Dynamic Programming
CSCI 532

Dynamic Programing Process

1. Identify optimal substructure:
   If we have an optimal solution of size X, what does that say about a sub-solution of X?

2. Recursively define value of optimal solution:
   $A_n =$ max or min of something related to A_{n-i}

3. Compute value of optimal solution.
   Fill out table for every optimal value $\leq A_n$

4. Construct optimal solution from computed information:
   Either backtrack or make a new table.

Edit Distance

Given two strings, how many edits are needed to turn one string into another?

SNOWY vs SUNNY

mismatch

```
S – N O W Y
S U N N – Y
```

match

blank inserted

Edit Distance

Given two strings, how many edits are needed to turn one string into another?

Input:

- Two strings
- Cost function
  (e.g., non-matches = +1)

```
S – N O W Y
S U N N – Y
```

Edit Distance

Given two strings, how many edits are needed to turn one string into another?

Input:

- Two strings
- Cost function
  (e.g., non-matches = +1)

S	–	N	O	W	Y
S	U	N	N	–	Y

cost = 3

Edit Distance

Given two strings, how many edits are needed to turn one string into another?

Input:

- Two strings
- Cost function
  (e.g., non-matches = +1)

–	S	N	O	W	–	Y
S	U	N	–	–	N	Y

cost = 5

Edit Distance

Given two strings, how many edits are needed to turn one string into another?

Input:

- Two strings
- Cost function
  (e.g., non-matches = +1)

–	S	N	O	W	–	Y
S	U	N	–	–	N	Y

cost = 5

Objective:

Find cheapest possible alignment.

Edit Distance

We want to align two strings, $x = [x_1, \ldots, x_n]$ and $y = [y_1, \ldots, y_m]$.

Edit Distance

We want to align two strings, $x = [x_1, \dots, x_n]$ and $y = [y_1, \dots, y_m]$.

$E(i, j)$ = optimal cost of aligning $[x_1, \dots, x_i]$ and $[y_1, \dots, y_j]$.

Edit Distance

We want to align two strings, $x = [x_1, \ldots, x_n]$ and $y = [y_1, \ldots, y_m]$.

$E(i, j)$ = optimal cost of aligning $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$.

Can we say anything about optimal alignment of $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$?

Specifically, how must the optimal alignment end?
(three possibilities).

Edit Distance

We want to align two strings, $x = [x_1, \dots, x_n]$ and $y = [y_1, \dots, y_m]$.

$E(i, j)$ = optimal cost of aligning $[x_1, \dots, x_i]$ and $[y_1, \dots, y_j]$.

Can we say anything about optimal alignment of $[x_1, \dots, x_i]$ and $[y_1, \dots, y_j]$?

Alignment	Cost
x_i y_j	
x_i $-$	
$-$ y_j	

Edit Distance

We want to align two strings, $x = [x_1, \dots, x_n]$ and $y = [y_1, \dots, y_m]$.

$E(i, j)$ = optimal cost of aligning $[x_1, \dots, x_i]$ and $[y_1, \dots, y_j]$.

Can we say anything about optimal alignment of $[x_1, \dots, x_i]$ and $[y_1, \dots, y_j]$?

Alignment	Cost
x_i y_j	
x_i $-$	
$-$ y_j	

$$\begin{array}{cccccc} \dots & x_{i-3} & x_{i-2} & x_{i-1} & - & x_i \\ \dots & y_{j-3} & - & y_{j-2} & y_{j-1} & y_j \end{array}$$

Edit Distance

We want to align two strings, $x = [x_1, \ldots, x_n]$ and $y = [y_1, \ldots, y_m]$.

$E(i, j)$ = optimal cost of aligning $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$.

Can we say anything about optimal alignment of $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$?

Alignment	Cost
x_i y_j	
x_i $-$	
$-$ y_j	

$$\begin{array}{cccccc} \ldots & x_{i-3} & x_{i-2} & x_{i-1} & - & x_i \\ \ldots & y_{j-3} & - & y_{j-2} & y_{j-1} & y_j \end{array}$$

Cost?

Edit Distance

We want to align two strings, $x = [x_1, \ldots, x_n]$ and $y = [y_1, \ldots, y_m]$.

