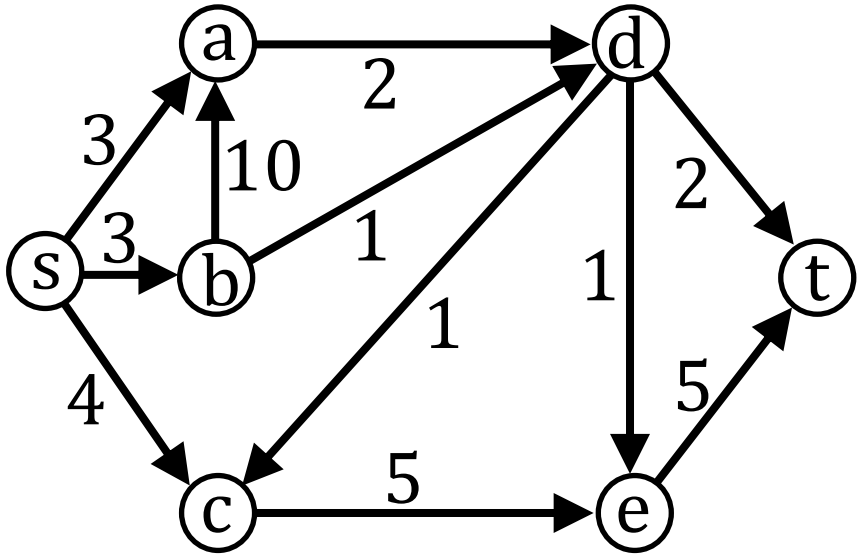
Flow Networks
CSCI 532

Test 1 Logistics

1. During class on Thursday 2/12.
2. You can bring your book and any notes you would like, but no electronic devices.
3. You may assume anything proven in class or on homework.
4. Three questions (12 points):
    1) Prove/disprove something related to MSTs (5 points).
    2) Identify recursive optimal substructure function for graph problem (5 points).
    3) Prove/disprove greedy algorithm's optimality (2 points).

Motivation

Suppose we have a directed graph that represent an oil pipeline network. Edge weight represent pipe capacity. How much oil can we transfer from source s to sink t?



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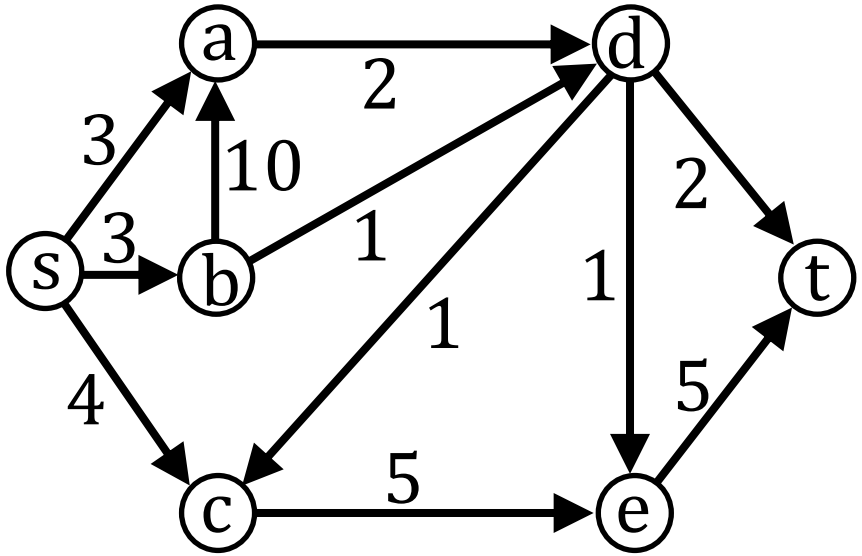

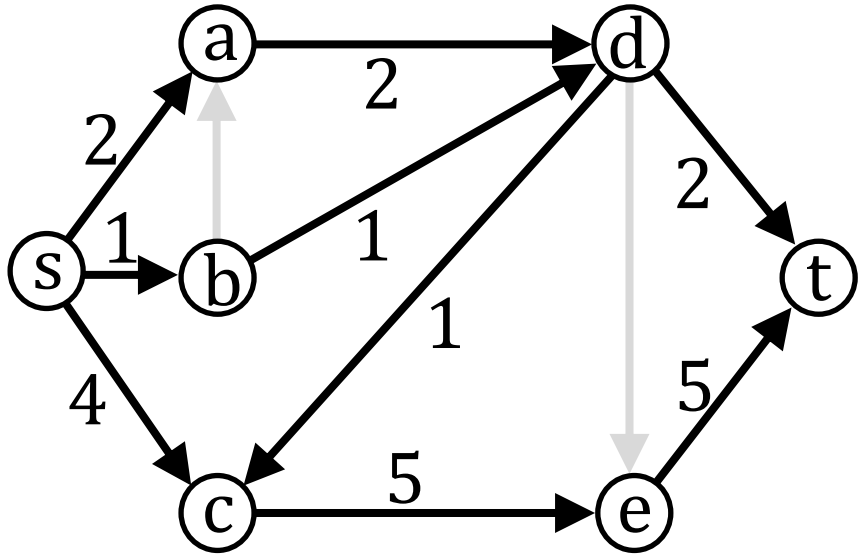

Total flow = 7

Flow Network

Flow Network:

An $s-t$ flow is a function f: $E \rightarrow \mathbb{R}^{+}$ such that:

Flow Network

Flow Network:

- Directed-edge graph, $G = (V, E)$.
- Finite positive edge capacity, c_e.
- Single source, s, without input edges.
- Single sink, t, without output edges.

We'll also sometimes use the assumption that the capacities are positive integer values.

