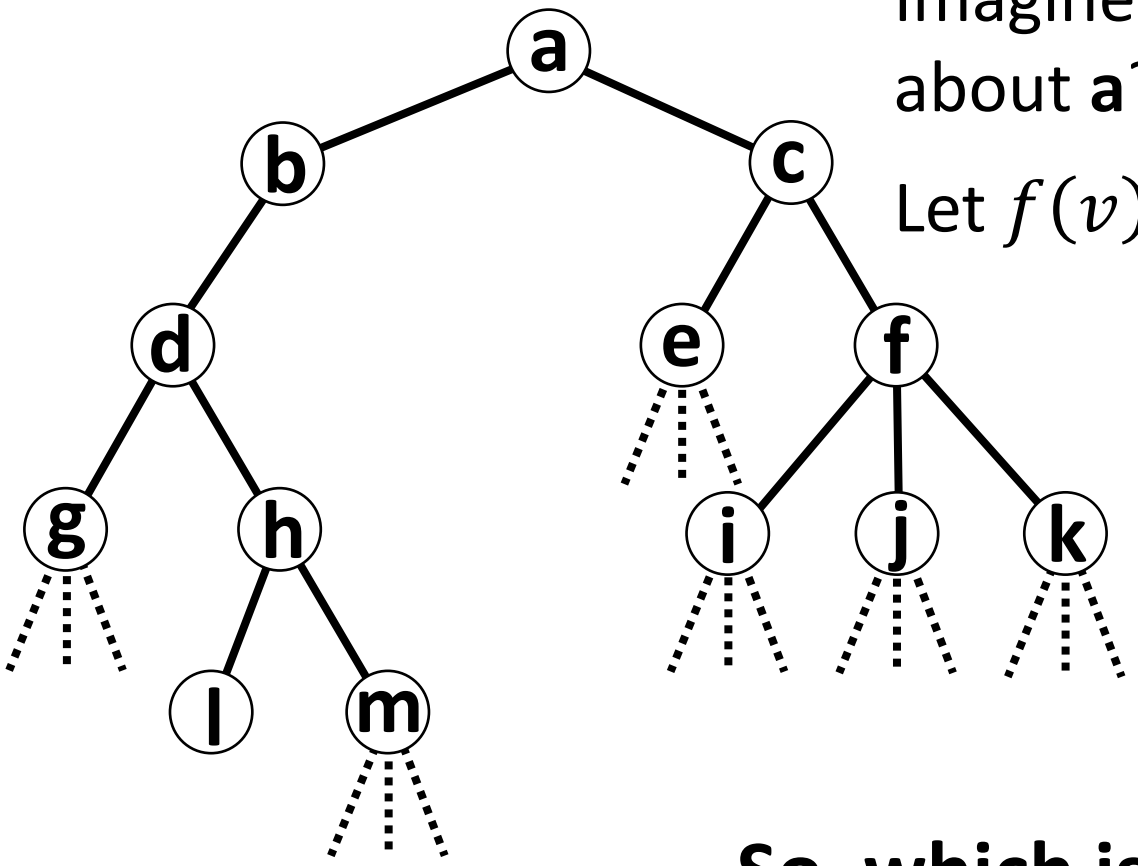
If a **is** in a minimum VC

$$f(a) = 1 + f(b) + f(c)$$

If a **is not** in a minimum VC

$$f(a) = 2 + f(d) + f(e) + f(f)$$

Vertex Cover in Trees



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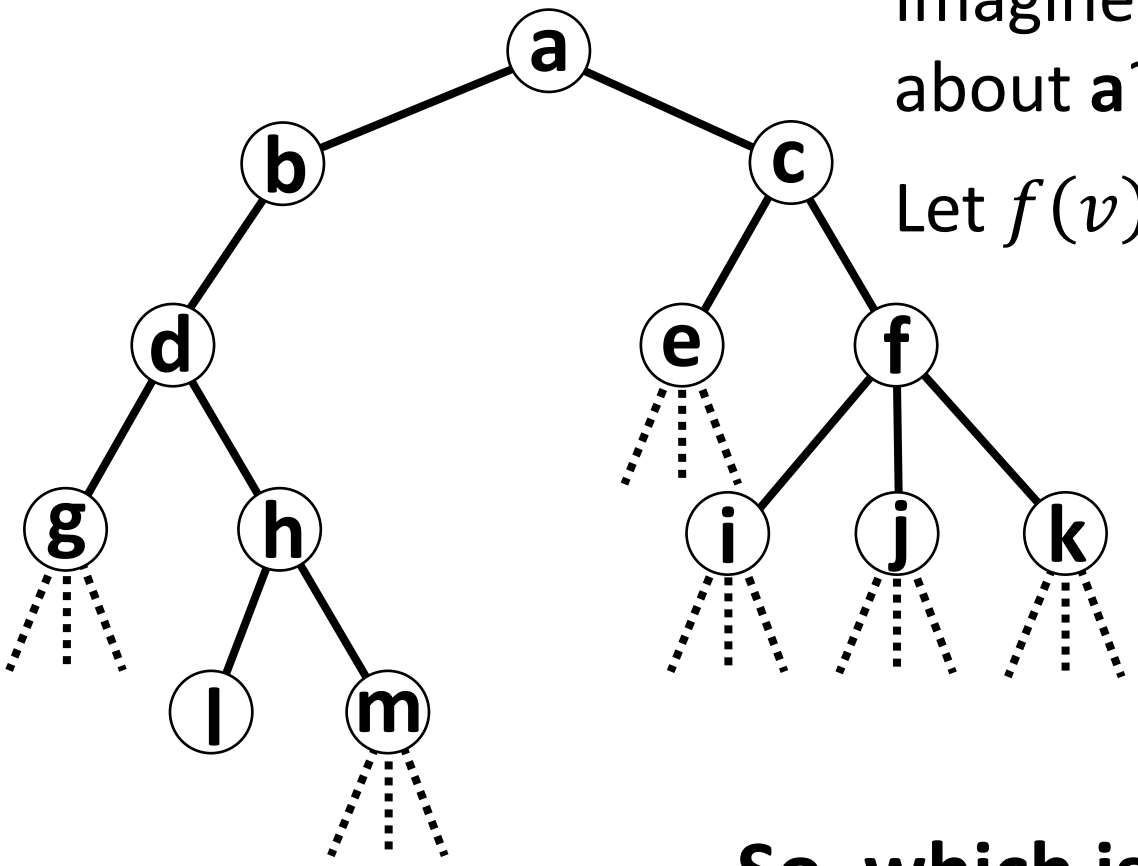
So, which is it?

Vertex Cover in Trees



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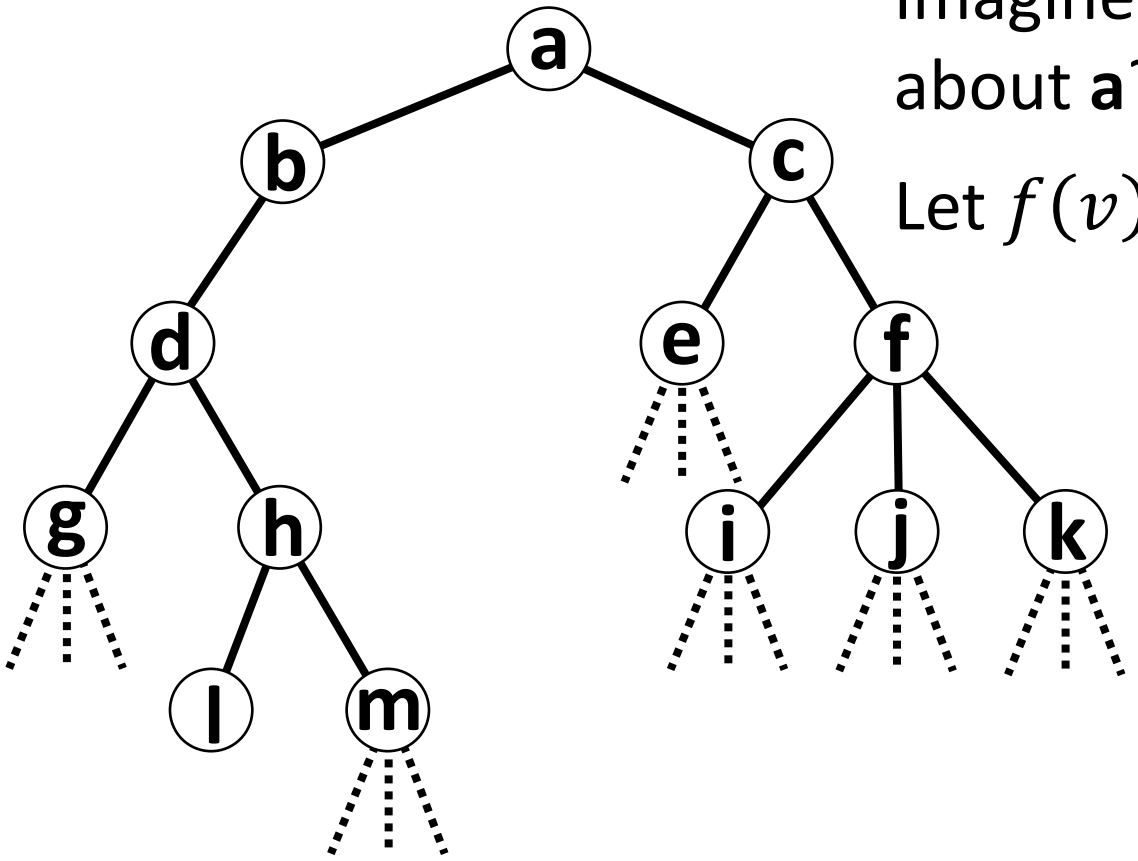
So, which is it?
The smallest!

Vertex Cover in Trees



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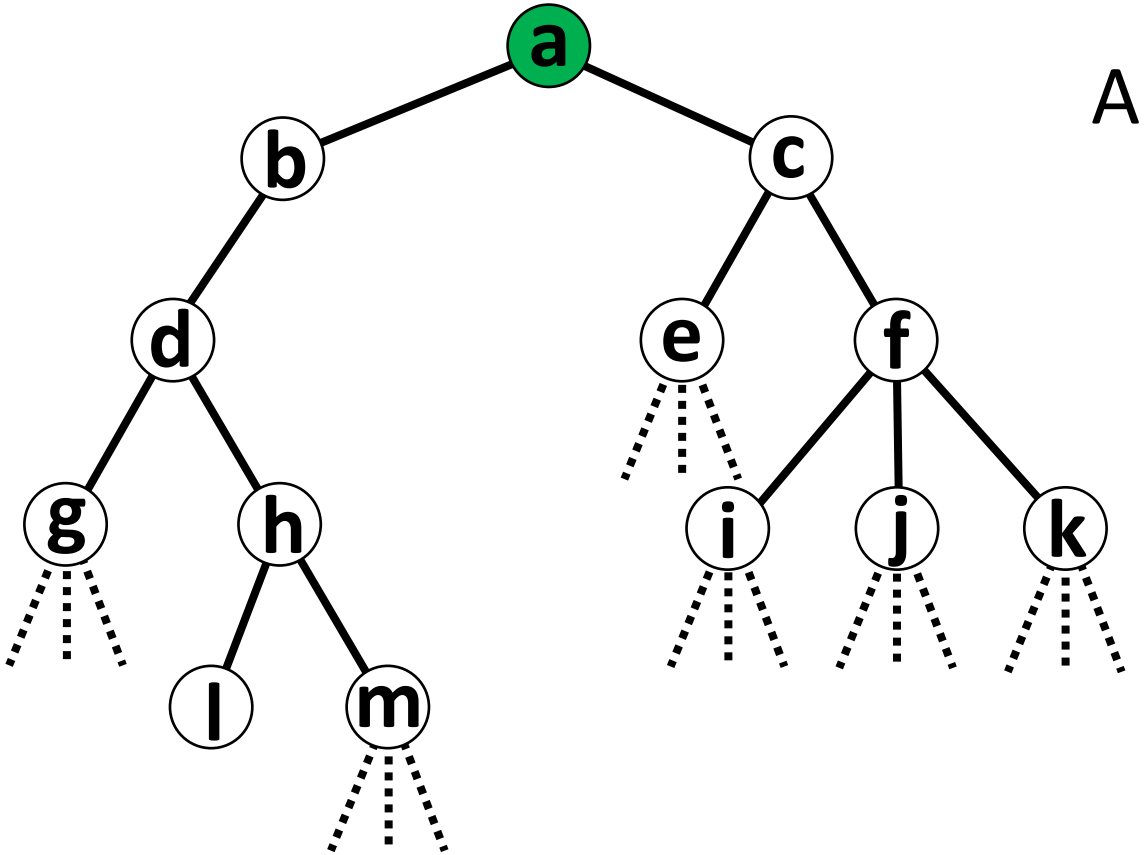
If a **is not** in a minimum VC

$$f(a) = 2 + f(d) + f(e) + f(f)$$

$$f(v) = \min\left\{1 + \sum_{u \in \text{child}(v)} f(u),\ |\text{child}(v)| + \sum_{w \in \text{grandchild}(v)} f(w)\right\}$$

Vertex Cover in Trees



Are we ready to calculate $f(v)$ **here**?

$$f(v) = \min\left\{1 + \sum_{u \in \text{child}(v)} f(u)\,,\ |\text{child}(v)| + \sum_{w \in \text{grandchild}(v)} f(w)\right\}$$

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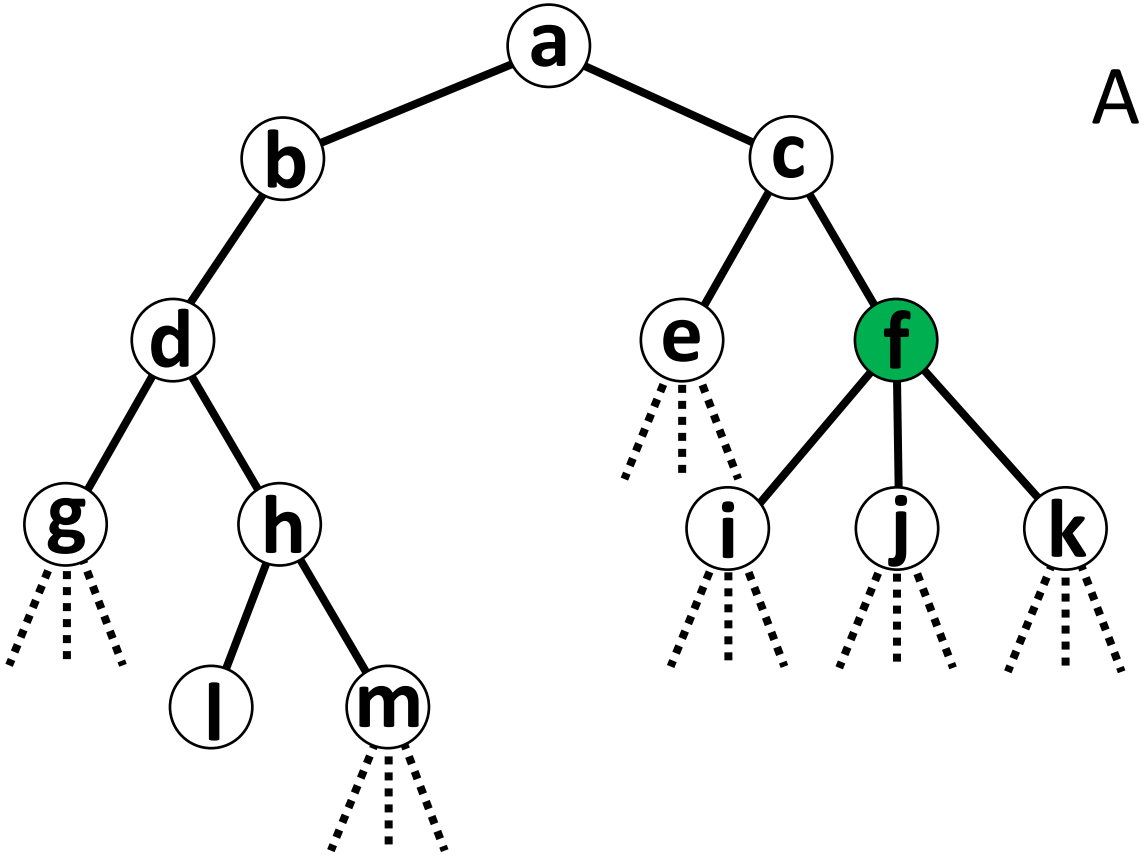

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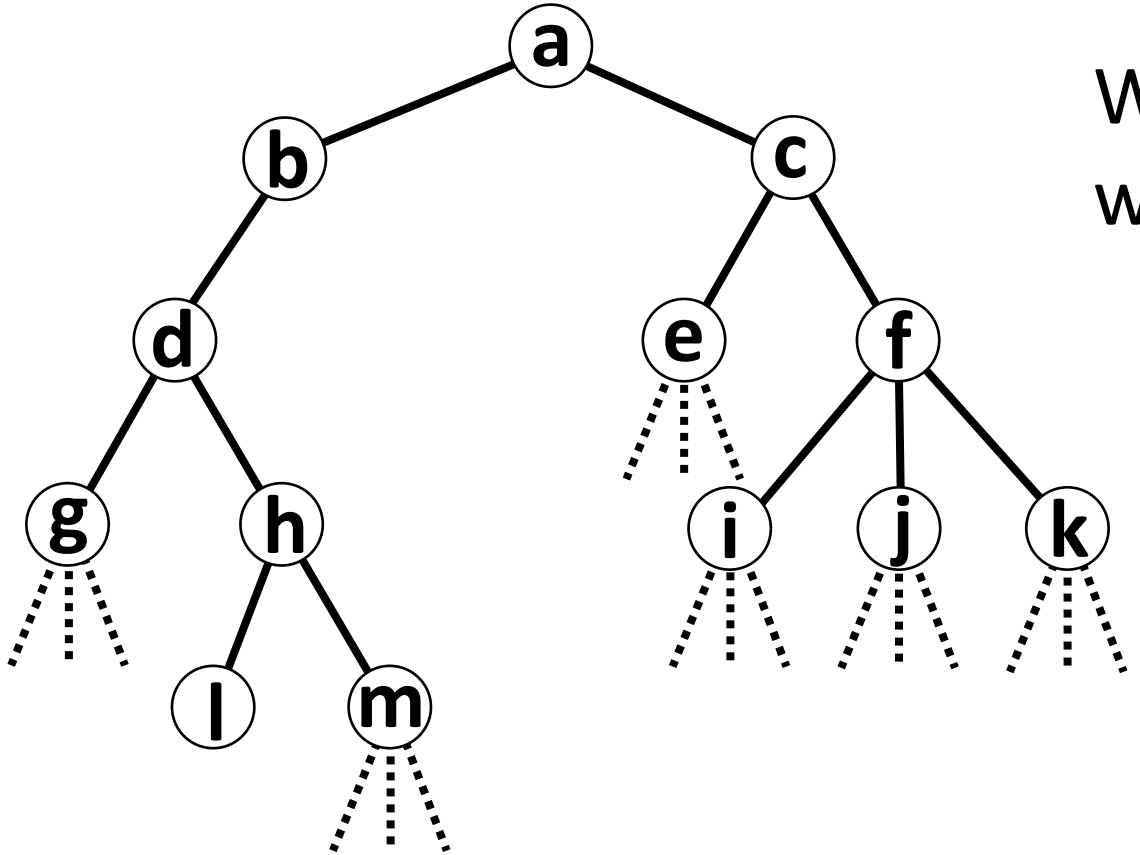

Where do we have all the information we need to calculate $f(v)$?

$$f(v) = \min\left\{1 + \sum_{u \in \text{child}(v)} f(u)\,,\ |\text{child}(v)| + \sum_{w \in \text{grandchild}(v)} f(w)\right\}$$