$E(i,j)$ = optimal cost of aligning $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$.

Can we say anything about optimal alignment of $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$?

Alignment	Cost
$\begin{matrix} x_i \\ y_j \end{matrix}$	$\{0,1\}$
$\begin{matrix} x_i \\ - \end{matrix}$	
$\begin{matrix} - \\ y_j \end{matrix}$	

$$\begin{matrix} \ldots & x_{i-3} & x_{i-2} & x_{i-1} & - & x_i \\ \ldots & y_{j-3} & - & y_{j-2} & y_{j-1} & y_j \end{matrix}$$

Cost?

Edit Distance

We want to align two strings, $x = [x_1, \ldots, x_n]$ and $y = [y_1, \ldots, y_m]$.

$E(i, j)$ = optimal cost of aligning $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$.

Can we say anything about optimal alignment of $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$?

Alignment	Cost
$\begin{matrix} x_i \\ y_j \end{matrix}$	$\{0,1\} +\ ???$
$\begin{matrix} x_i \\ - \end{matrix}$	
$\begin{matrix} - \\ y_j \end{matrix}$	

$$\begin{matrix} \ldots & x_{i-3} & x_{i-2} & x_{i-1} & - & x_i \\ \ldots & y_{j-3} & - & y_{j-2} & y_{j-1} & y_j \end{matrix}$$

Cost?

Edit Distance

We want to align two strings, $x = [x_1, \ldots, x_n]$ and $y = [y_1, \ldots, y_m]$.

$E(i, j)$ = optimal cost of aligning $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$.

Can we say anything about optimal alignment of $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$?

Alignment	Cost
x_i y_j	$\{0,1\} +$???
x_i $-$	
$-$ y_j	

$$\begin{array}{cccccc} \ldots & x_{i-3} & x_{i-2} & x_{i-1} & - & x_i \\ \ldots & y_{j-3} & - & y_{j-2} & y_{j-1} & y_j \end{array}$$

$E(i-1, j-1)$:
Alignment of $[x_1, \ldots, x_{i-1}]$ with $[y_1, \ldots, y_{j-1}]$

Edit Distance

We want to align two strings, $x = [x_1, \dots, x_n]$ and $y = [y_1, \dots, y_m]$.

$E(i, j)$ = optimal cost of aligning $[x_1, \dots, x_i]$ and $[y_1, \dots, y_j]$.

Can we say anything about optimal alignment of $[x_1, \dots, x_i]$ and $[y_1, \dots, y_j]$?

Alignment	Cost
x_i y_j	$\{0,1\} + E(i-1, j-1)$
x_i $-$	
$-$ y_j	

$$\begin{array}{cccccc} \dots & x_{i-3} & x_{i-2} & x_{i-1} & - & x_i \\ \dots & y_{j-3} & - & y_{j-2} & y_{j-1} & y_j \end{array}$$

$E(i-1, j-1)$:
Alignment of $[x_1, \dots, x_{i-1}]$ with $[y_1, \dots, y_{j-1}]$

Edit Distance

We want to align two strings, $x = [x_1, \ldots, x_n]$ and $y = [y_1, \ldots, y_m]$.

$E(i, j)$ = optimal cost of aligning $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$.

Can we say anything about optimal alignment of $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$?

Alignment	Cost
x_i y_j	$\{0,1\} + E(i-1, j-1)$
x_i $-$	
$-$ y_j	

$$\begin{array}{cccccc} \ldots & x_{i-3} & x_{i-2} & x_{i-1} & - & x_i \\ \ldots & y_{j-2} & - & y_{j-1} & y_j & - \end{array}$$

Edit Distance

We want to align two strings, $x = [x_1, \ldots, x_n]$ and $y = [y_1, \ldots, y_m]$.

$E(i, j)$ = optimal cost of aligning $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$.

Can we say anything about optimal alignment of $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$?