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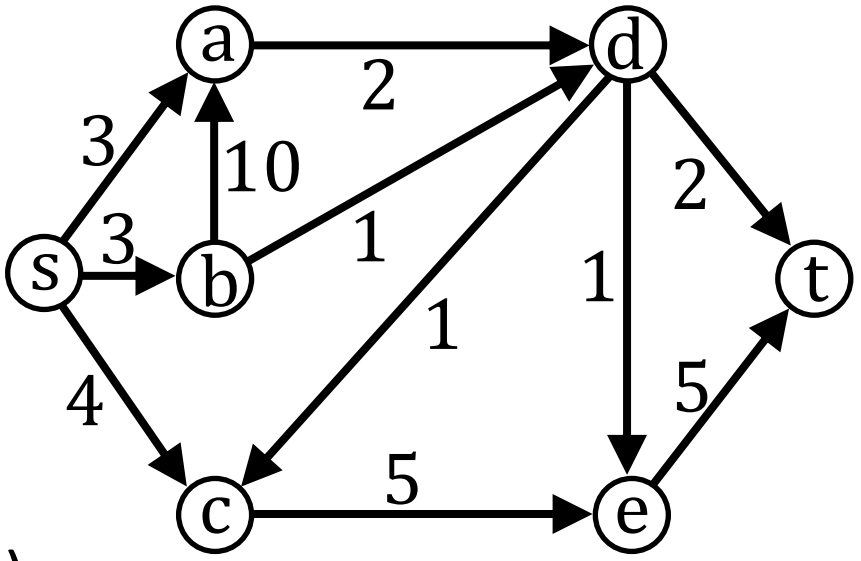

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- $0 \leq f(e) \leq c_e, \forall e \in E$. (capacity constraint)

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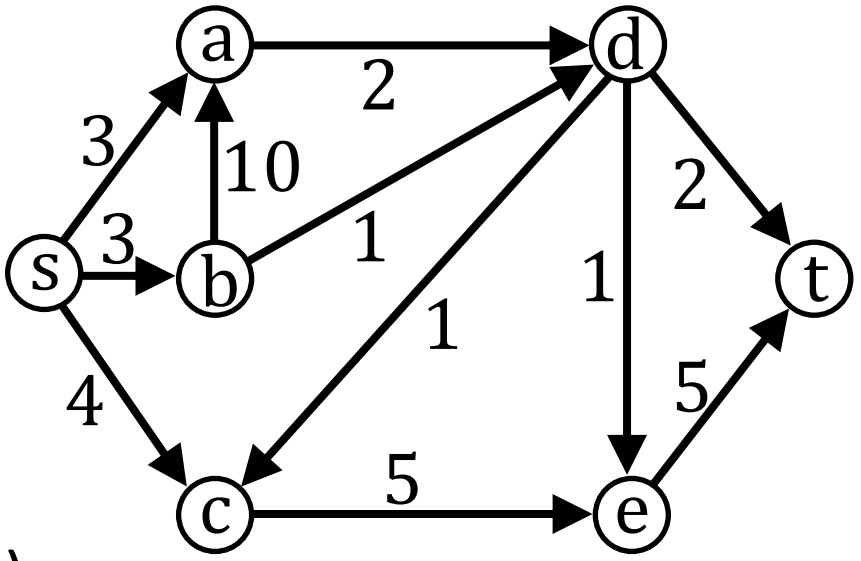

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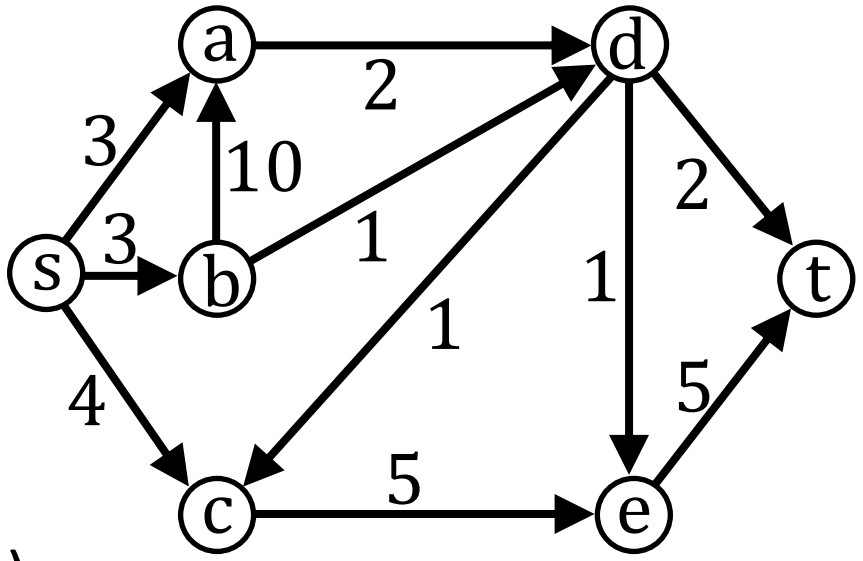

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- Value of flow = $val(f)$ = $\sum_{e \epsilon \text{output}(s)} f(e)$ = $\sum_{e \epsilon \text{input}(t)} f(e)$

Flow Network

Maximum Flow Problem:
Given a flow network, find the maximum possible value of flow.

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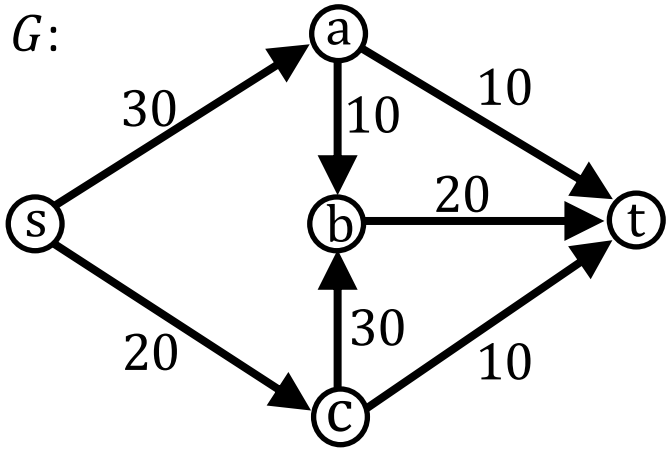
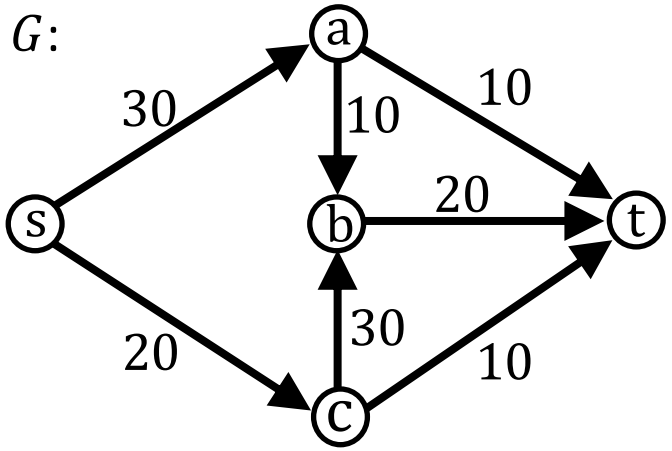
Max Flow Algorithm