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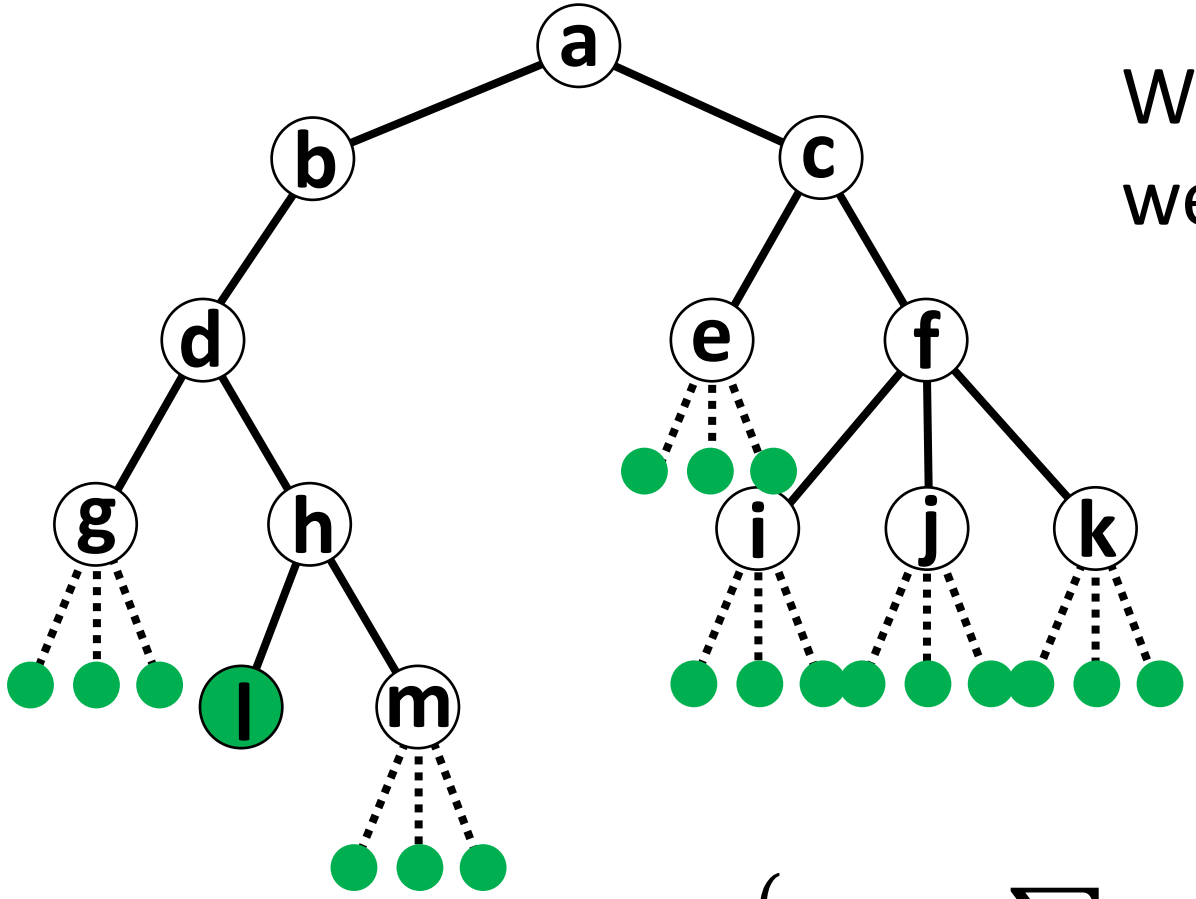


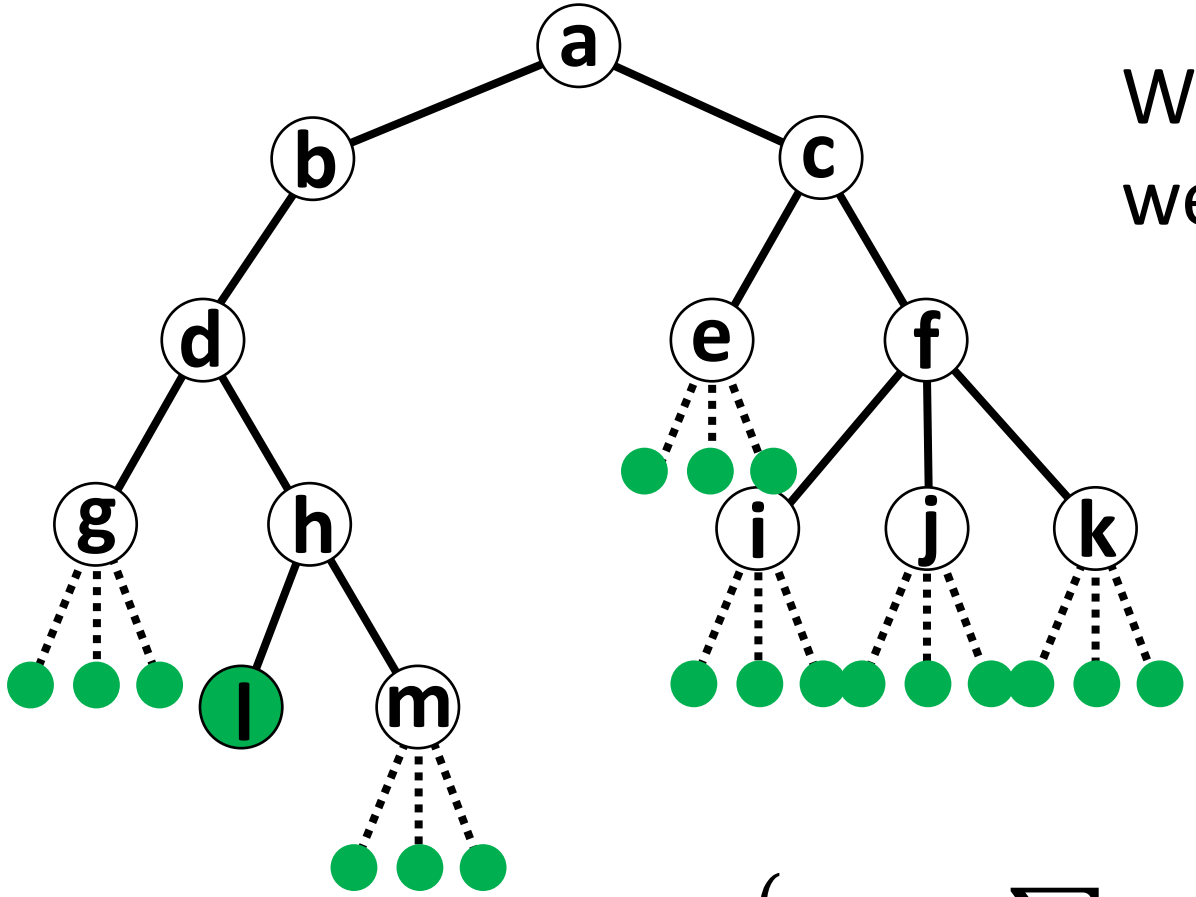
Where do we have all the information we need to calculate $f(v)$?

At the leaves!!

$$f(v) = \min\left\{1 + \sum_{u \in \text{child}(v)} f(u)\,,\ |\text{child}(v)| + \sum_{w \in \text{grandchild}(v)} f(w)\right\}$$

Vertex Cover in Trees

Let $f(v)$ = Size of minimum vertex cover rooted at v.



Where do we have all the information we need to calculate $f(v)$?

At the leaves!!

If v is a leaf, $f(v) = ??$

$$f(v) = \min\left\{1 + \sum_{u \in \text{child}(v)} f(u),\ |\text{child}(v)| + \sum_{w \in \text{grandchild}(v)} f(w)\right\}$$

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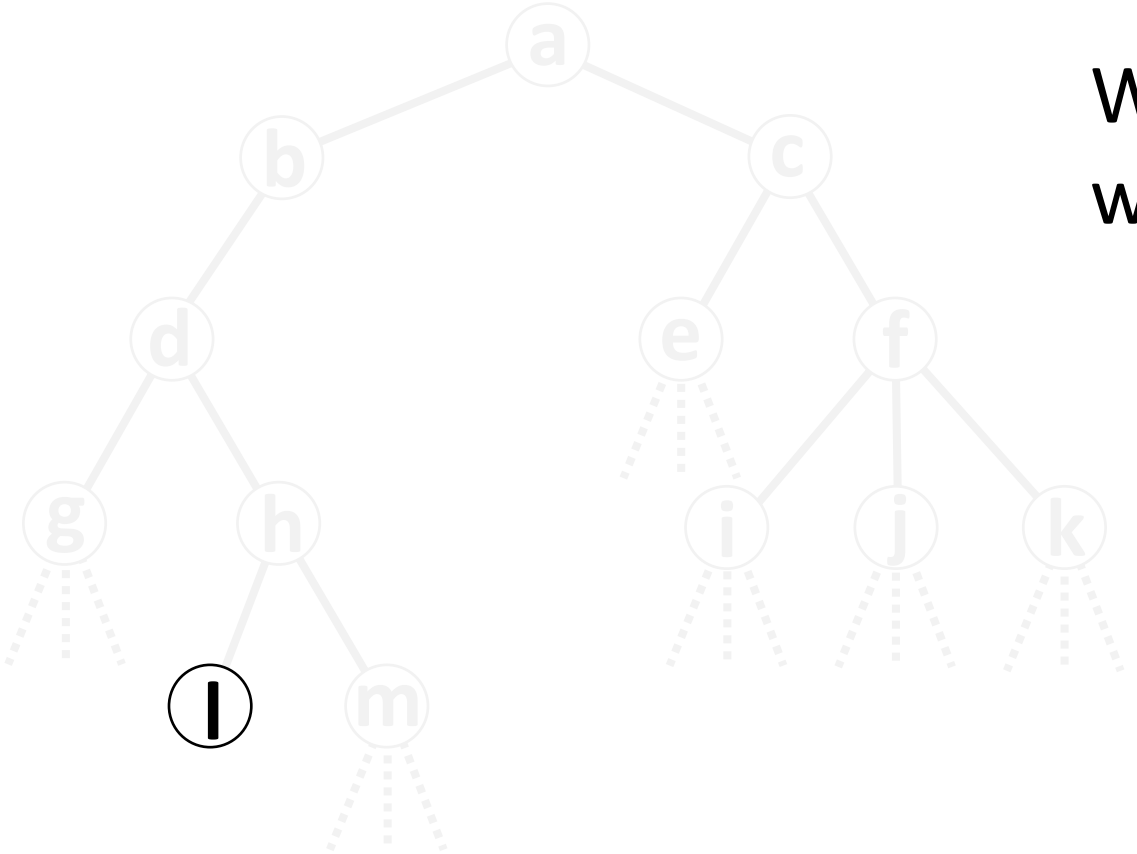

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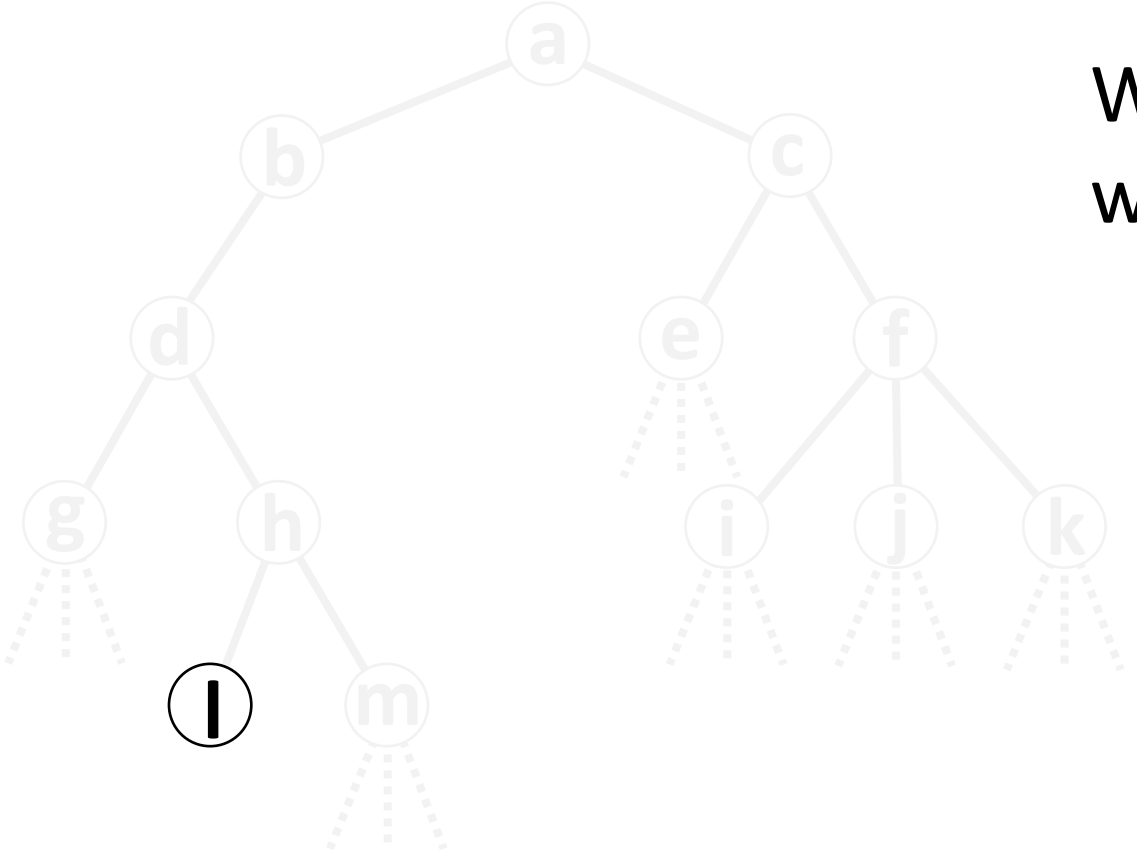


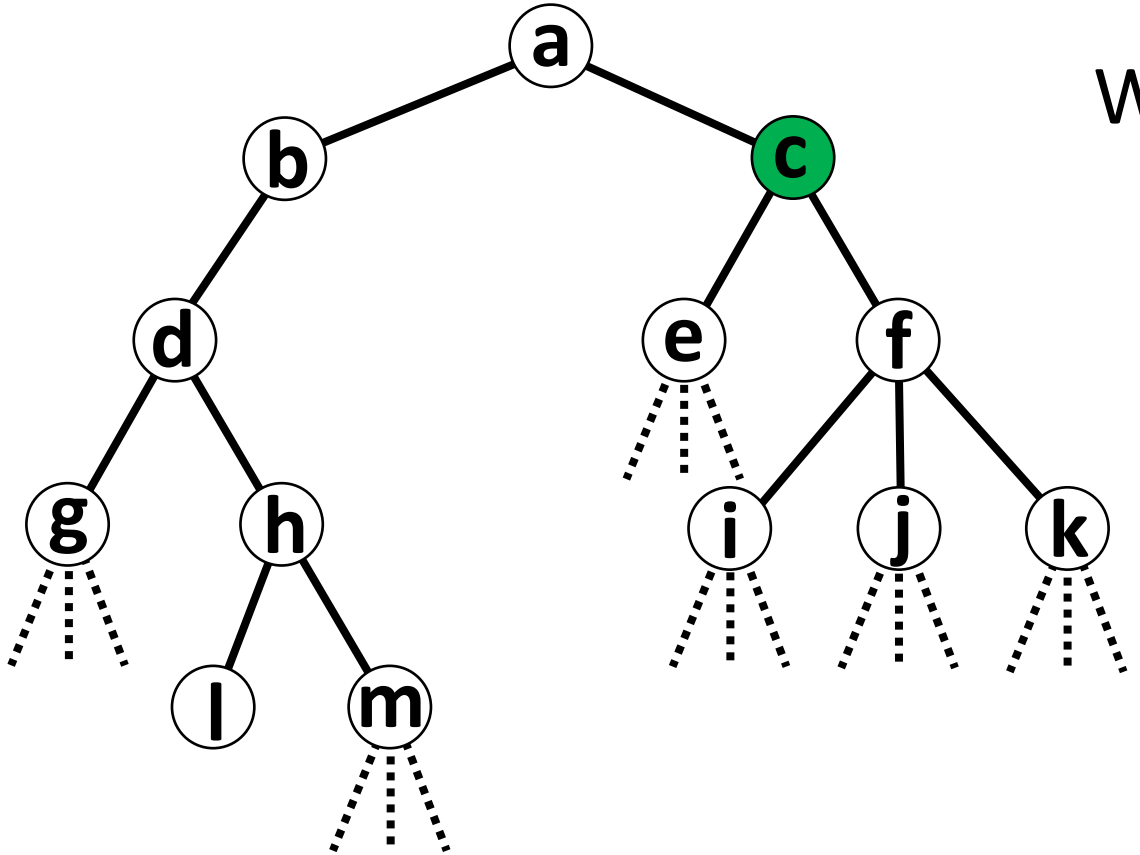
Where do we have all the information we need to calculate $f(v)$?

At the leaves!!