Alignment	Cost
x_i y_j	$\{0,1\} + E(i-1, j-1)$
x_i $-$	1
$-$ y_j	

$$
\begin{array}{cccccc}
\ldots & x_{i-3} & x_{i-2} & x_{i-1} & - & x_i \\
\ldots & y_{j-2} & - & y_{j-1} & y_j & -
\end{array}
$$

Cost = 1

Edit Distance

We want to align two strings, $x = [x_1, \dots, x_n]$ and $y = [y_1, \dots, y_m]$.

$E(i, j)$ = optimal cost of aligning $[x_1, \dots, x_i]$ and $[y_1, \dots, y_j]$.

Can we say anything about optimal alignment of $[x_1, \dots, x_i]$ and $[y_1, \dots, y_j]$?

Alignment	Cost
x_i y_j	$\{0,1\} + E(i-1, j-1)$
x_i $-$	$1 +$???
$-$ y_j	

$$\begin{array}{cccccc} \dots & x_{i-3} & x_{i-2} & x_{i-1} & - & x_i \\ \dots & y_{j-2} & - & y_{j-1} & y_j & - \end{array}$$

Cost?

Edit Distance

We want to align two strings, $x = [x_1, \dots, x_n]$ and $y = [y_1, \dots, y_m]$.

$E(i,j)$ = optimal cost of aligning $[x_1, \dots, x_i]$ and $[y_1, \dots, y_j]$.

Can we say anything about optimal alignment of $[x_1, \dots, x_i]$ and $[y_1, \dots, y_j]$?

Alignment	Cost
$\begin{matrix} x_i \\ y_j \end{matrix}$	$\{0,1\} + E(i-1, j-1)$
$\begin{matrix} x_i \\ - \end{matrix}$	$1 + E(i-1, j)$
$\begin{matrix} - \\ y_j \end{matrix}$	

$$\begin{matrix} \dots & x_{i-3} & x_{i-2} & x_{i-1} & - & x_i \\ \dots & y_{j-2} & - & y_{j-1} & y_j & - \end{matrix}$$

$E(i-1, j)$:
Alignment of $[x_1, \dots, x_{i-1}]$ with $[y_1, \dots, y_j]$

Edit Distance

We want to align two strings, $x = [x_1, \dots, x_n]$ and $y = [y_1, \dots, y_m]$.

$E(i, j)$ = optimal cost of aligning $[x_1, \dots, x_i]$ and $[y_1, \dots, y_j]$.

Can we say anything about optimal alignment of $[x_1, \dots, x_i]$ and $[y_1, \dots, y_j]$?

Alignment	Cost
x_i y_j	$\{0,1\} + E(i-1, j-1)$
x_i $-$	$1 + E(i-1, j)$
$-$ y_j	$1 + E(i, j-1)$

$$\begin{array}{cccccc} \dots & x_{i-2} & x_{i-1} & x_i & - & - \\ \dots & y_{j-3} & - & y_{j-2} & y_{j-1} & y_j \end{array}$$

Edit Distance

We want to align two strings, $x = [x_1, \ldots, x_n]$ and $y = [y_1, \ldots, y_m]$.

$E(i, j)$ = optimal cost of aligning $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$.

Can we say anything about optimal alignment of $[x_1, \ldots, x_i]$ and $[y_1, \ldots, y_j]$?

Alignment	Cost
$\begin{matrix} x_i \\ y_j \end{matrix}$	$\{0,1\} + E(i-1, j-1)$
$\begin{matrix} x_i \\ - \end{matrix}$	$1 + E(i-1, j)$
$\begin{matrix} - \\ y_j \end{matrix}$	$1 + E(i, j-1)$

$E(i, j) = ??$

Edit Distance

We want to align two strings, $x = [x_1, \dots, x_n]$ and $y = [y_1, \dots, y_m]$.

$E(i,j)$ = optimal cost of aligning $[x_1, \dots, x_i]$ and $[y_1, \dots, y_j]$.

Can we say anything about optimal alignment of $[x_1, \dots, x_i]$ and $[y_1, \dots, y_j]$?