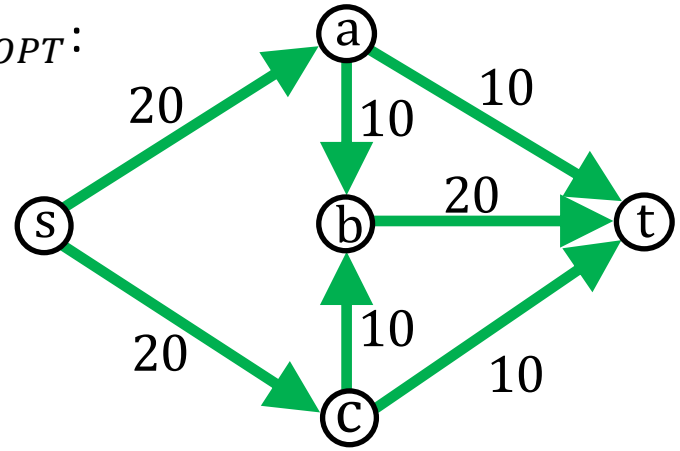
Max Flow?

Max Flow Algorithm

G:



f_{OPT}:



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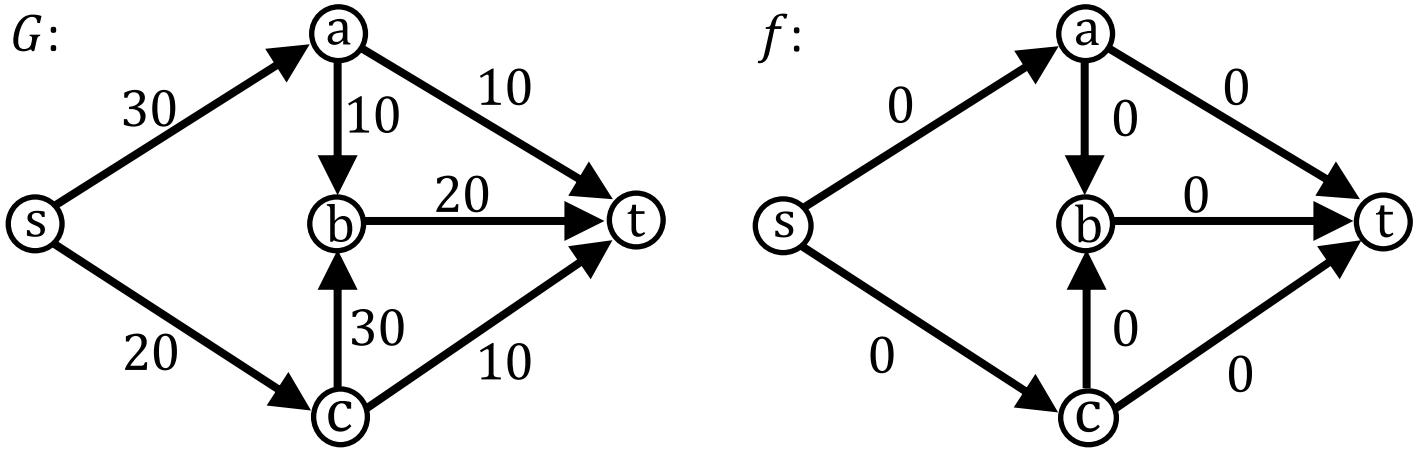
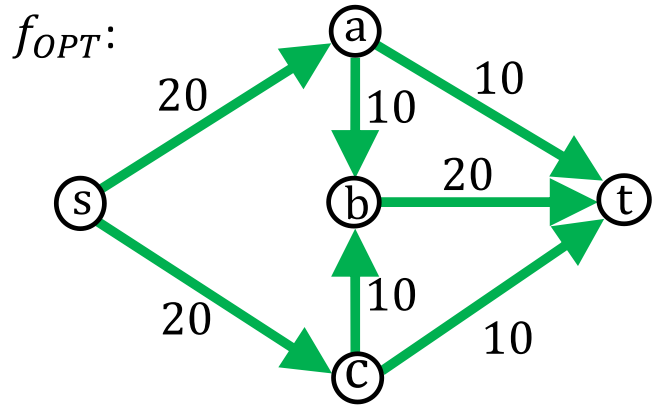


Ideas?