If v is a leaf, $f(v) = 0$

$$f(v) = \min\left\{1 + \sum_{u \in \text{child}(v)} f(u),\ |\text{child}(v)| + \sum_{w \in \text{grandchild}(v)} f(w)\right\}$$

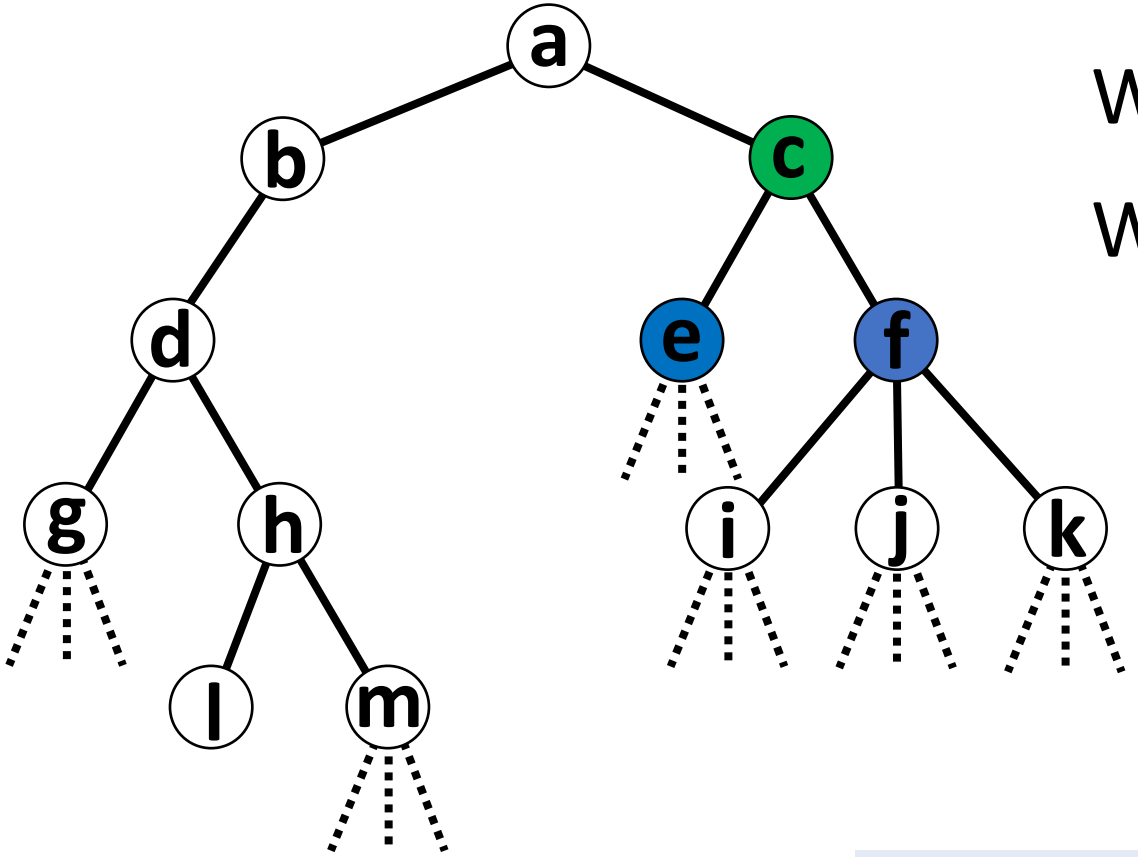
Vertex Cover in Trees



When are we ready to calculate $f(c)$?

$$f(v) = \min\left\{1 + \sum_{u \in \text{child}(v)} f(u)\,,\ |\text{child}(v)| + \sum_{w \in \text{grandchild}(v)} f(w)\right\}$$

Vertex Cover in Trees

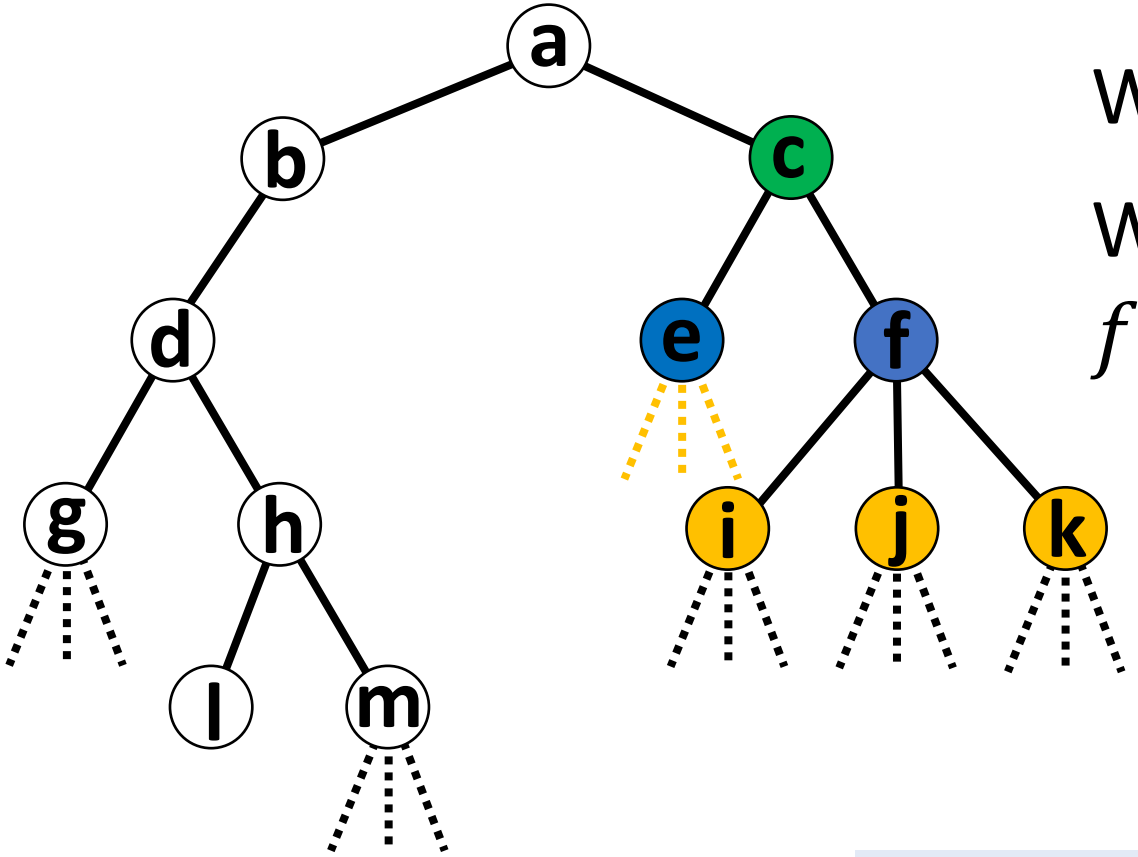


When are we ready to calculate $f(c)$?

When we have $f(e)$ and $f(f)$…

$$f(v) = \min\left\{1 + \sum_{u \in \text{child}(v)} f(u),\ |\text{child}(v)| + \sum_{w \in \text{grandchild}(v)} f(w)\right\}$$

Vertex Cover in Trees



When are we ready to calculate $f(c)$?

When we have $f(e)$ and $f(f)$ and $f(v)$ for all of c's grandchildren.

$$f(v) = \min\left\{1 + \sum_{u \in \text{child}(v)} f(u),\ |\text{child}(v)| + \sum_{w \in \text{grandchild}(v)} f(w)\right\}$$

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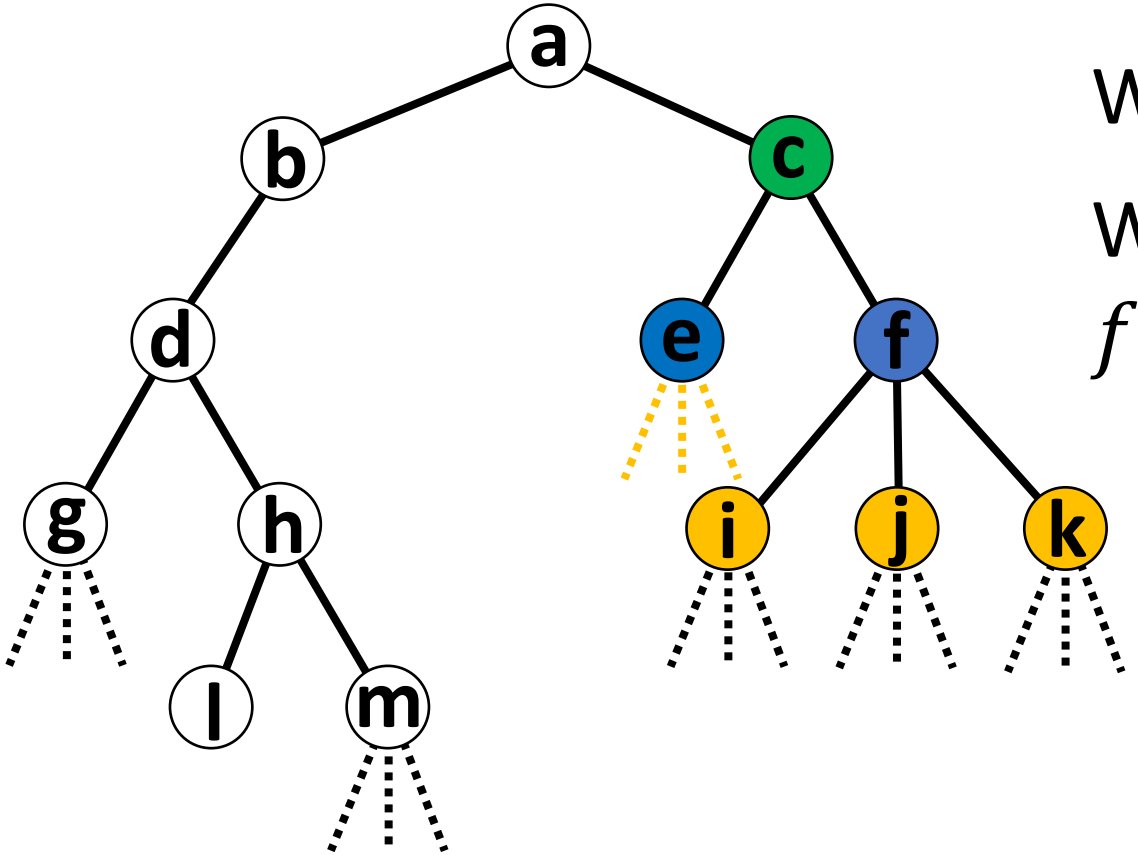


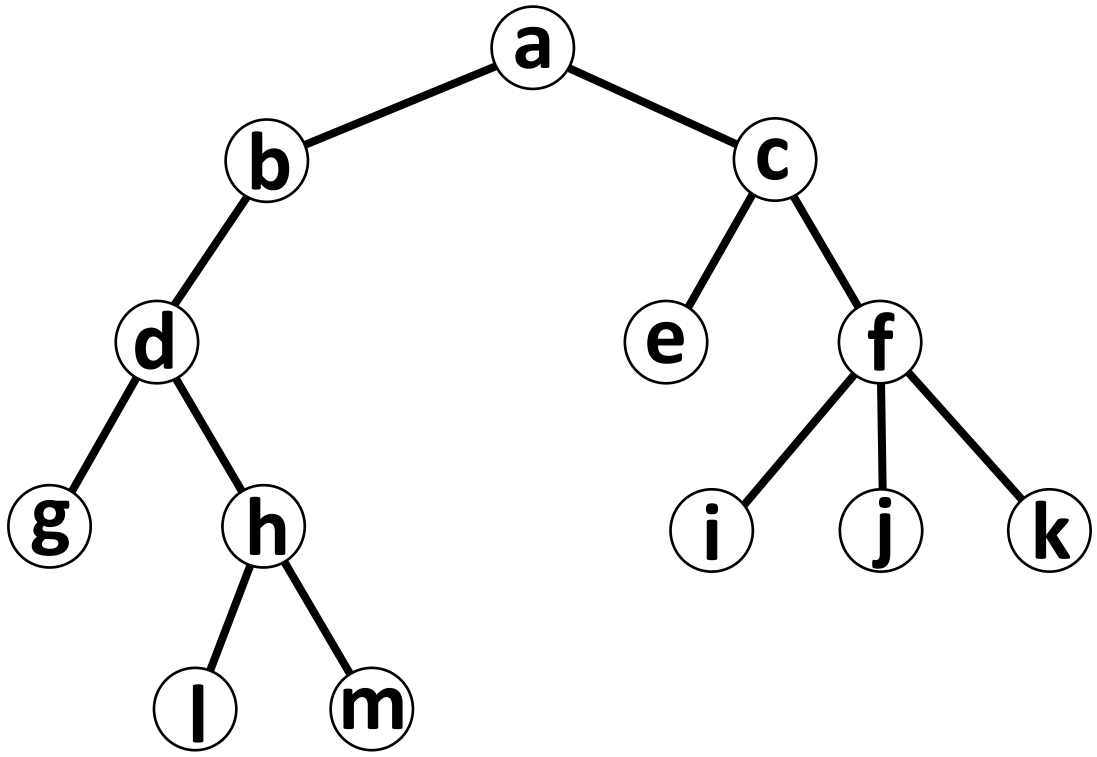
When are we ready to calculate $f(c)$?

When we have $f(e)$ and $f(f)$ and $f(v)$ for all of c's grandchildren.

How can we do this?

$$f(v) = \min\left\{1 + \sum_{u \in \text{child}(v)} f(u),\ |\text{child}(v)| + \sum_{w \in \text{grandchild}(v)} f(w)\right\}$$

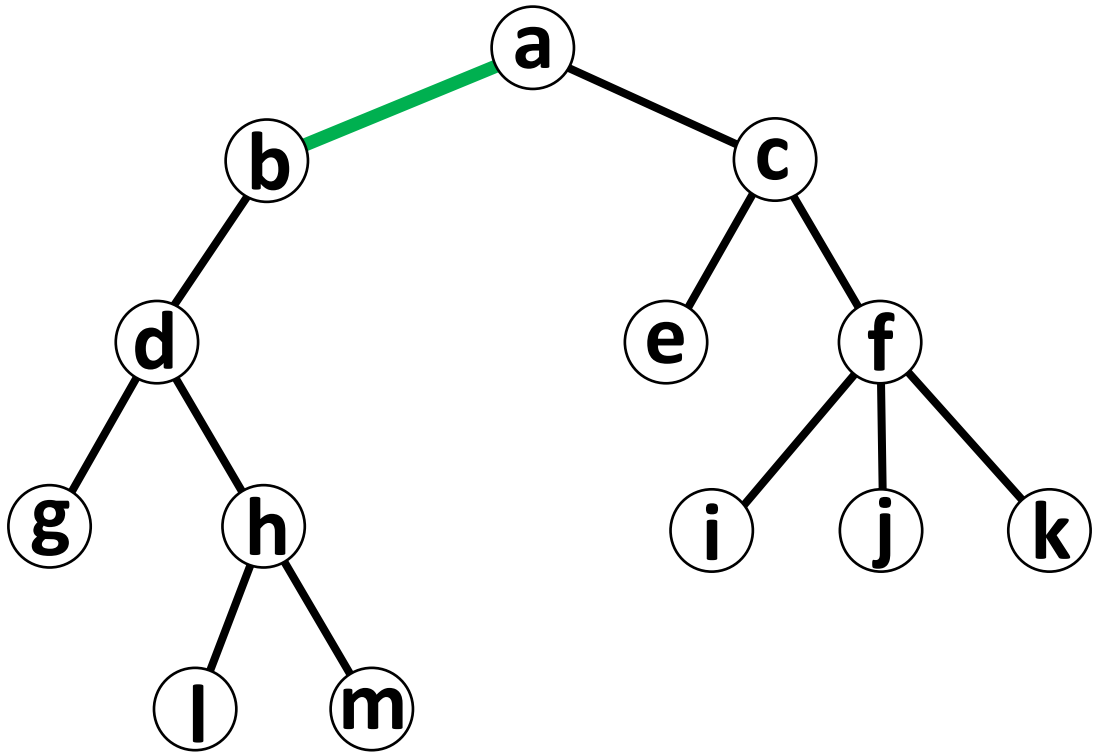
Tree Traversals



Breadth First: Start at a node. Visit all of its children, then all of its grandchildren, then great-grandchildren,…

Order visited: a, b, c, d, e, f, g, h, i, j, k, l ,m

Tree Traversals



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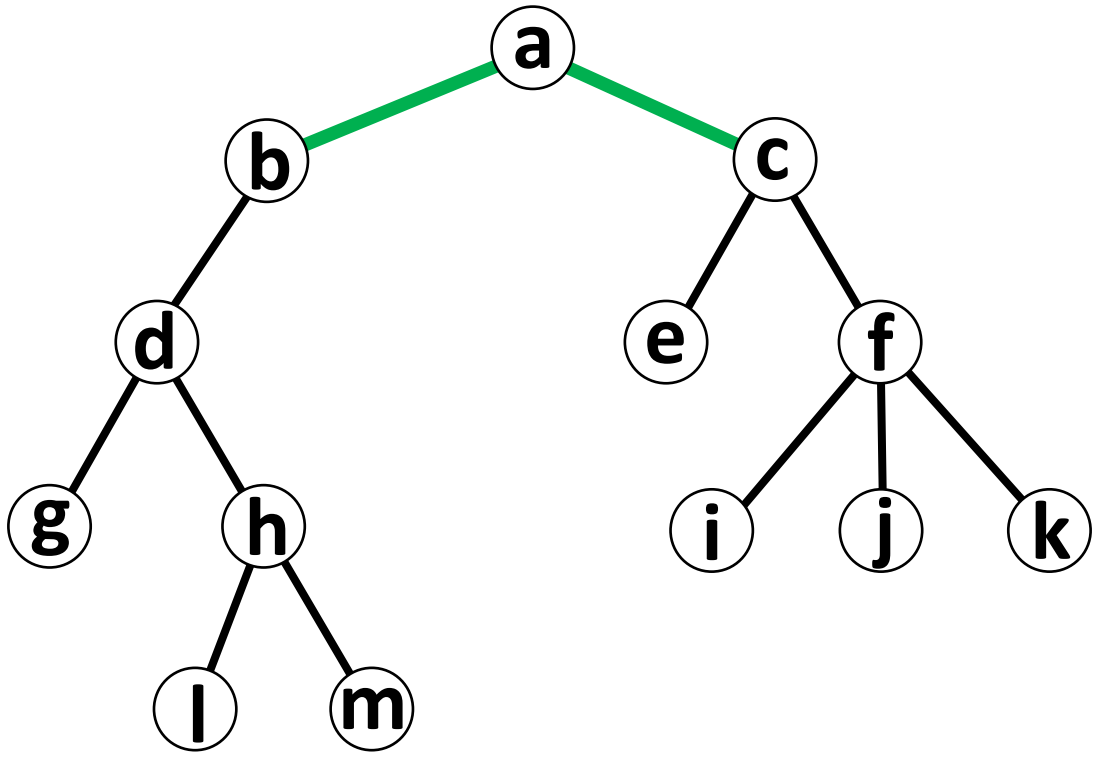

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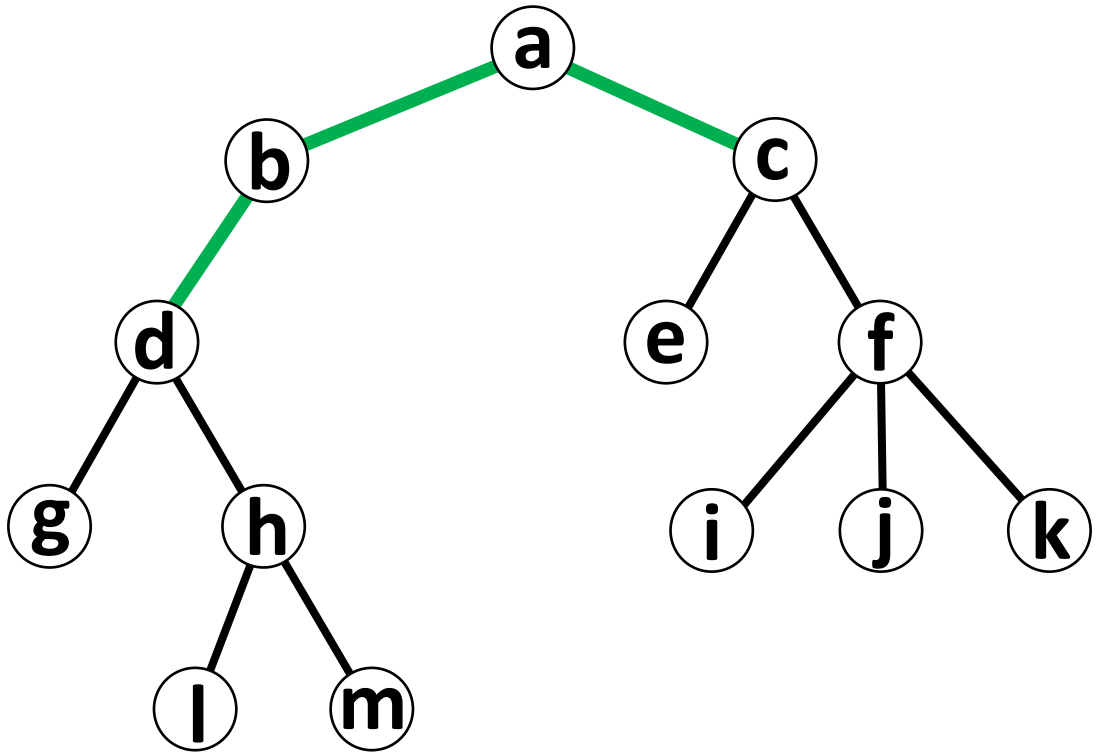