Alignment	Cost
$\begin{matrix} x_i \\ y_j \end{matrix}$	$\{0,1\} + E(i-1, j-1)$
$\begin{matrix} x_i \\ - \end{matrix}$	$1 + E(i-1, j)$
$\begin{matrix} - \\ y_j \end{matrix}$	$1 + E(i, j-1)$

$$E(i,j) = \min \begin{cases} \mathrm{diff}(i,j) + E(i-1, j-1) \\ 1 + E(i-1, j) \\ 1 + E(i, j-1) \end{cases}$$

$$\text{where } \mathrm{diff}(i,j) = \begin{cases} 0, & x_i = y_j \\ 1, & x_i \neq y_j \end{cases}$$

Edit Distance

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i] = y[j] \\ 1, & x[i] \neq y[j] \end{cases}$$

How should we find $E(n,m)$?

Edit Distance

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i] = y[j] \\ 1, & x[i] \neq y[j] \end{cases}$$

How should we find $E(n,m)$?

- Find all the other $E(i,j)$'s for $i < n, j < m$.

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0					
1	S						
2	N						
3	O						
4	W						
5	Y						

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i] = y[j] \\ 1, & x[i] \neq y[j] \end{cases}$$

How should we find $E(n,m)$?

- Find all the other $E(i,j)$'s for $i < n, j < m$.
- Store intermediate $E(i,j)$ values in a 2d array.

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0					
1	S						
2	N						
3	O					●	
4	W						
5	Y						

$E(3,4)$

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i] = y[j] \\ 1, & x[i] \neq y[j] \end{cases}$$

Where can we start?

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0					
1	S						
2	N						
3	O						
4	W						
5	Y						

$E(3,4)$

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i] = y[j] \\ 1, & x[i] \neq y[j] \end{cases}$$

Where can we start?

$E(0,1)$ or $E(1,0)$

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0					
1	S						
2	N						
3	O						
4	W						
5	Y						

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i]=y[j] \\ 1, & \text{otherwise} \end{cases}$$

$$E(0,1) = \min \begin{cases} E(-1,1)+1 \\ E(0,0)+1 \\ E(-1,0)+1 \end{cases} = ?$$

Edit Distance

i	j	0	1	2	3	4	5
			S	U	N	N	Y
0		0					
1	S						
2	N						
3	O						
4	W						
5	Y						

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i] = y[j] \\ 1, & \text{otherwise} \end{cases}$$

$$E(0,1) = \min \begin{cases} \cancel{E(-1,1)+1} \\ E(0,0)+1 \quad = ? \\ \cancel{E(-1,0)+1} \end{cases}$$

Edit Distance

i \ j		0	1	2	3	4	5
			S	U	N	N	Y
0		0	1				
1	S						
2	N						
3	O						
4	W						
5	Y						

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i] = y[j] \\ 1, & \text{otherwise} \end{cases}$$

$$E(0,1) = \min \begin{cases} \cancel{E(-1,1)+1} \\ E(0,0)+1 \quad = 1 \\ \cancel{E(-1,0)+1} \end{cases}$$

Corresponds to aligning S from SUNNY with inserted – from SNOWY.

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1				
1	S						
2	N						
3	O						
4	W						
5	Y						

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i]=y[j] \\ 1, & \text{otherwise} \end{cases}$$

$$E(1,1) = \min \begin{cases} E(0,1)+1 \\ E(1,0)+1 =? \\ E(0,0)+0 \end{cases}$$

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1				
1	S						
2	N						
3	O						
4	W						
5	Y						

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i]=y[j] \\ 1, & \text{otherwise} \end{cases}$$

$$E(1,1) = \min \begin{cases} E(0,1)+1 \\ \boldsymbol{E(1,0)}+1 =? \\ E(0,0)+0 \end{cases}$$

Not calculated yet!

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1				
1	S						
2	N						
3	O						
4	W						
5	Y						

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i]=y[j] \\ 1, & \text{otherwise} \end{cases}$$

Need upper left-hand corner filled out before we can progress.