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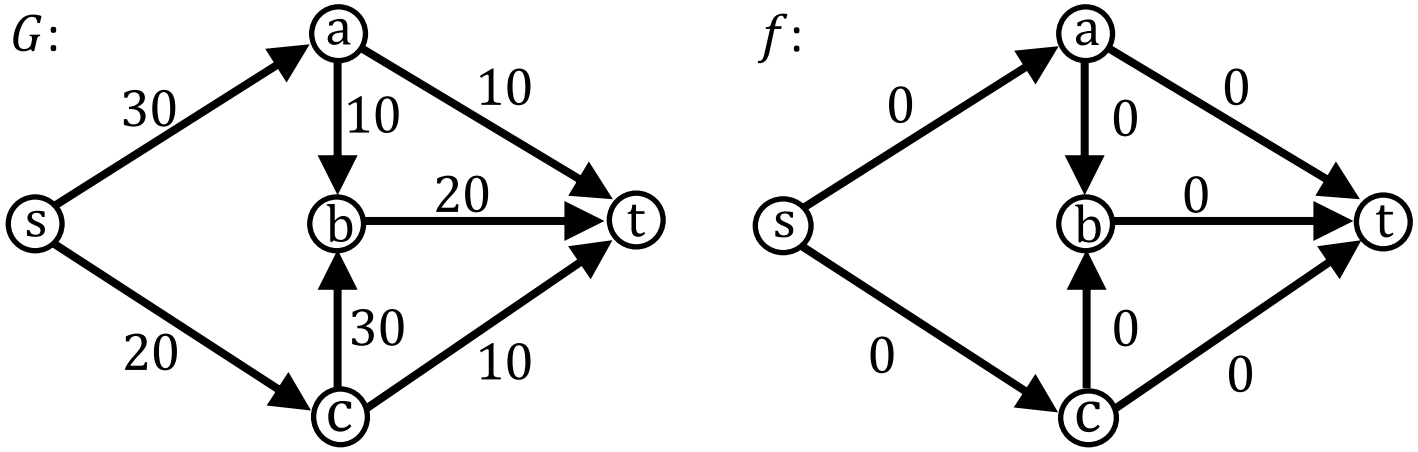
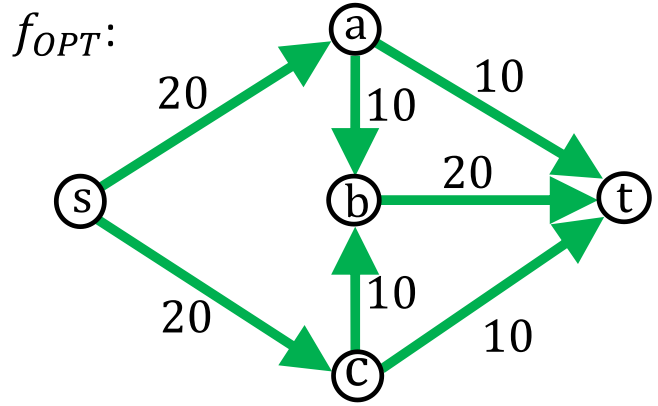


Ideas?

Somehow, we are going to have to put flow on edges. Should we select edges or paths?



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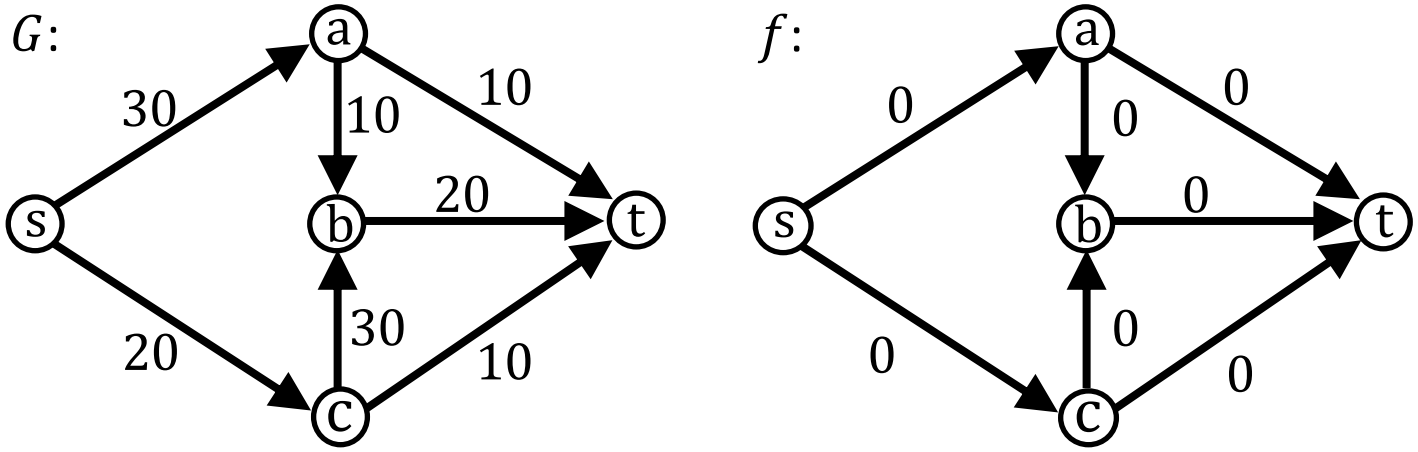
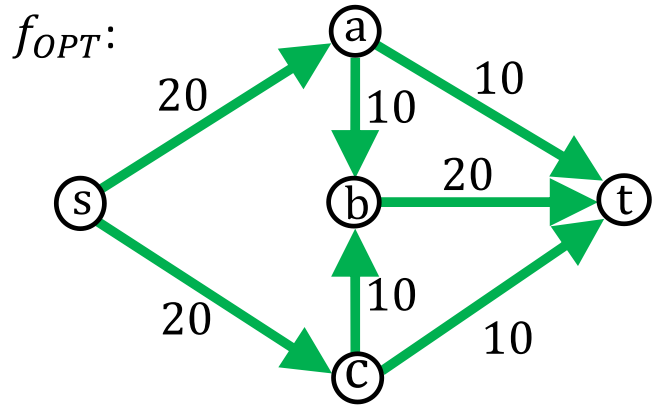


Ideas?