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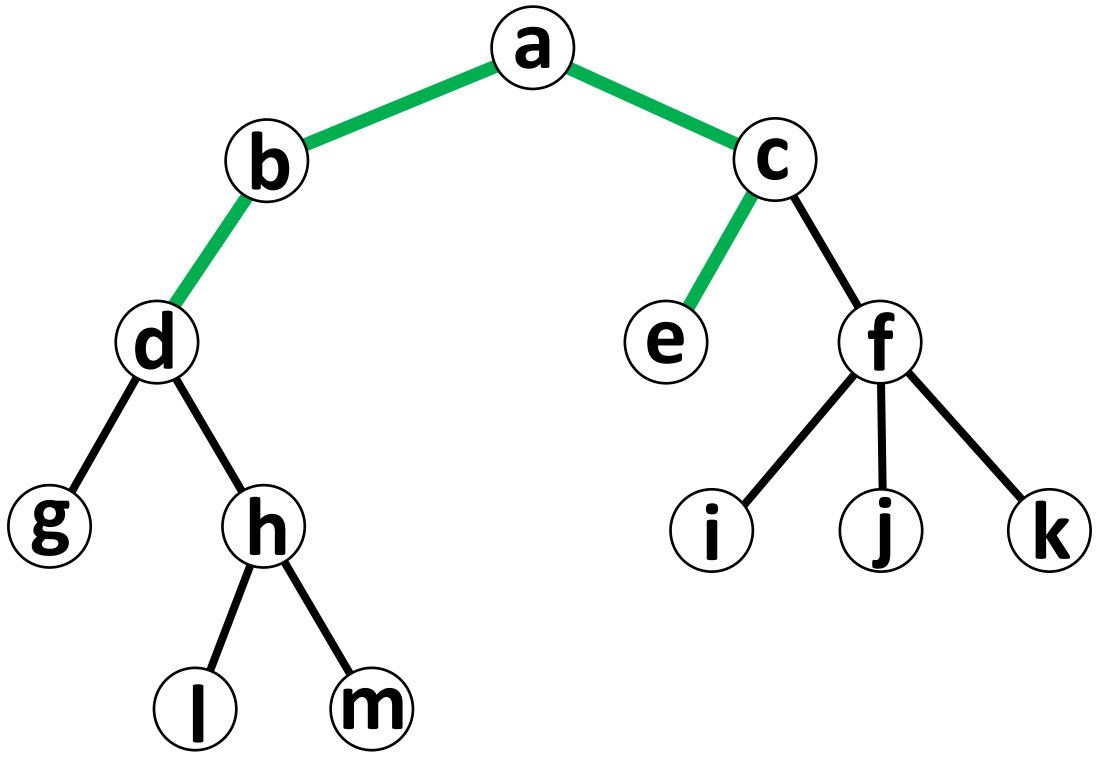

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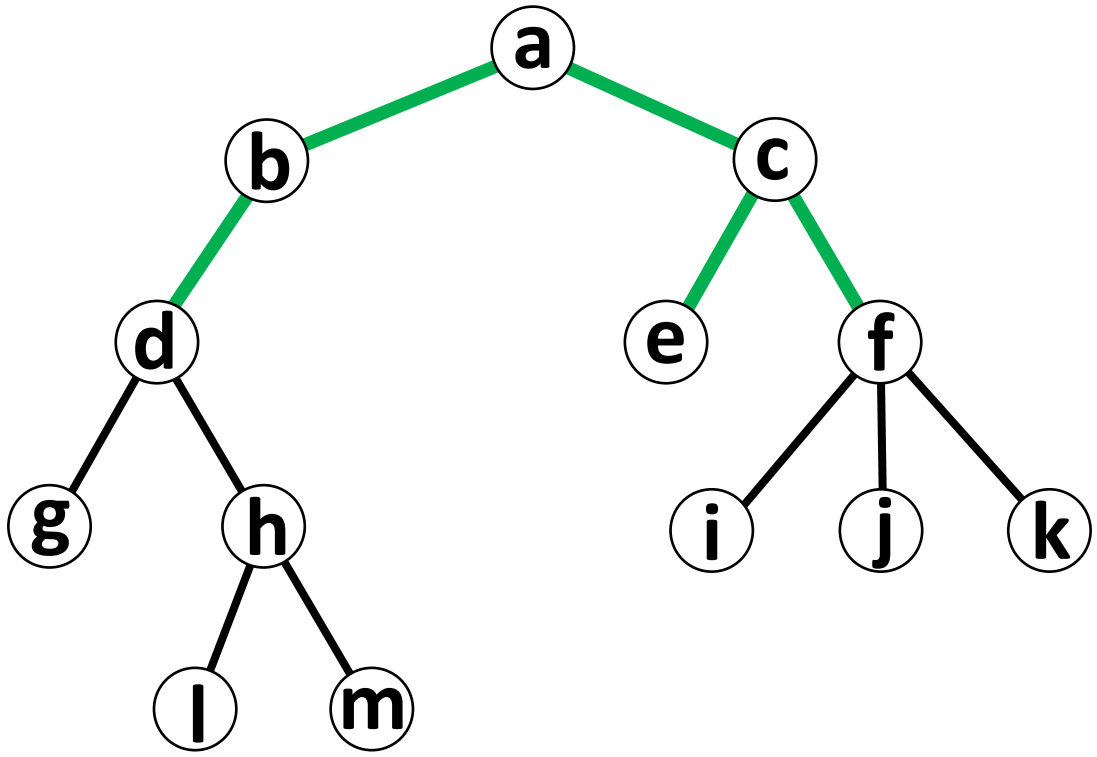

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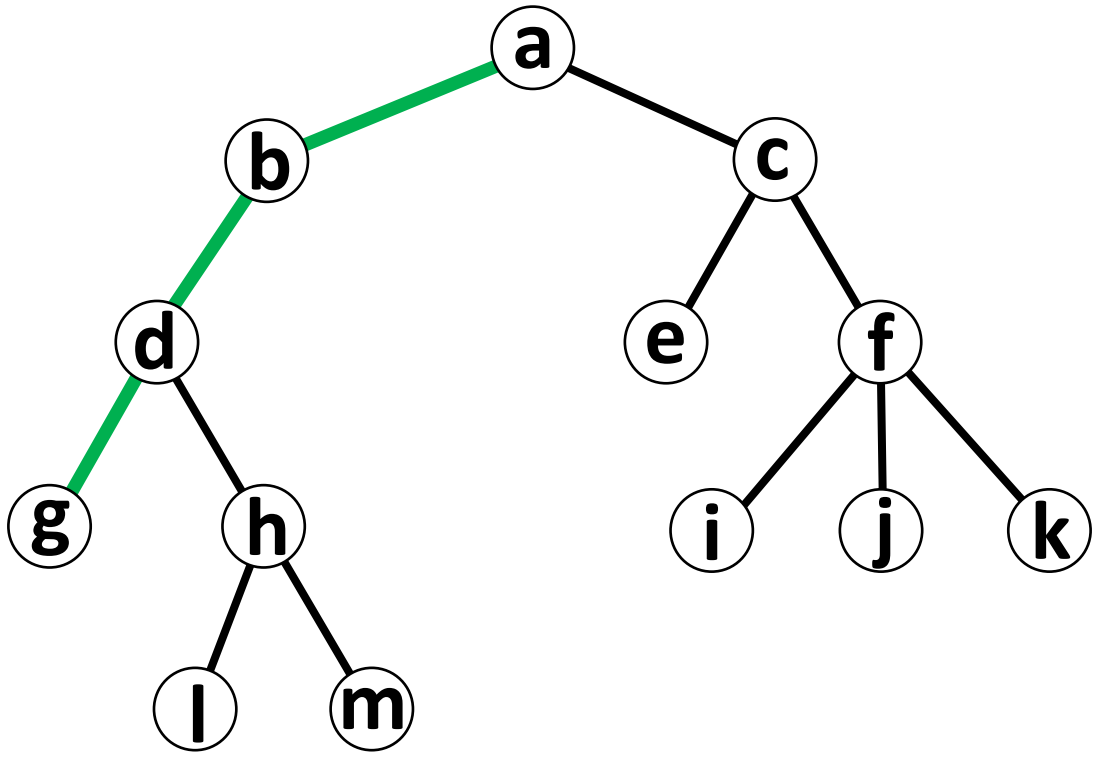

Depth First: Start at a node. Follow children until leaf is reached. Back track until unvisited child is available. Repeat…

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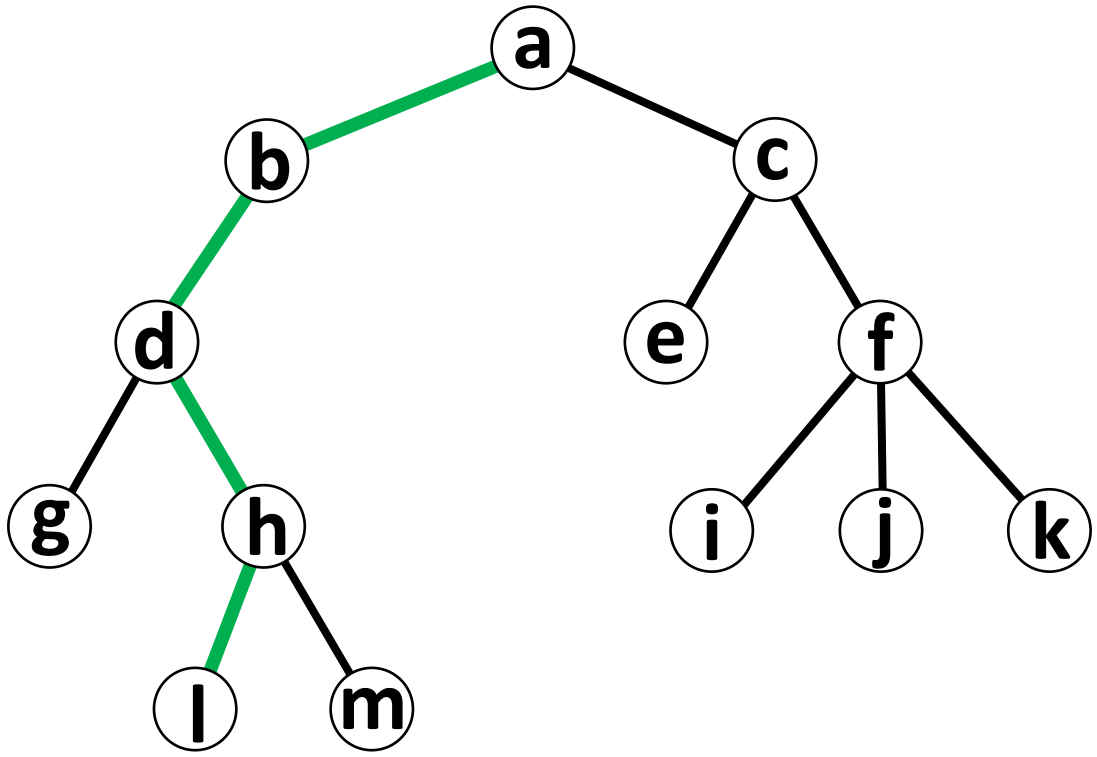

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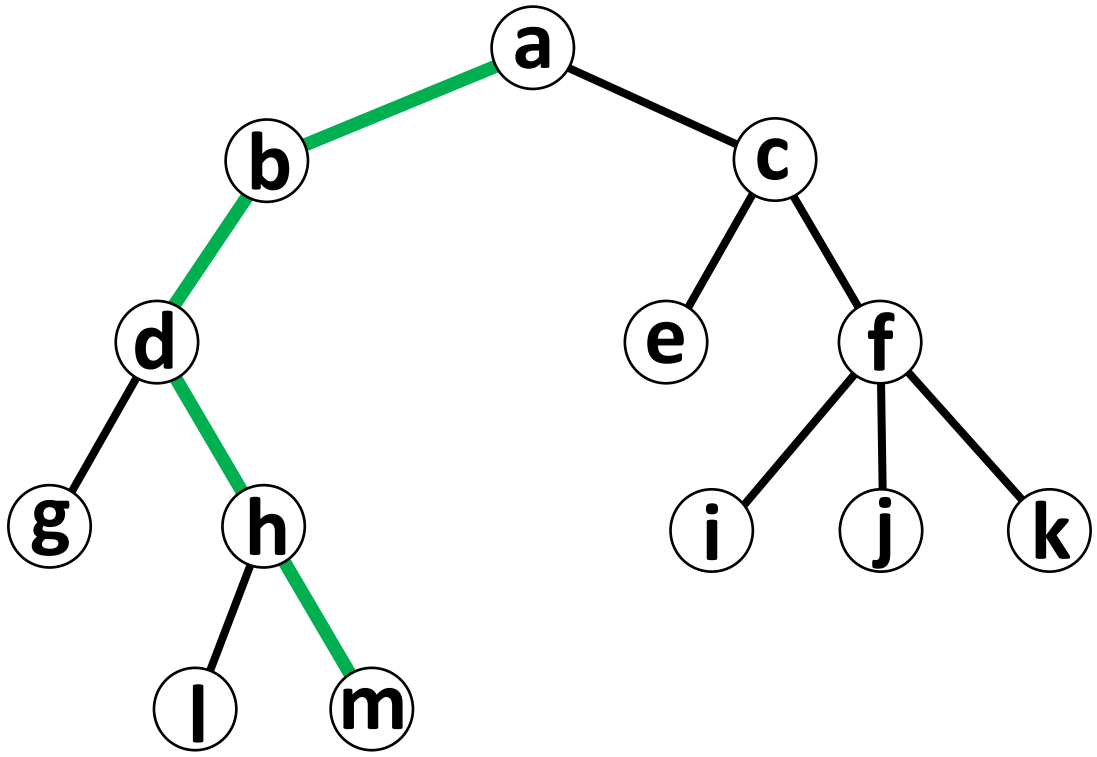

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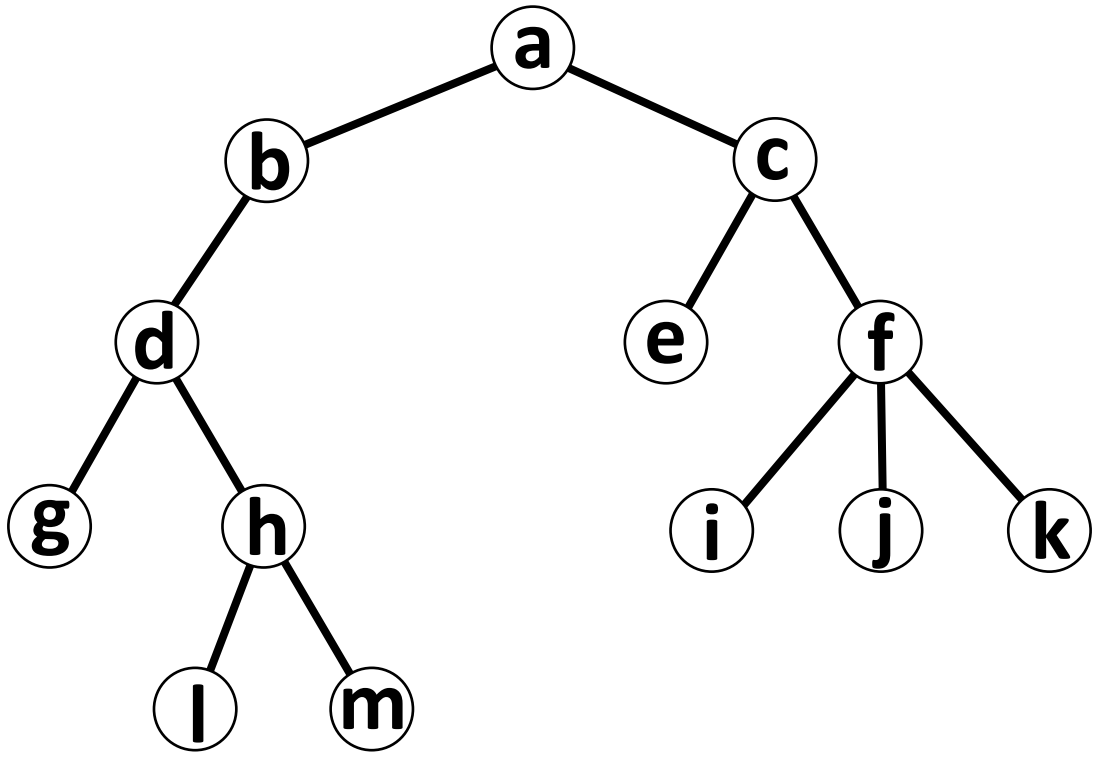

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```
preorder(Vertex v):
    if v not null:
        print(v.value())
        for u in child(v):
            preorder(u)
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postorder(Vertex v):
    if v not null:
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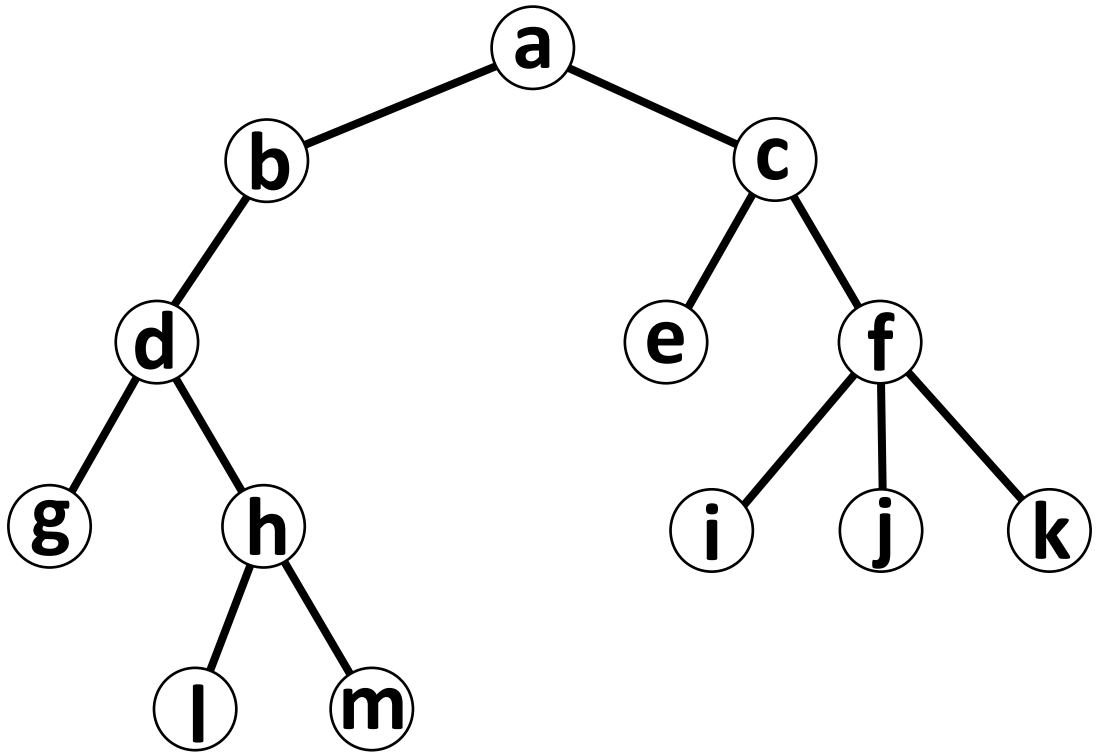
Tree Traversals