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2			
1	S						
2	N						
3	O						
4	W						
5	Y						

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i]=y[j] \\ 1, & \text{otherwise} \end{cases}$$

$$E(0,2) = \min \begin{cases} E(-1,2)+1 \\ E(0,1)+1 \\ E(-1,1)+1 \end{cases} = 2$$

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2			
1	S	1					
2	N						
3	O						
4	W						
5	Y						

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i] = y[j] \\ 1, & \text{otherwise} \end{cases}$$

$$E(1,0) = \min \begin{cases} E(0,0)+1 \\ E(1,-1)+1 = 1 \\ E(0,-1)+1 \end{cases}$$

Edit Distance

i \ j		0	1	2	3	4	5
			S	U	N	N	Y
0		0	1	2			
1	S	1	0				
2	N						
3	O						
4	W						
5	Y						

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i]=y[j] \\ 1, & \text{otherwise} \end{cases}$$

$$E(1,1) = \min \begin{cases} E(0,1)+1 \\ E(1,0)+1 = 0 \\ E(0,0)+0 \end{cases}$$

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i] = y[j] \\ 1, & \text{otherwise} \end{cases}$$

Running Time?

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i] = y[j] \\ 1, & \text{otherwise} \end{cases}$$

Running Time?

Fill out $n \times m$ table with constant operations: $O(nm)$

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	**3**

Edit distance = **3**.

How can we recreate the actual alignment?

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	**3**

Edit distance = **3**.

How can we recreate the actual alignment?

Backtracking.

Ask the question: "How did we get here?"

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	**3**

How did we get to $E(5,5)$?

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	← **3**

How did we get to $E(5,5)$?

From $E(5,4)$?

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	← **3**

How did we get to $E(5,5)$?

From $E(5,4)$? – No. Can never go down in cost.

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	**3**

How did we get to $E(5,5)$?

From $E(5,4)$? – No. Can never go down in cost.

From $E(4,5)$?

Edit Distance

i \ j		0	1	2	3	4	5
			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	**3**

How did we get to $E(5,5)$?

From $E(5,4)$? – No. Can never go down in cost.

From $E(4,5)$? – No. Need +1 to move that direction.

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	**3**

How did we get to $E(5,5)$?

From $E(5,4)$? – No. Can never go down in cost.

From $E(4,5)$? – No. Need +1 to move that direction.

From $E(4,4)$?

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

How did we get to $E(5,5)$?

From $E(5,4)$? – No. Can never go down in cost.

From $E(4,5)$? – No. Need +1 to move that direction.

From $E(4,4)$? – Yes. Match Y's.

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\mathrm{diff}(i,j) \end{cases}$$

Edit Distance

Continuing the process yields all of the optimal solutions.

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

Diagonal move indicates ?

Vertical move indicates ?

Horizontal move indicates ?

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\mathrm{diff}(i,j) \end{cases}$$

Edit Distance

i \ j		0	1	2	3	4	5
			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

Continuing the process yields all of the optimal solutions.

Diagonal move indicates match.

Vertical move indicates ?

Horizontal move indicates ?

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

Edit Distance

i	j	0	1	2	3	4	5
			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

S	–	N	O	W	Y
S	U	N	N	–	Y

Continuing the process yields all of the optimal solutions.

Diagonal move indicates match.

Vertical move indicates space inserted in j.

Horizontal move indicates ?

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\mathrm{diff}(i,j) \end{cases}$$

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

S	–	N	O	W	Y
S	U	N	N	–	Y

Continuing the process yields all of the optimal solutions.

Diagonal move indicates match.

Vertical move indicates space inserted in j.

Horizontal move indicates space inserted in i.

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\mathrm{diff}(i,j) \end{cases}$$

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

Diagonal move indicates match.

Vertical move indicates space inserted in j.

Horizontal move indicates space inserted in i.

S – N O W Y

S U N N – Y

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

Diagonal move indicates match.

Vertical move indicates space inserted in j.

Horizontal move indicates space inserted in i.

Alignment?

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

Diagonal move indicates match.

Vertical move indicates space inserted in j.

Horizontal move indicates space inserted in i.

S – N O W Y

S U N – N Y

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

Diagonal move indicates match.

Vertical move indicates space inserted in j.

Horizontal move indicates space inserted in i.