1. Select a path P.



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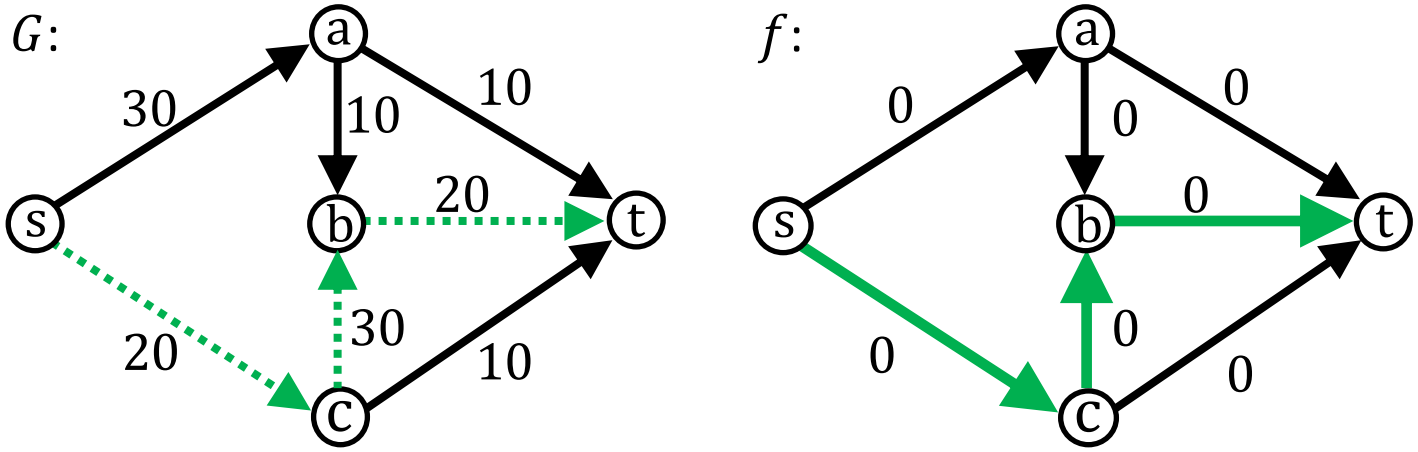
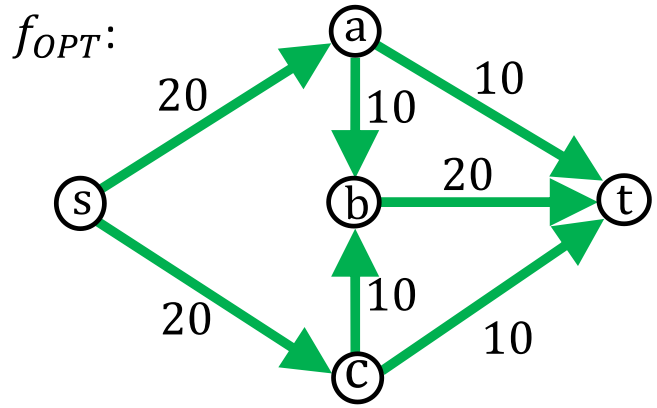


Ideas?

1. Select a path P.

How much flow should we push?



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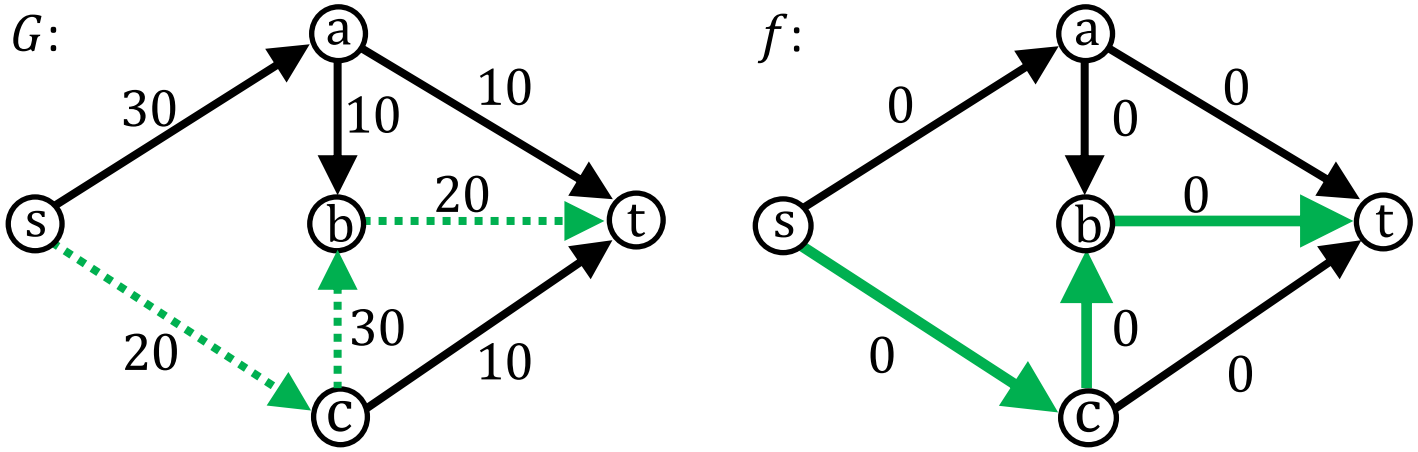
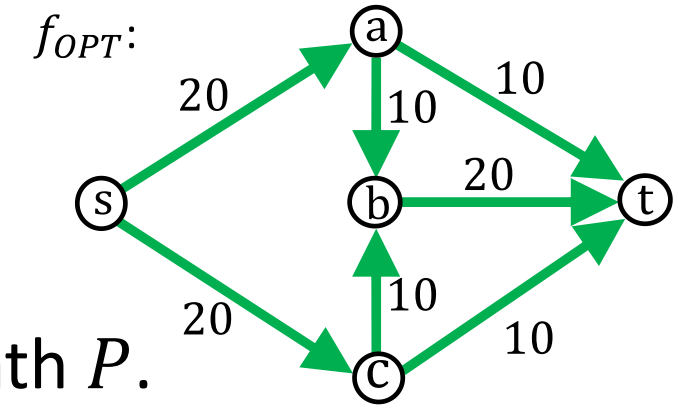


Ideas?