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2. Process children

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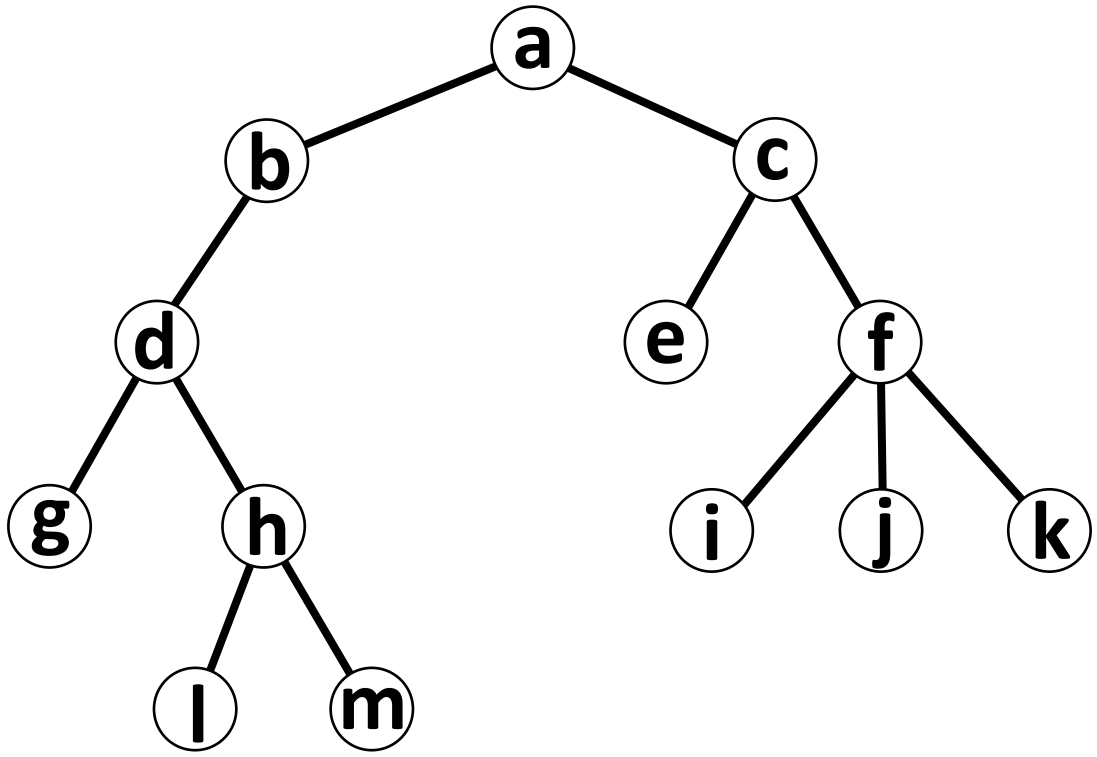

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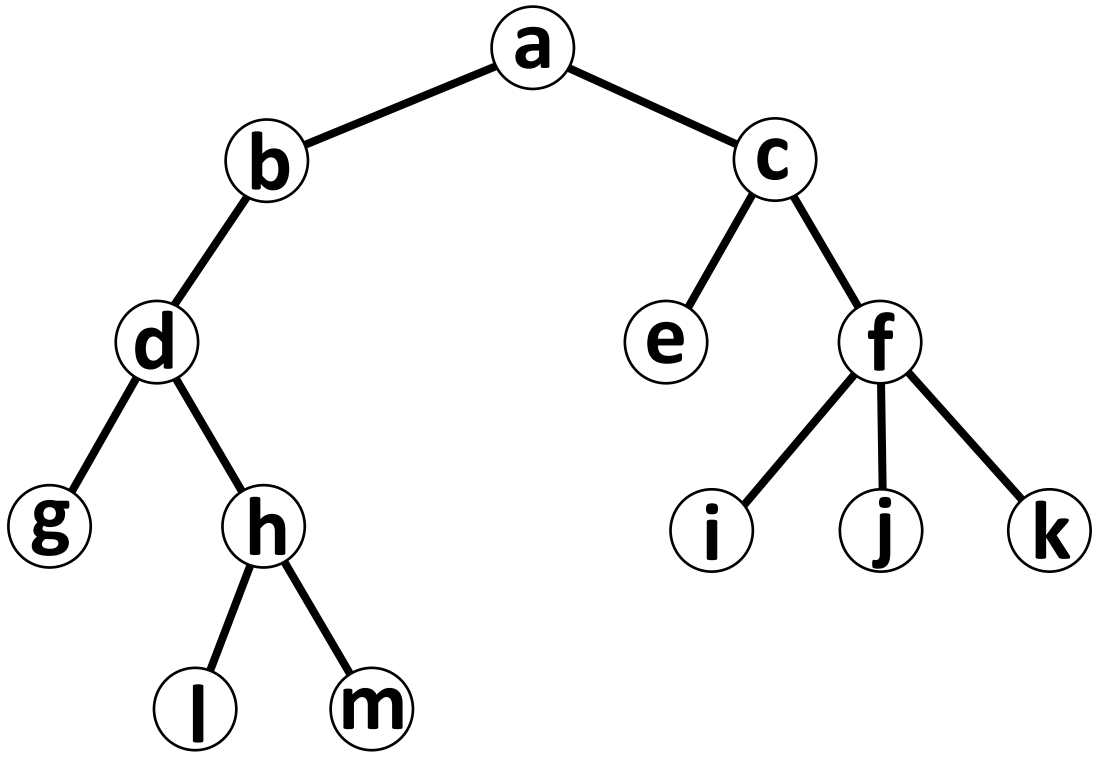

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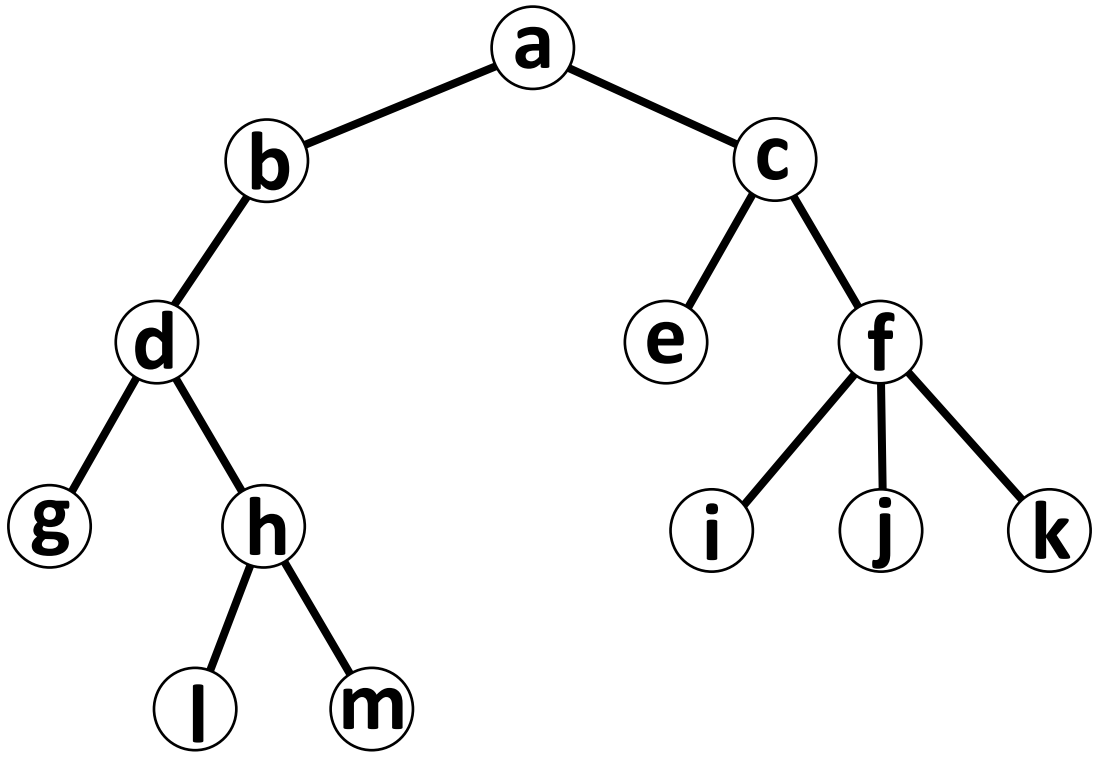

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Which one do we want?

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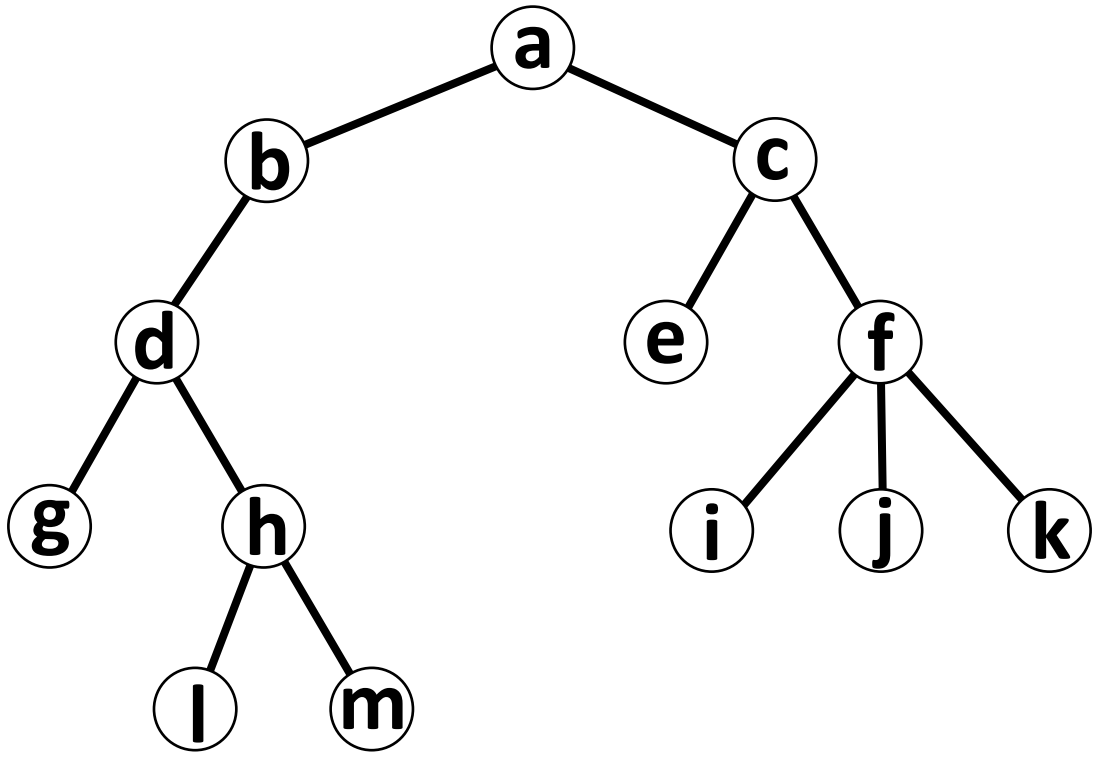
Tree Traversals



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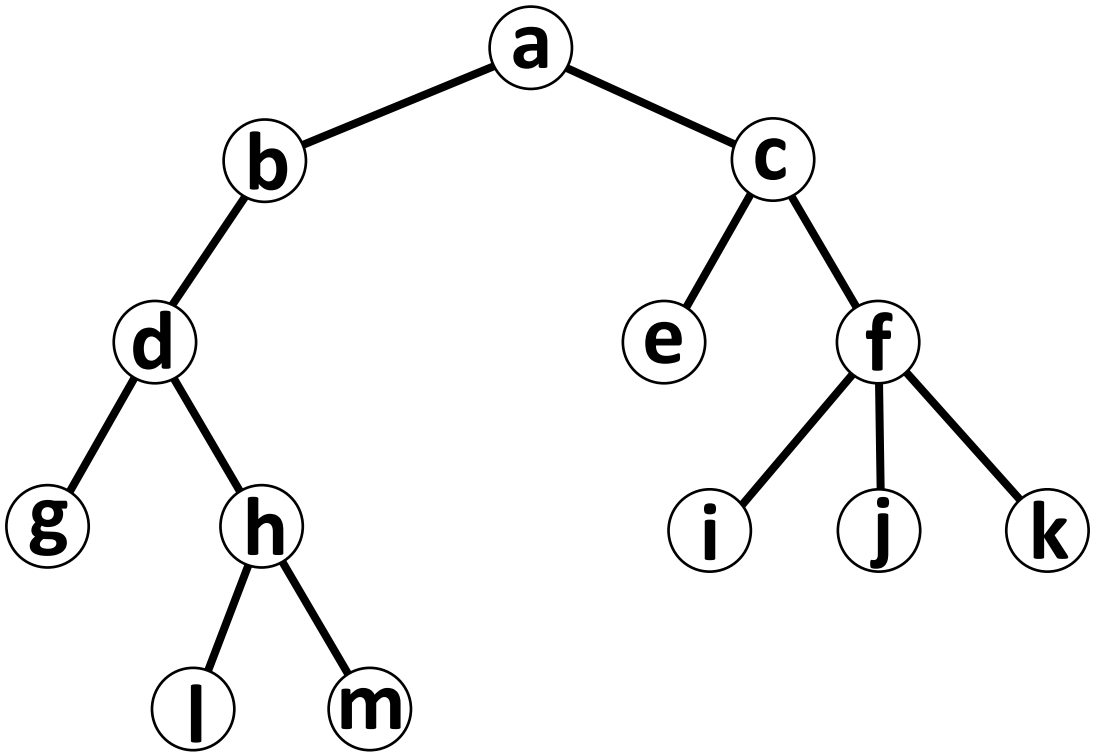
Preorder visited: a, b, d, g, h, l, m, c, e, f, i, j, k

Postorder visited: g, l, m, h, d, b, e, i, j, k, f, c, a

```
min_vc(Vertex v):
```

Vertex Cover in Trees



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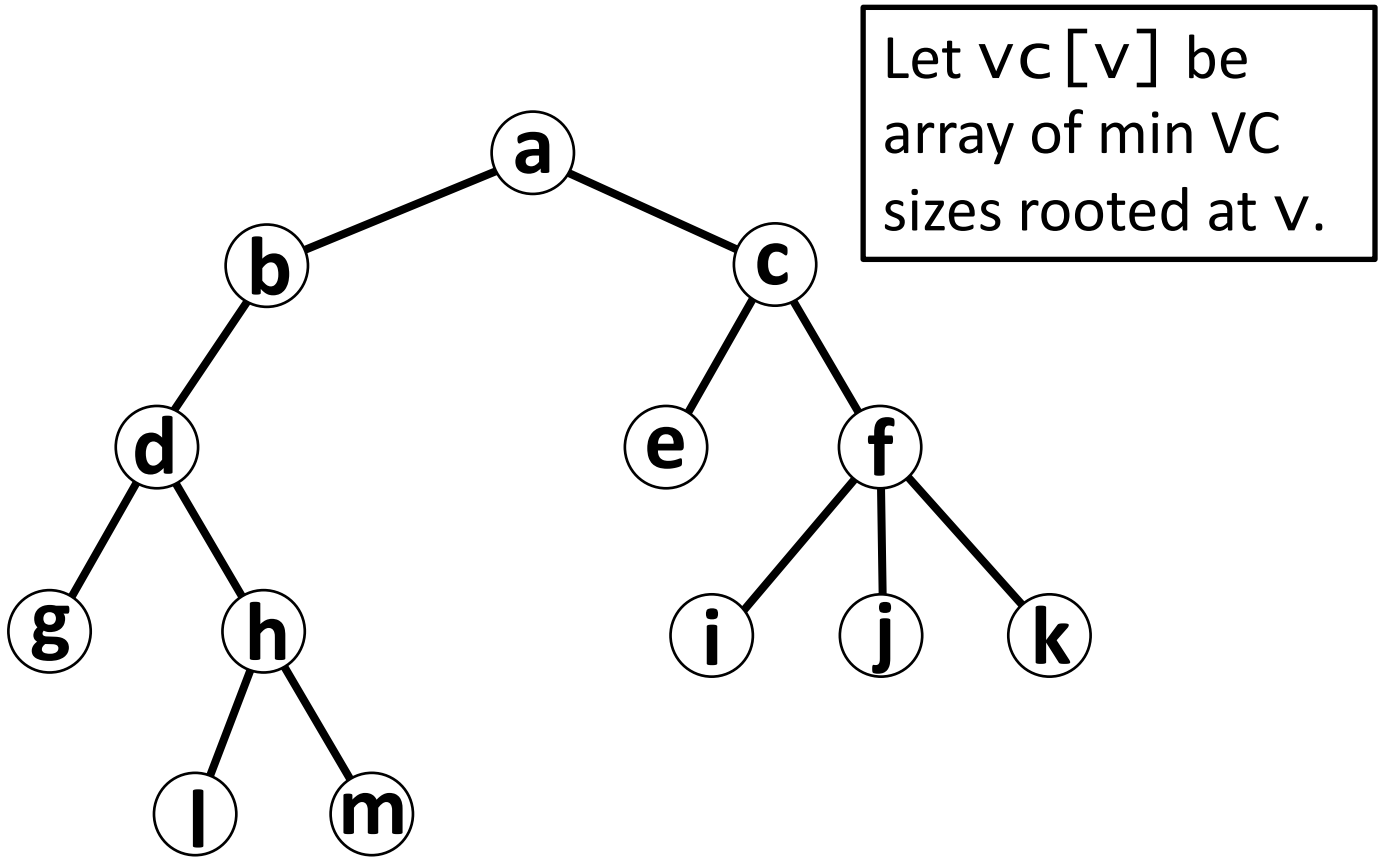

Let `vc[v]` be array of min VC sizes rooted at `v`.

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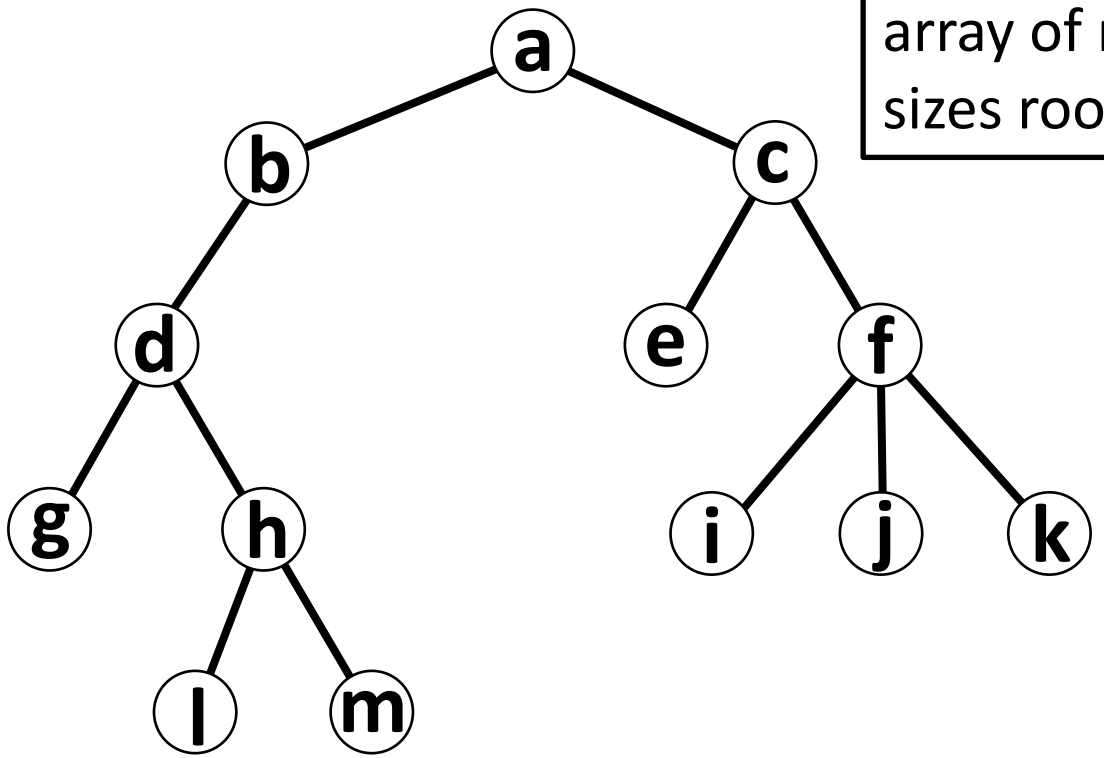
Vertex Cover in Trees

```
min_vc(Vertex v):
    if v is leaf:

    else:
```

Let `vc[v]` be array of min VC sizes rooted at `v`.