S N O W Y

S U N N Y

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

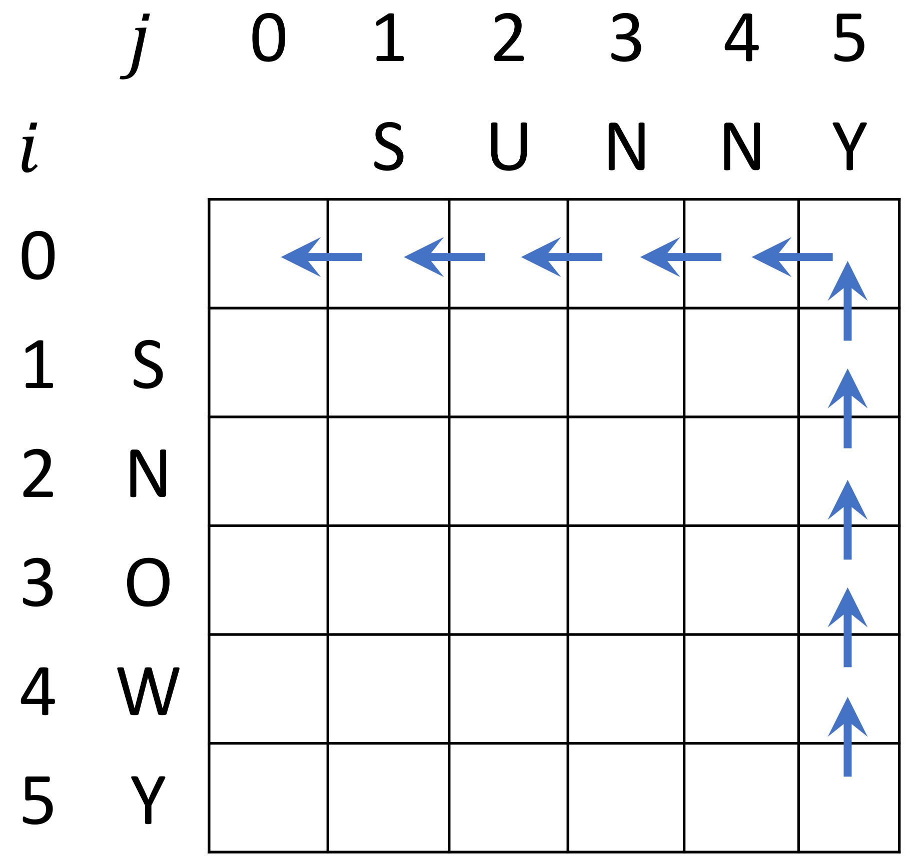
Diagonal move indicates match.

Vertical move indicates space inserted in j.

Horizontal move indicates space inserted in i.

Running time for backtracking?

Edit Distance



Diagonal move indicates match.

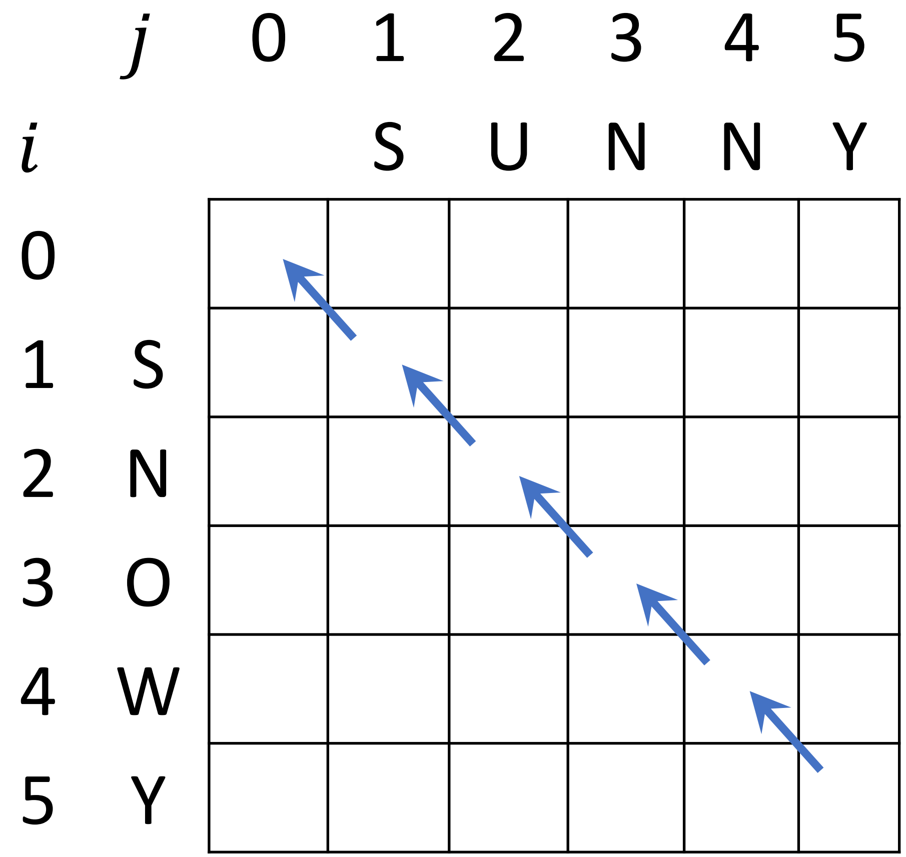
Running time for backtracking?