1. Select a path P.

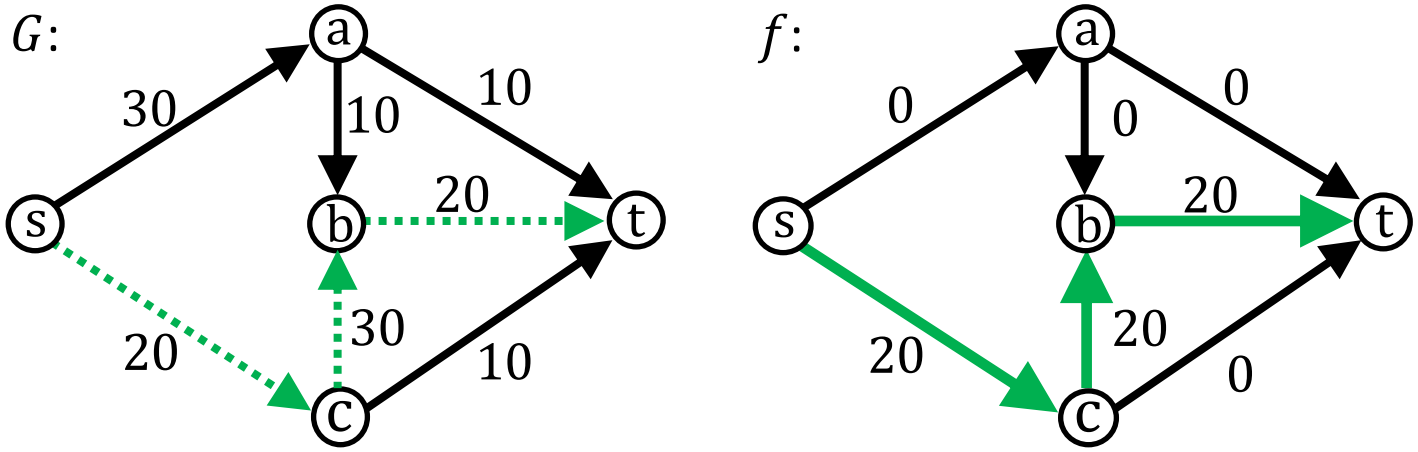
How much flow should we push?

As much as possible.



`bottleneck(P)` = minimum capacity on any edge in path P.

Max Flow Algorithm



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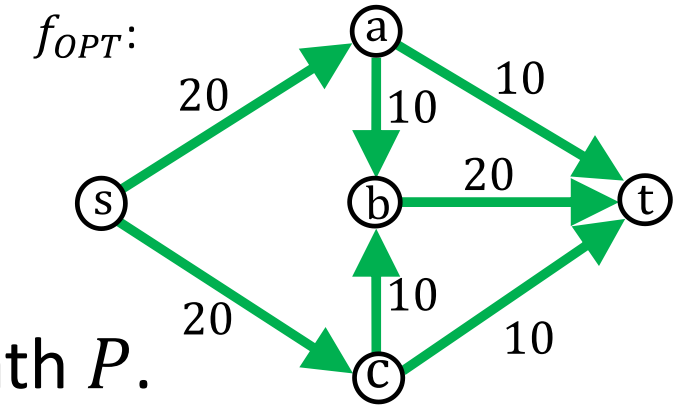

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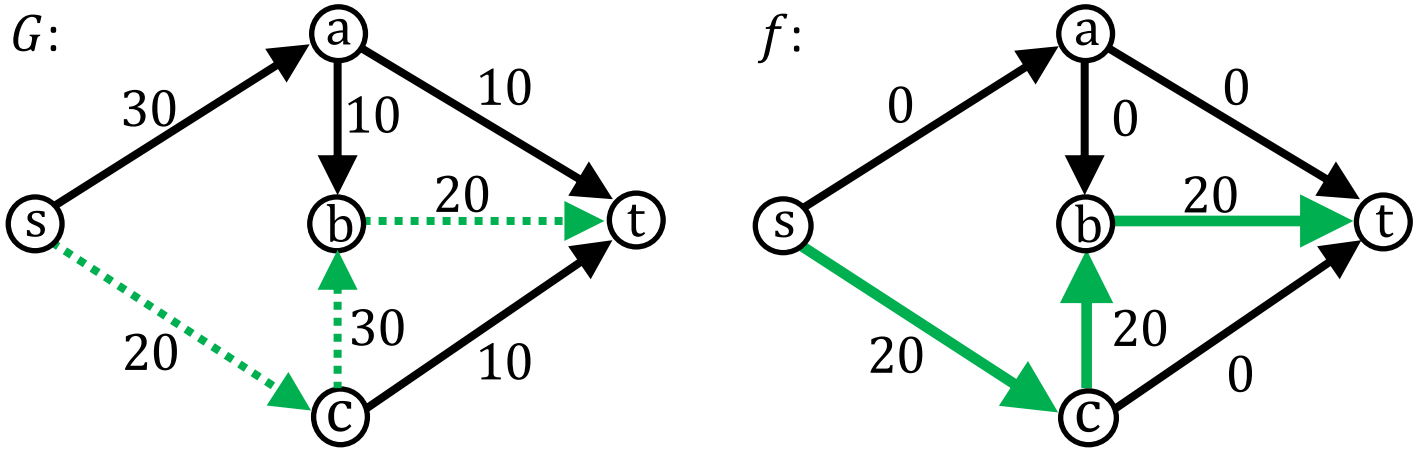

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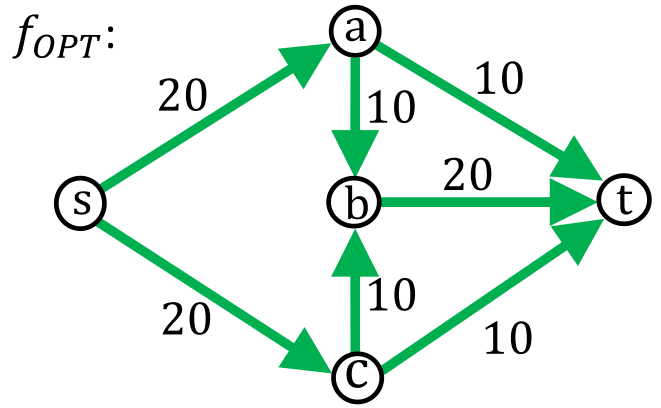