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Vertex Cover in Trees

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min_vc(Vertex v):
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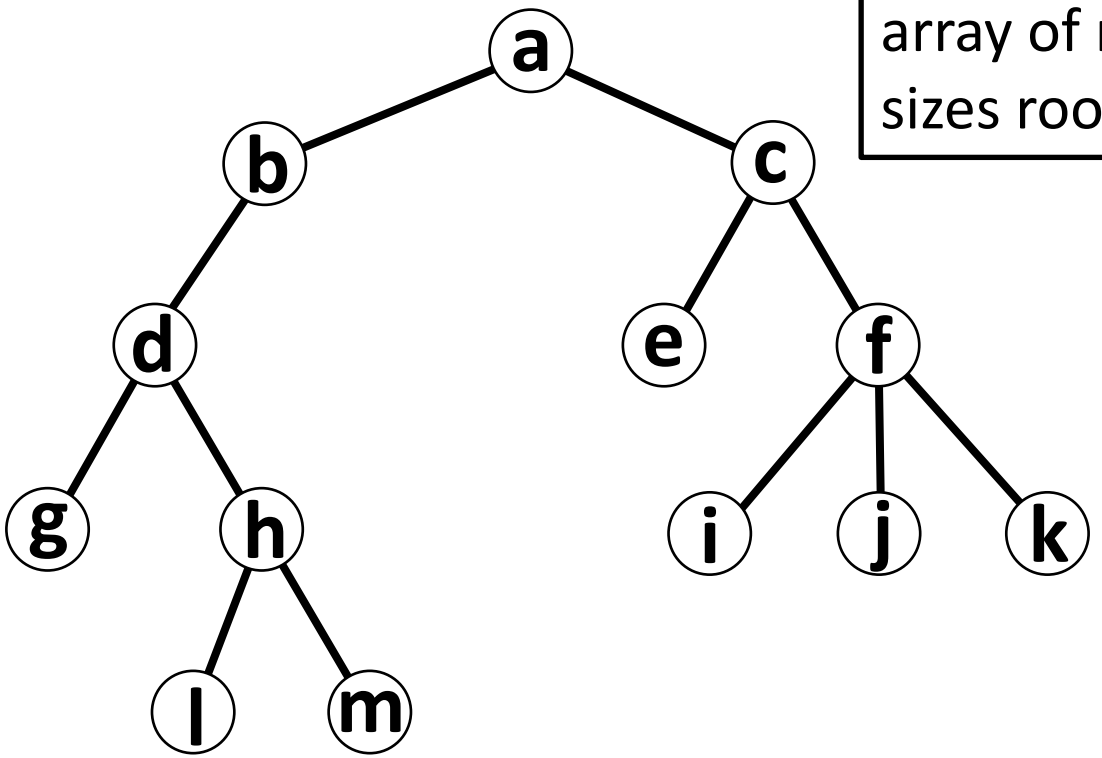
Let `vc[v]` be array of min VC sizes rooted at `v`.



$$f(v) = \min\left\{1 + \sum_{u \in \text{child}(v)} f(u),\ |\text{child}(v)| + \sum_{w \in \text{grandchild}(v)} f(w)\right\}$$

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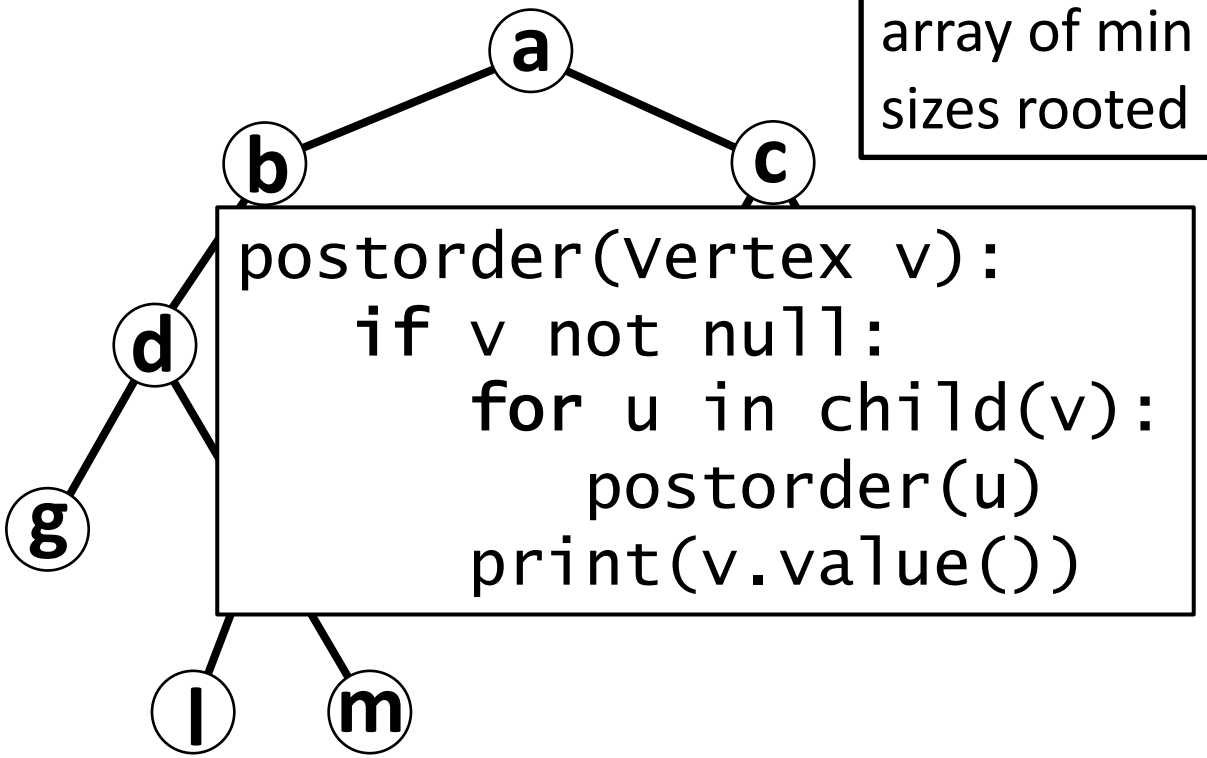


```
postorder(Vertex v):
   if v not null:
      for u in child(v):
         postorder(u)
      print(v.value())
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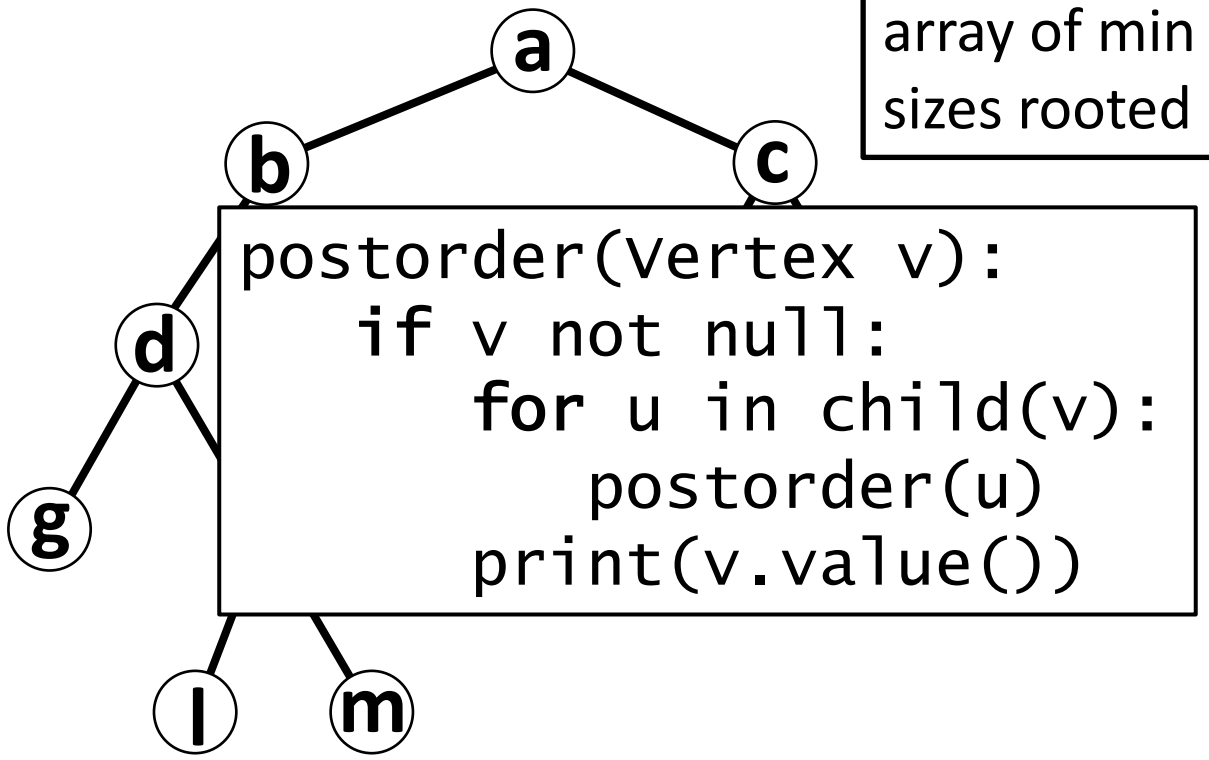


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min_vc(Vertex v):
   if v is leaf:
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      // min VC includes v

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      vc[v] = min(???, ???)
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Vertex Cover in Trees

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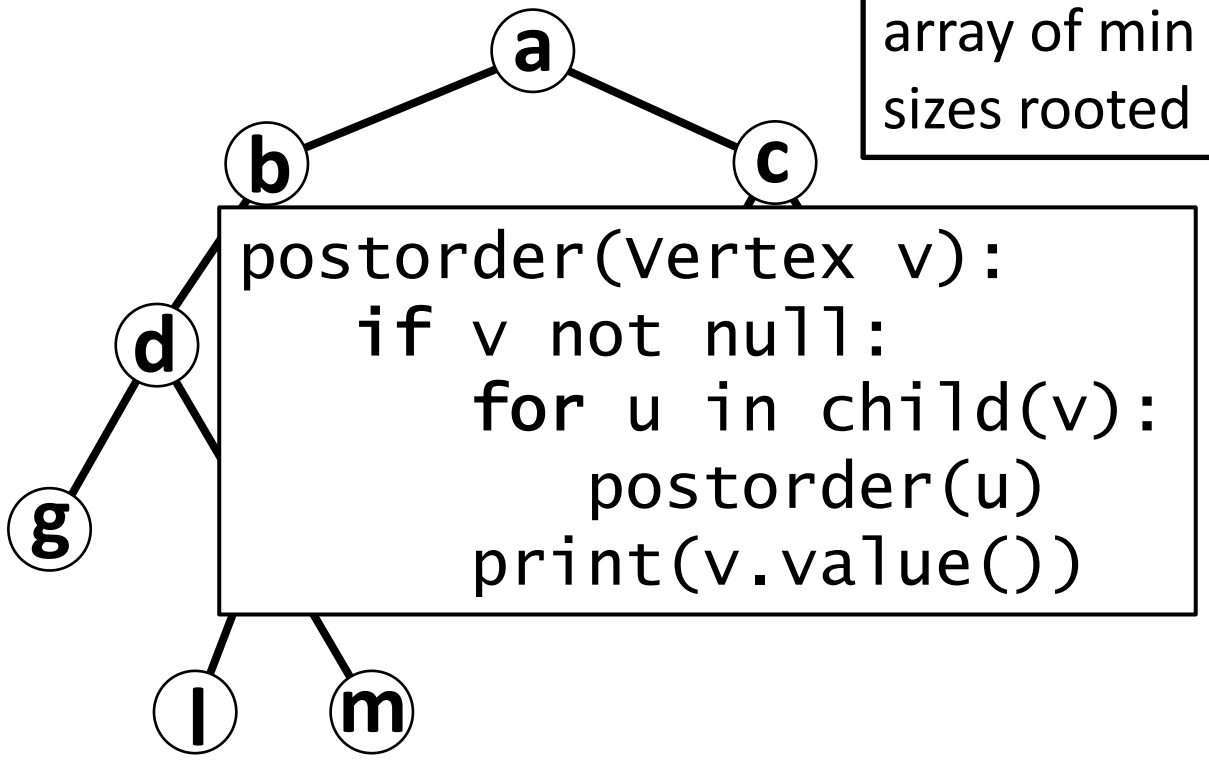


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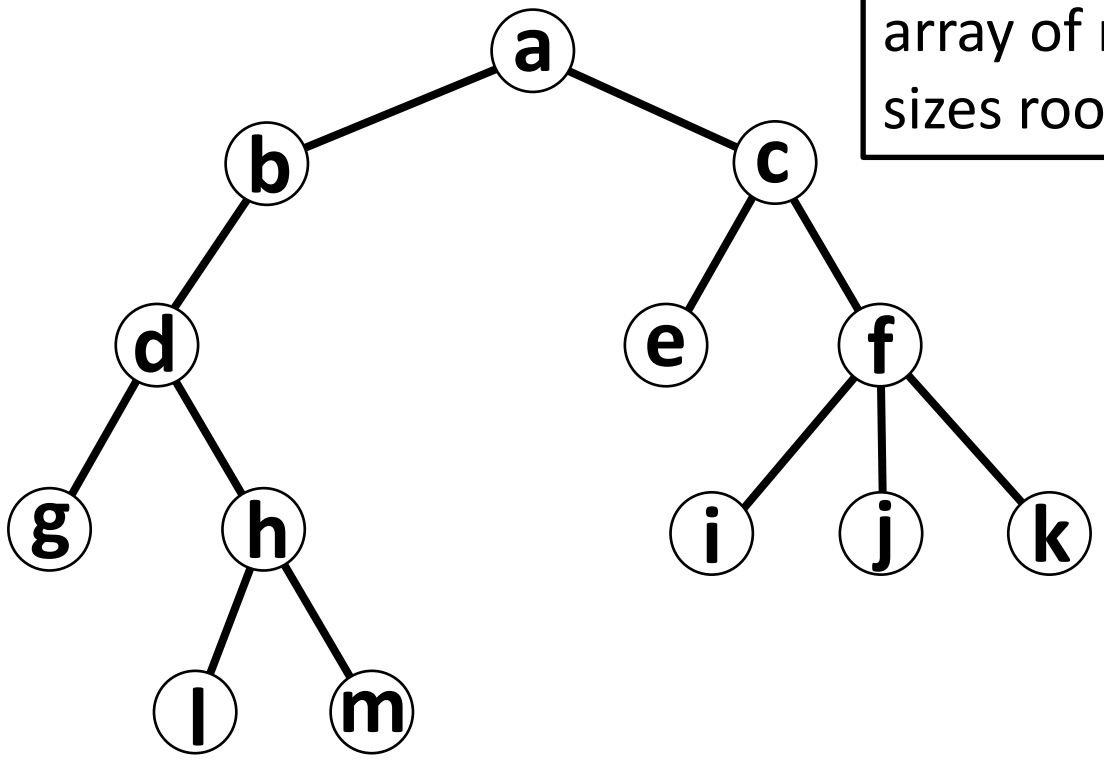

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        inc = 1



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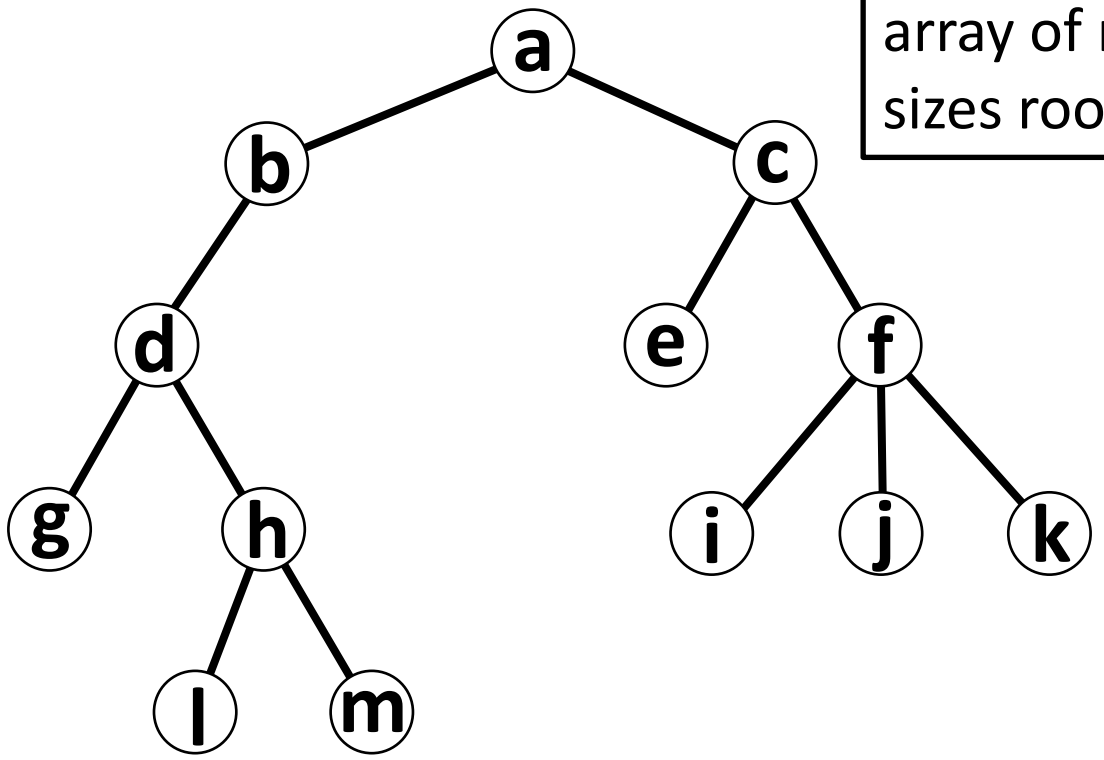
```

Vertex Cover in Trees



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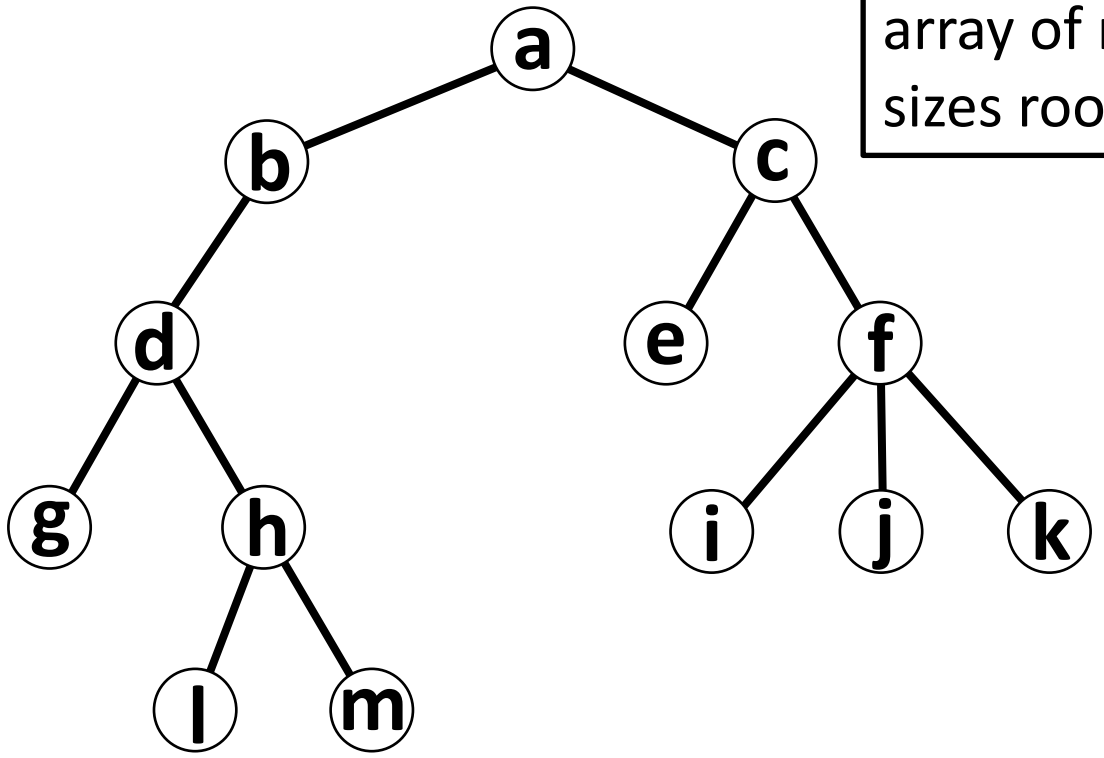
Vertex Cover in Trees