At worst, only match to insertions:

$$O(2\max(n,m)) = O(\max(n,m))$$

–	–	–	–	–	S	N	O	W	Y
S	U	N	N	Y	–	–	–	–	–

Edit Distance



Diagonal move indicates match.

Running time for backtracking?

At worst, only match to insertions:

$$O(2\max(n,m)) = O(\max(n,m))$$

At best, no insertions in larger string:

$$O(\max(n,m))$$

S N O W Y

S U N N Y

Edit Distance

i		j: 0	1	2	3	4	5
			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

Diagonal move indicates match.

Running time for backtracking?

$$O(\max(n, m))$$

At worst, only match to insertions:

$$O(2\max(n, m)) = O(\max(n, m))$$

At best, no insertions in larger string:

$$O(\max(n, m))$$

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i]=y[j] \\ 1, & \text{otherwise} \end{cases}$$

Space Requirements?

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
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Space Requirements?

$n \times m$ table: $O(nm)$

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

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$$\text{diff}(i,j) = \begin{cases} 0, & x[i]=y[j] \\ 1, & \text{otherwise} \end{cases}$$

Space Requirements?

$n \times m$ table: $O(nm)$

Can we do it in $O(\min(n,m))$ space?

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

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0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
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3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

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$$\text{diff}(i,j) = \begin{cases} 0, & x[i] = y[j] \\ 1, & \text{otherwise} \end{cases}$$

Space Requirements?

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Can we do it in $O(\min(n,m))$ space?

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

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Space Requirements?

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Edit Distance

i \ j		0	1	2	3	4	5
			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
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Space Requirements?

$n \times m$ table: $O(nm)$

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Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
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Space Requirements?

$n \times m$ table: $O(nm)$

Can we do it in $O(\min(n,m))$ space?

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

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$$\text{diff}(i,j) = \begin{cases} 0, & x[i] = y[j] \\ 1, & \text{otherwise} \end{cases}$$

Space Requirements?

$n \times m$ table: $O(nm)$

Can we do it in $O(\min(n,m))$ space?

Edit Distance

j		0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

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$$\text{diff}(i,j) = \begin{cases} 0, & x[i] = y[j] \\ 1, & \text{otherwise} \end{cases}$$

Space Requirements?

$n \times m$ table: $O(nm)$

Can we do it in $O(\min(n,m))$ space?

Backtracking?

Edit Distance

	j	0	1	2	3	4	5
i			S	U	N	N	Y
0		0	1	2	3	4	5
1	S	1	0	1	2	3	4
2	N	2	1	1	1	2	3
3	O	3	2	2	2	2	3
4	W	4	3	3	3	3	3
5	Y	5	4	4	4	4	3

$$E(i,j) = \min \begin{cases} E(i-1,j)+1 \\ E(i,j-1)+1 \\ E(i-1,j-1)+\text{diff}(i,j) \end{cases}$$

$$\text{diff}(i,j) = \begin{cases} 0, & x[i] = y[j] \\ 1, & \text{otherwise} \end{cases}$$

Space Requirements?

$n \times m$ table: $O(nm)$

Can we do it in $O(\min(n,m))$ space?

Backtracking?

Ill advised.