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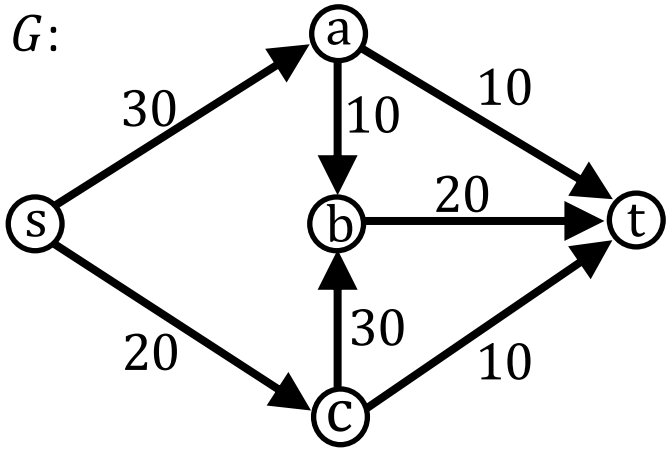

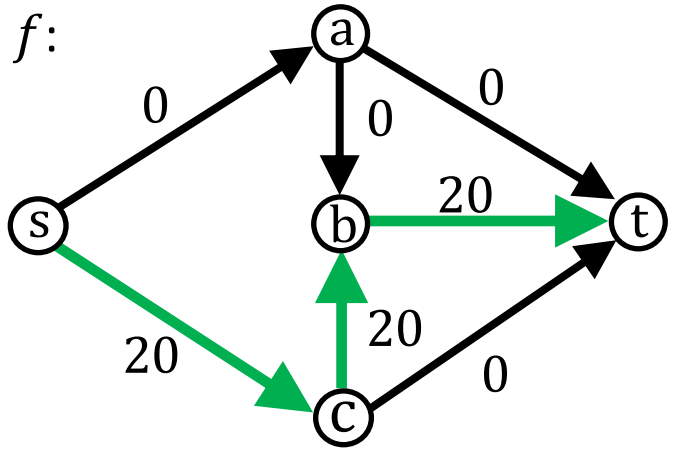

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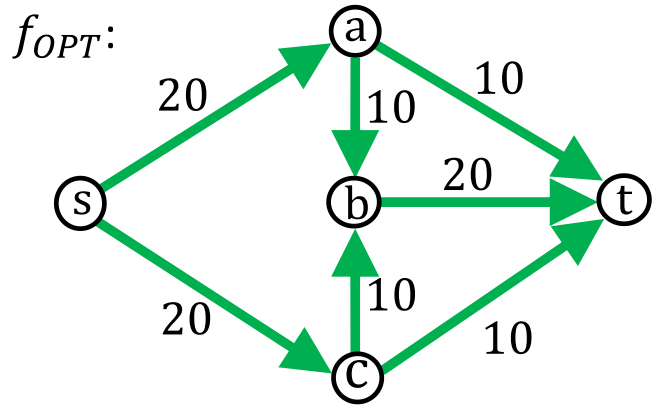
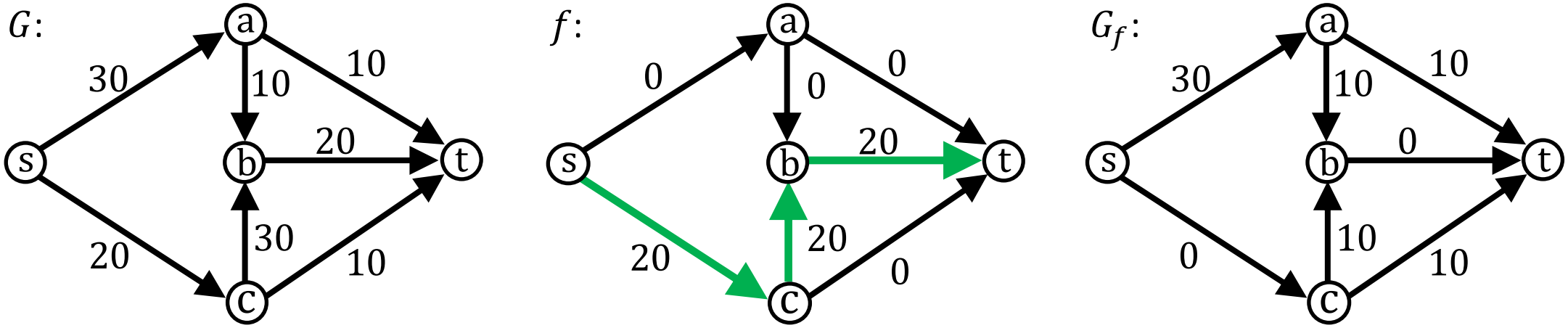
So far:

- **Guaranteed to meet conservation of flow constraints.**
- **Guaranteed to meet capacity constraints.**



Max Flow Algorithm

Residual Capacity



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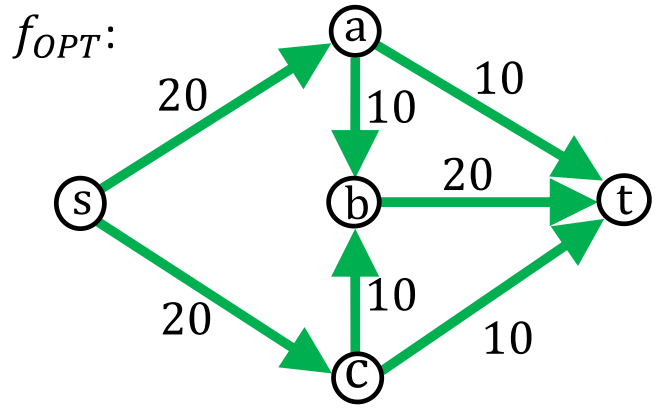