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        exc = |child(v)|

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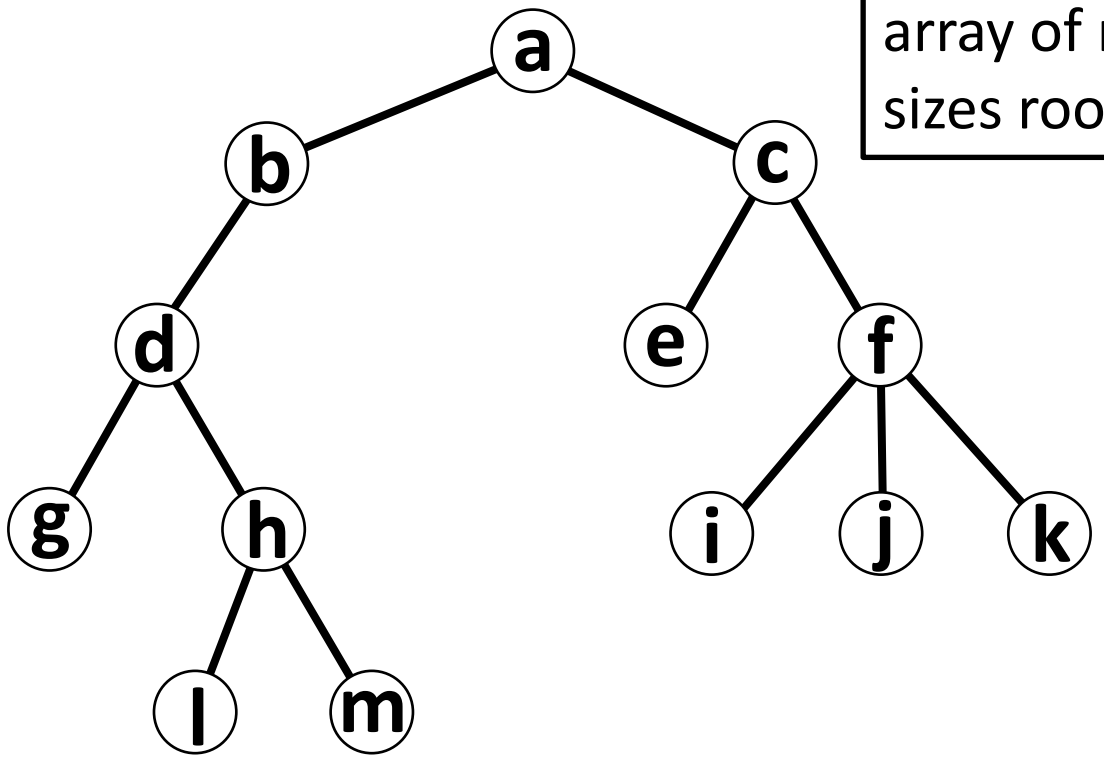
Vertex Cover in Trees

Let vc[v] be array of min VC sizes rooted at v.



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Vertex Cover in Trees

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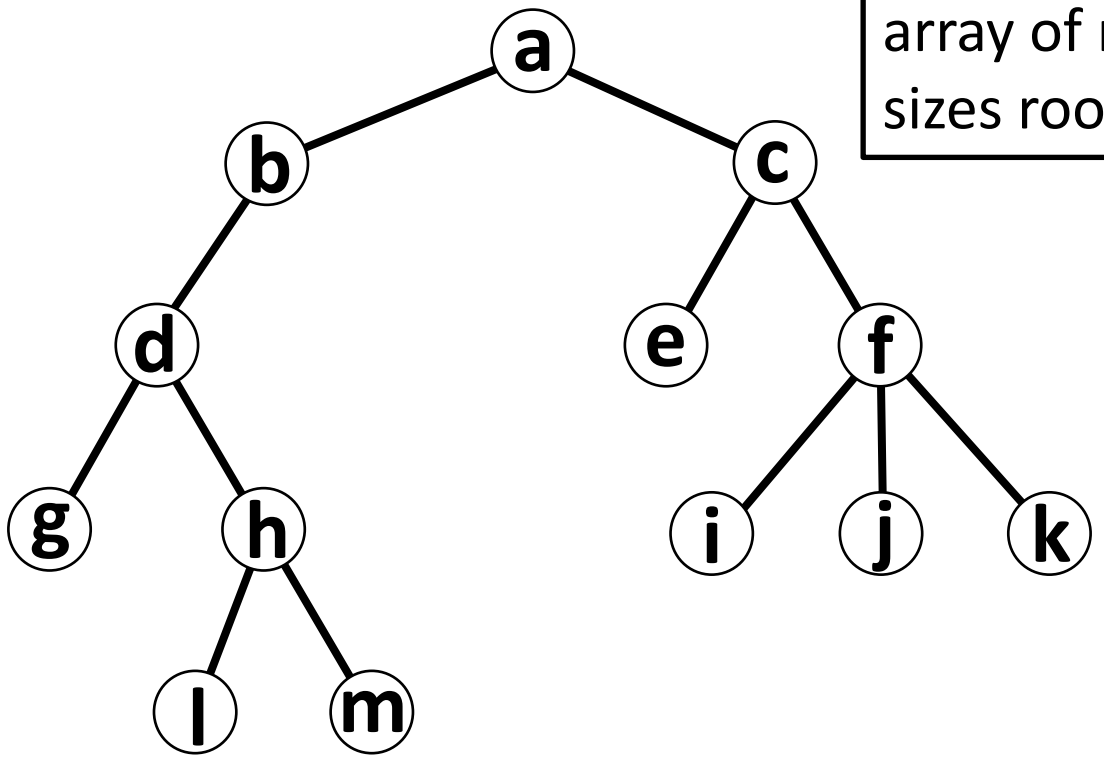

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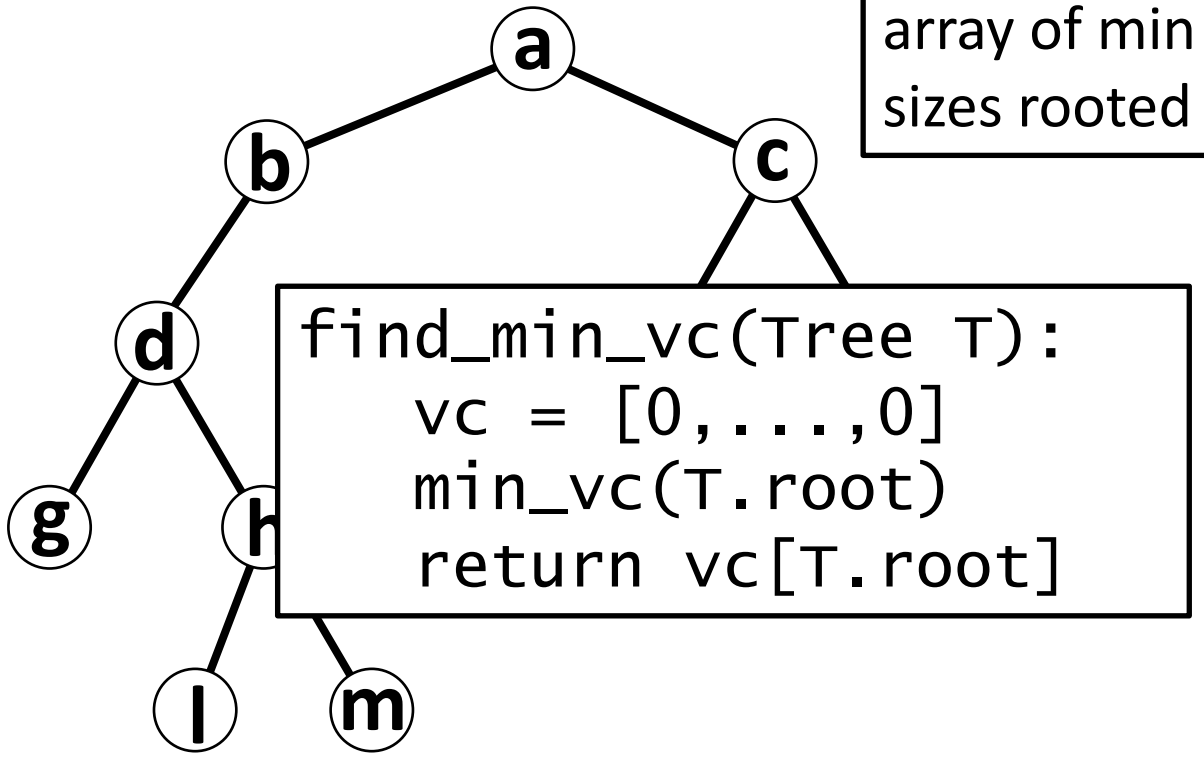

Let `vc[v]` be array of min VC sizes rooted at v.

```
find_min_vc(Tree T):
    vc = [0,...,0]
    min_vc(T.root)
    return vc[T.root]
```

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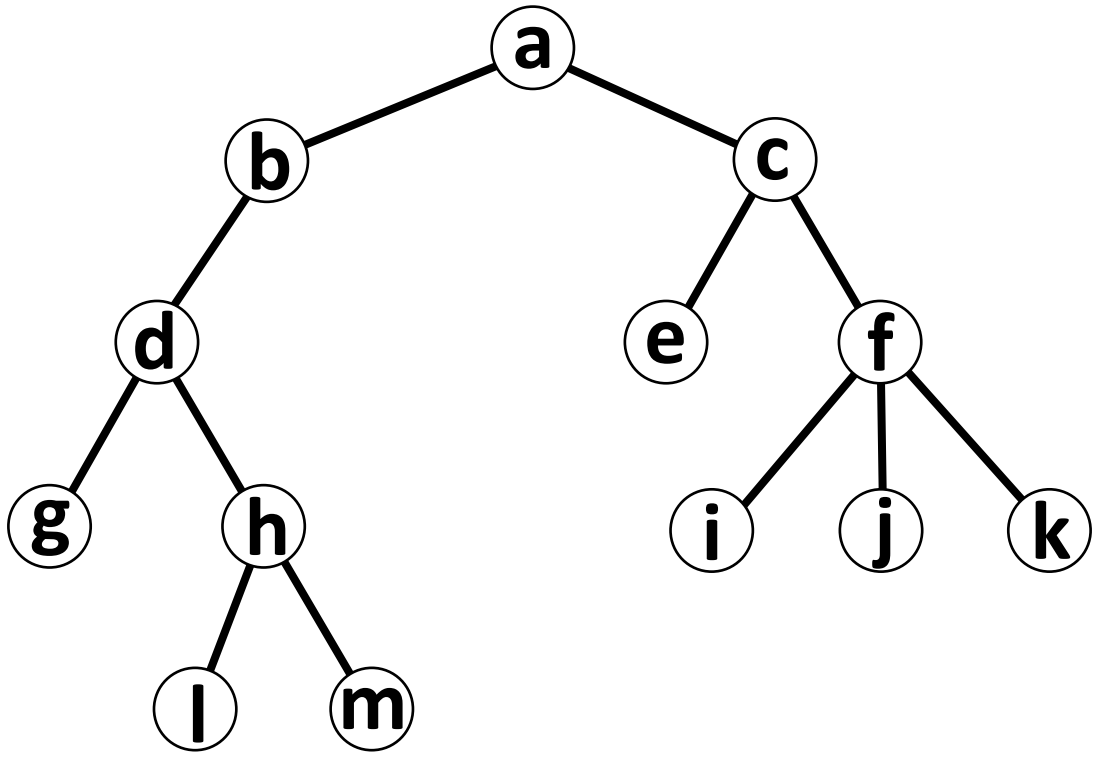

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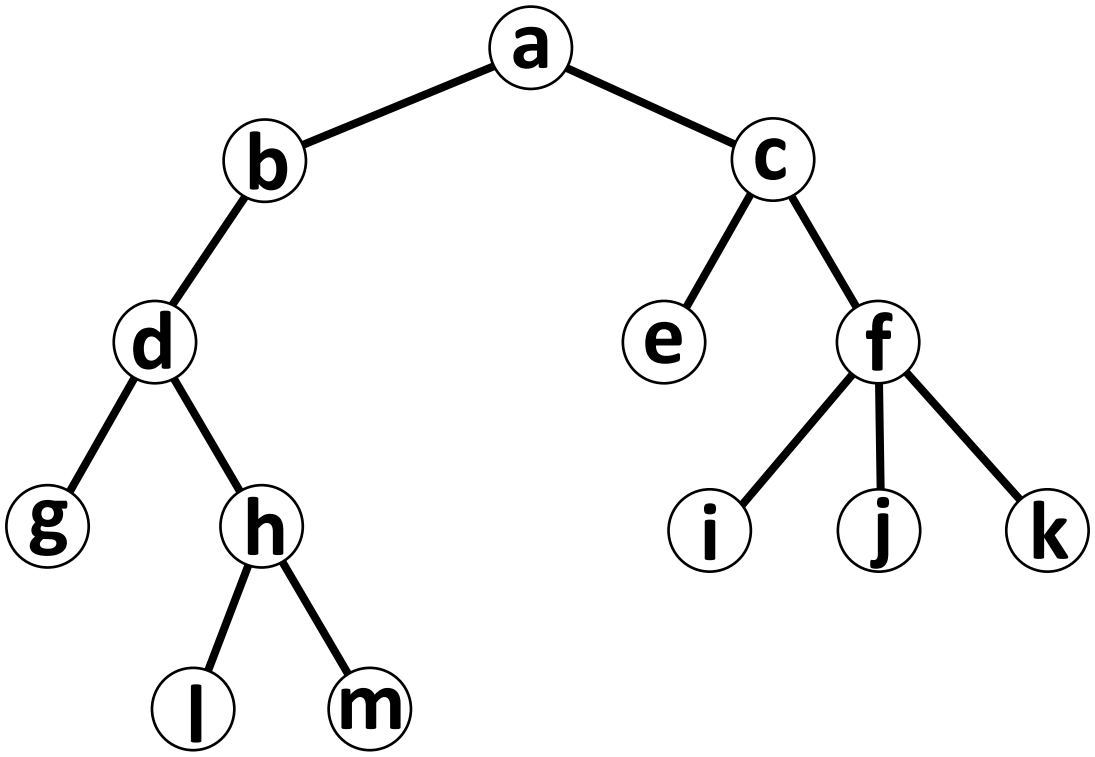

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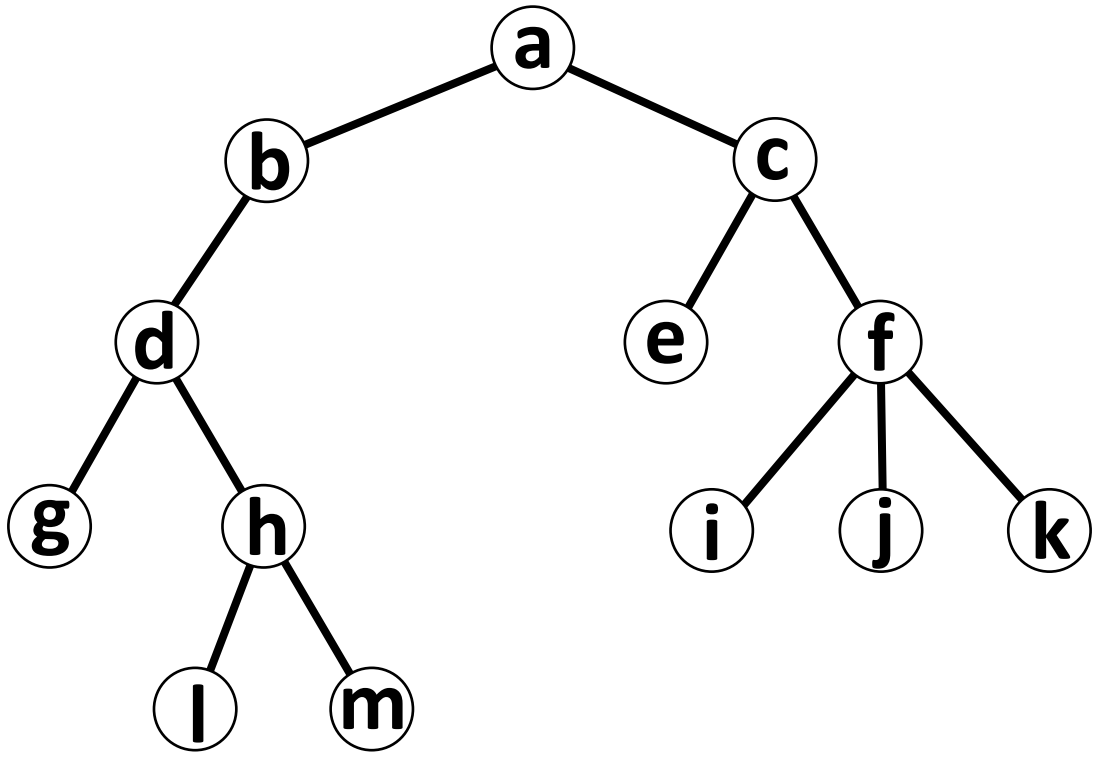

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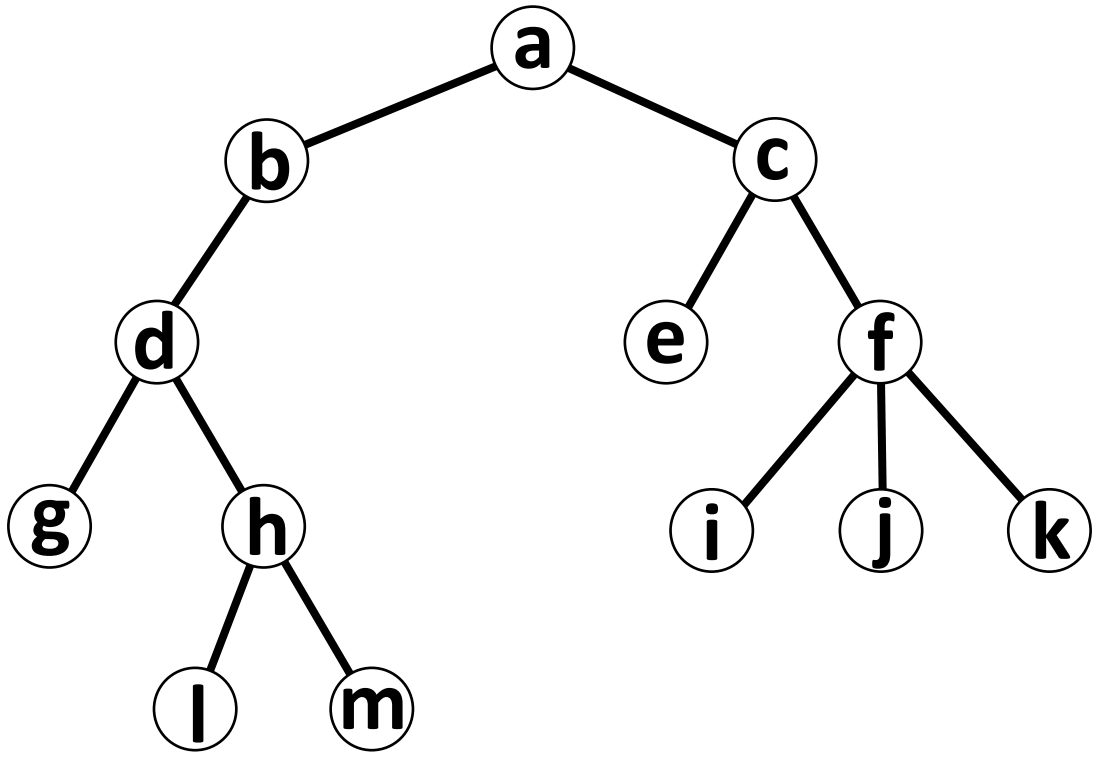
Visit each node once

Vertex Cover in Trees



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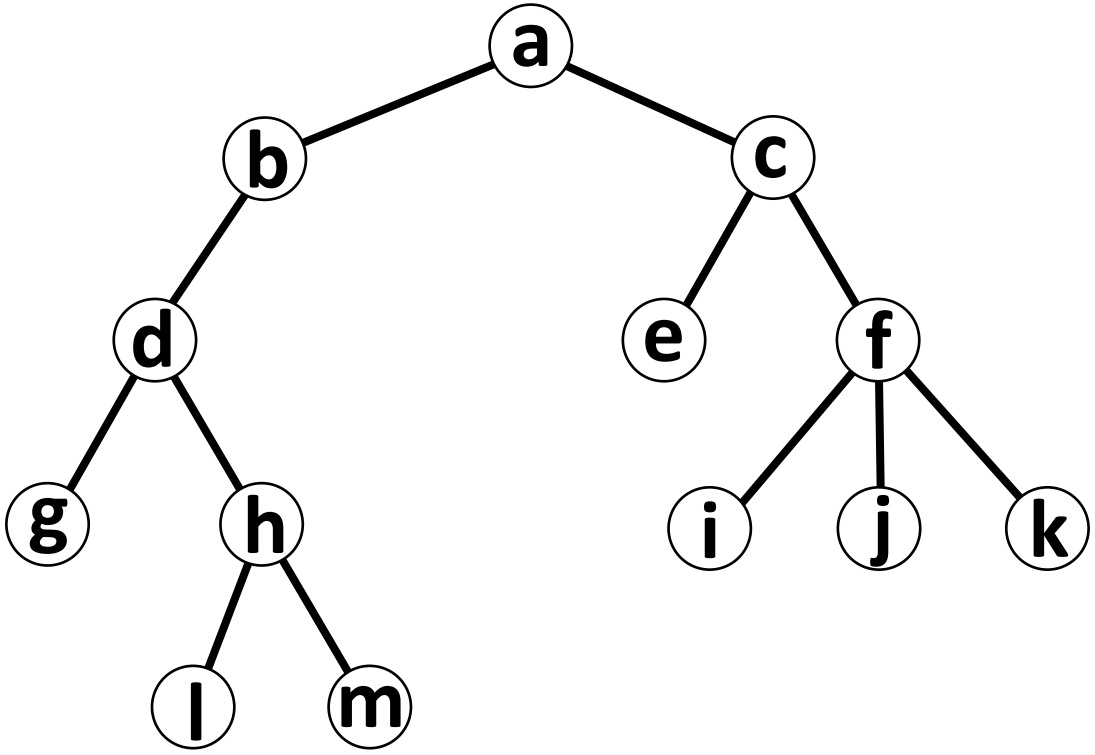

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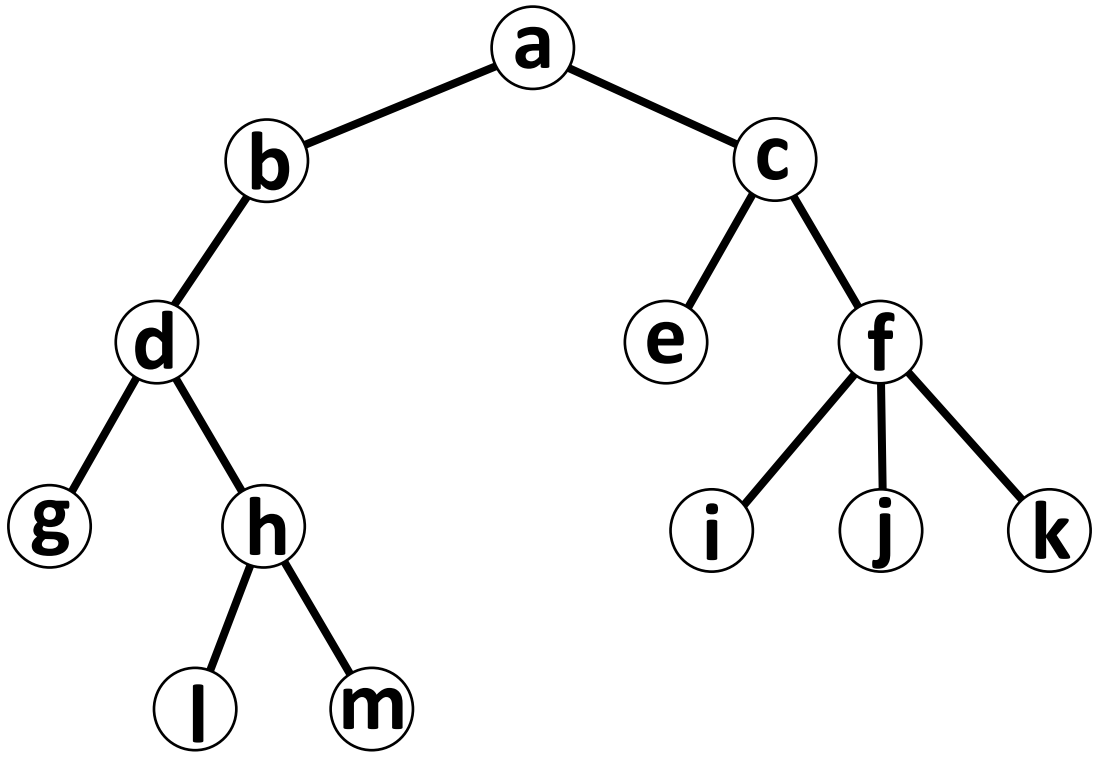

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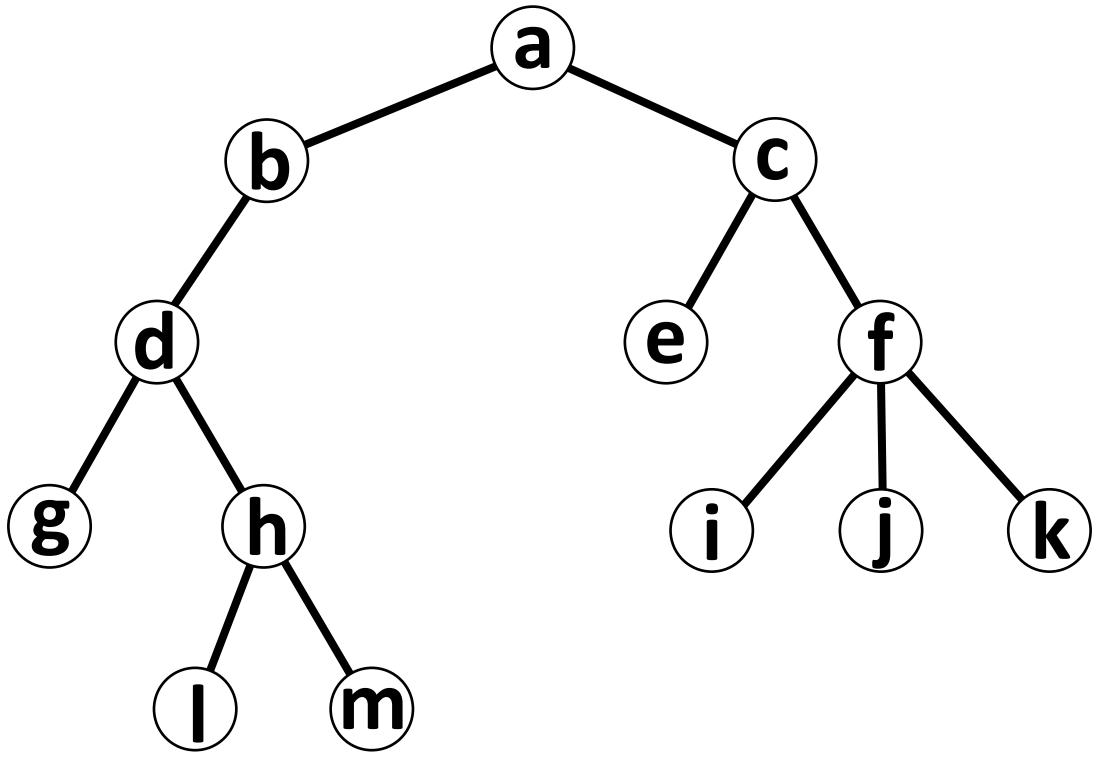

Running Time: $O(n)$

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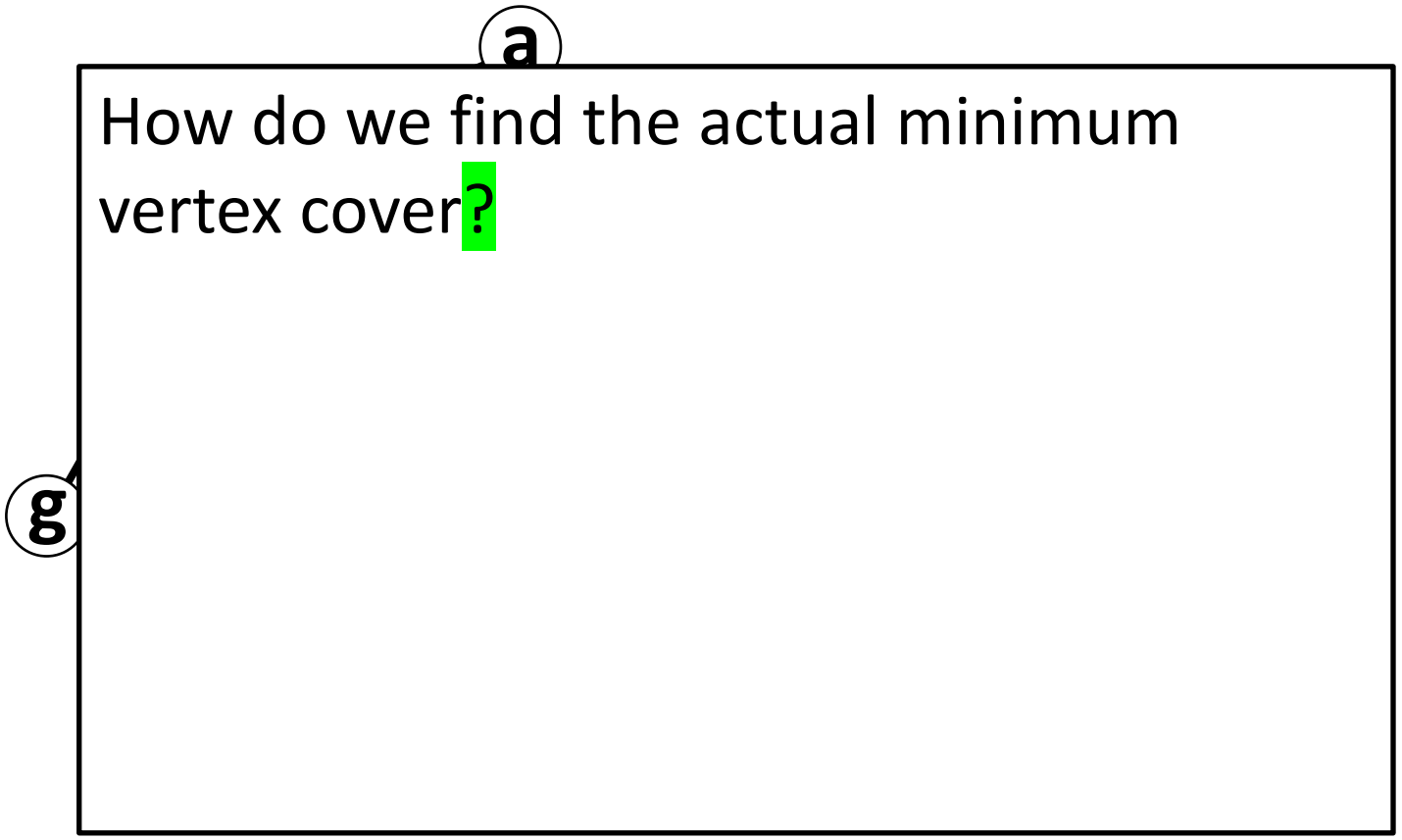

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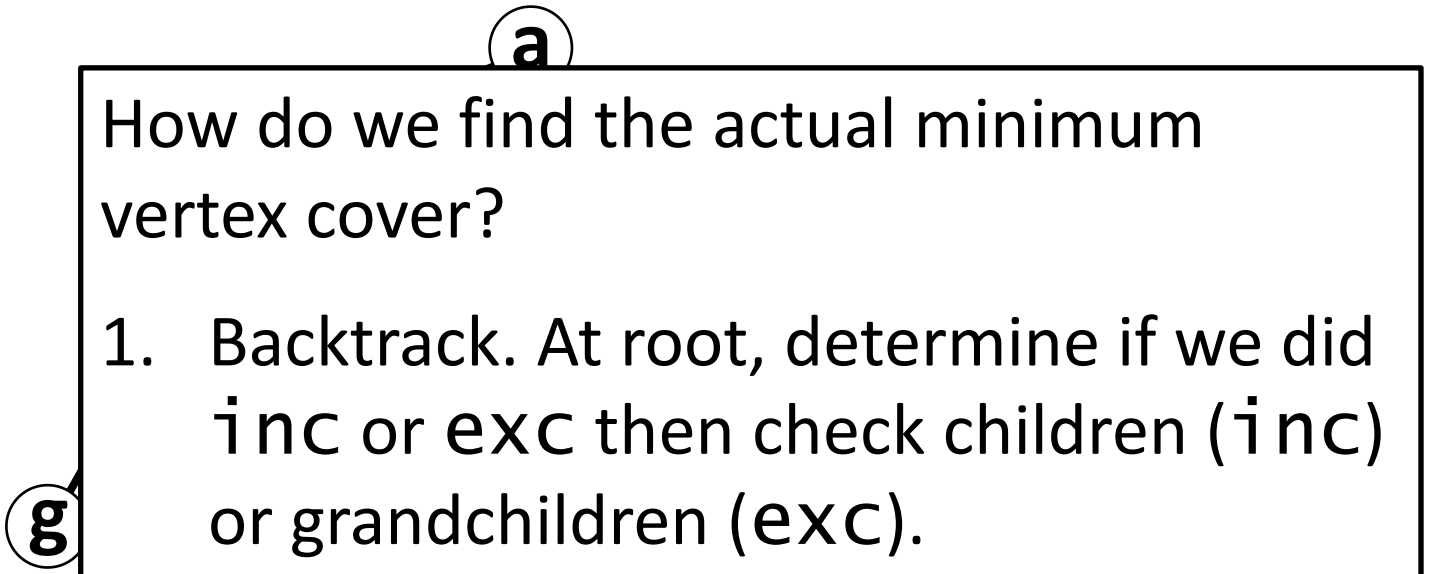

How do we find the actual minimum vertex cover?

1. Backtrack. At root, determine if we did `inc` or `exc` then check children (`inc`) or grandchildren (`exc`).
2. Save Results. When the minimum of (`inc`, `exc`) is selected, save whether `v` or children of `v` are part of the cover.

Running Time: $O(n)$

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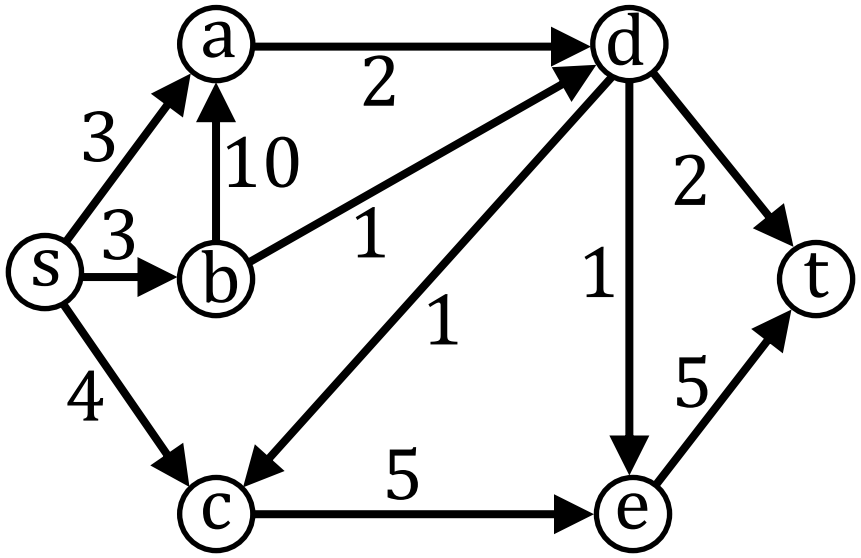
Motivation

Suppose we have a directed graph that represent an oil pipeline network. Edge weight represent pipe capacity. How much oil can we transfer from source s to sink t?



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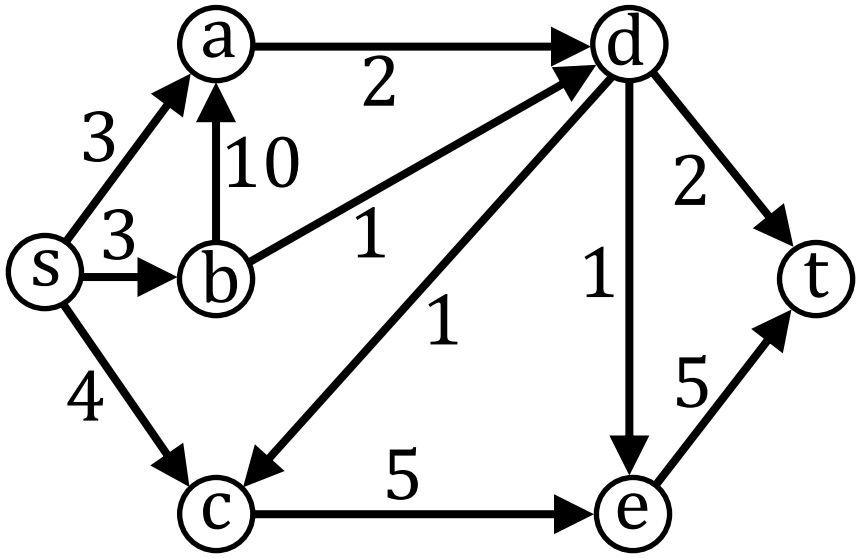

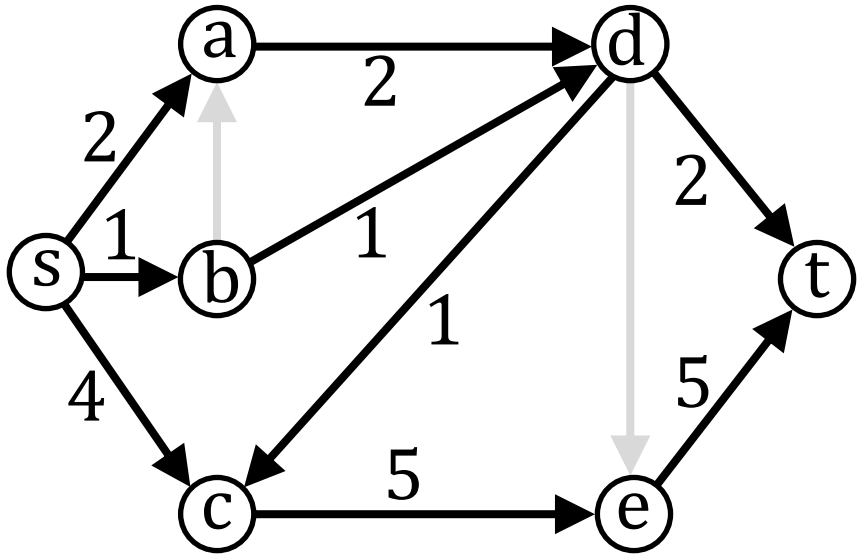

Total flow = 7