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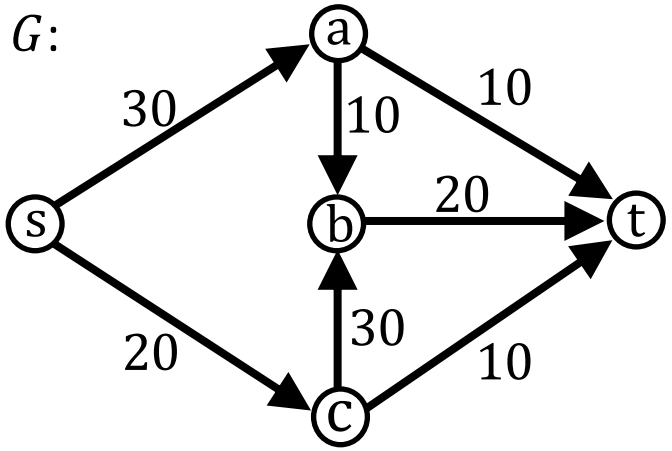

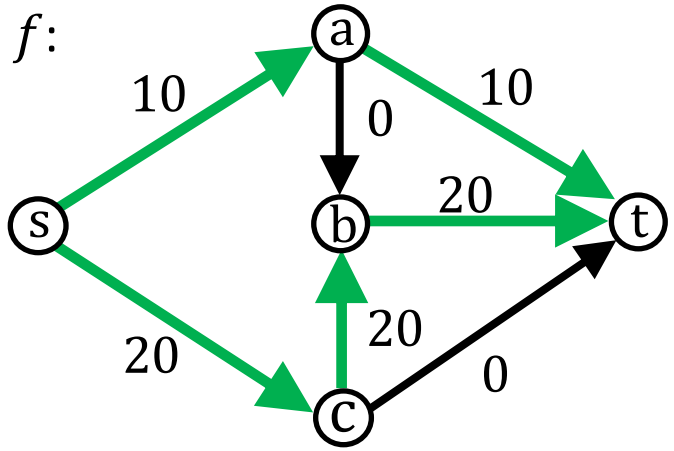

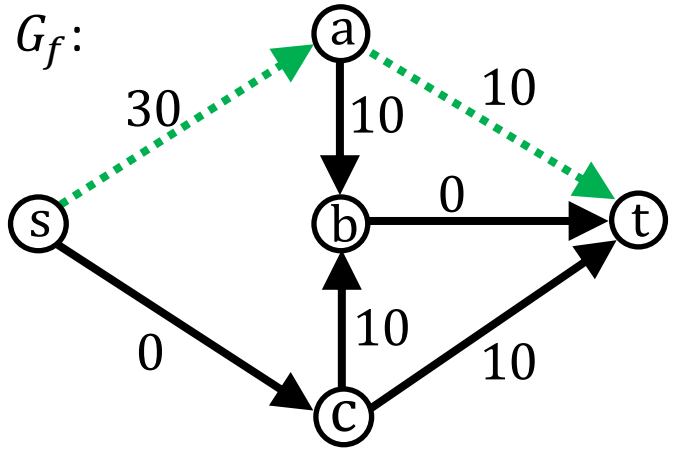

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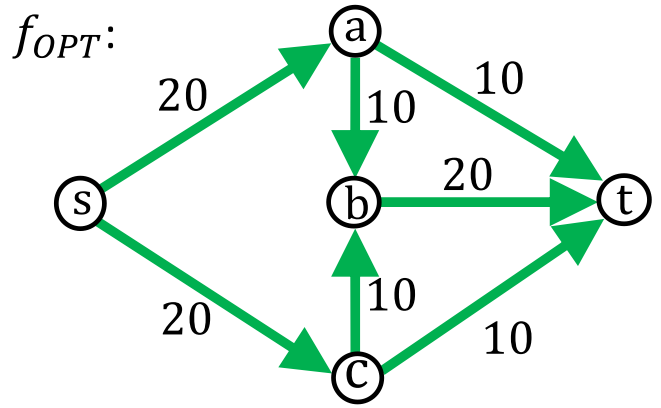

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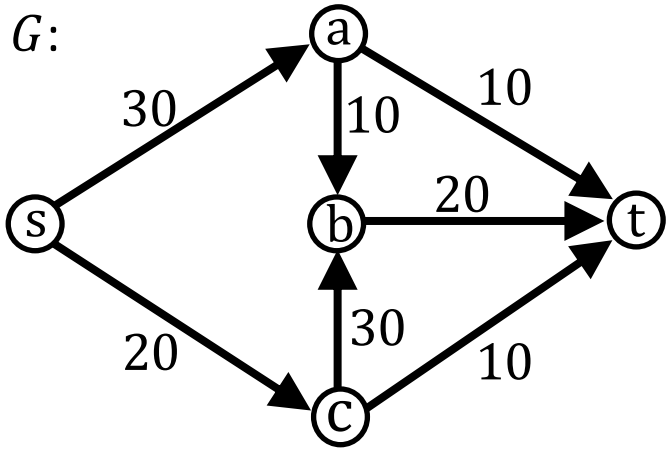

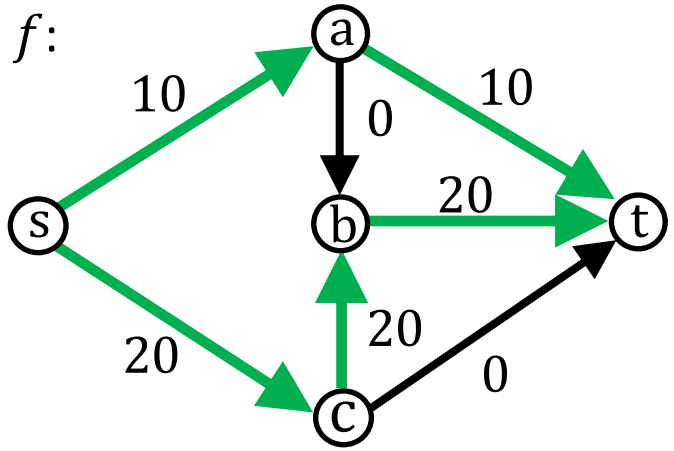

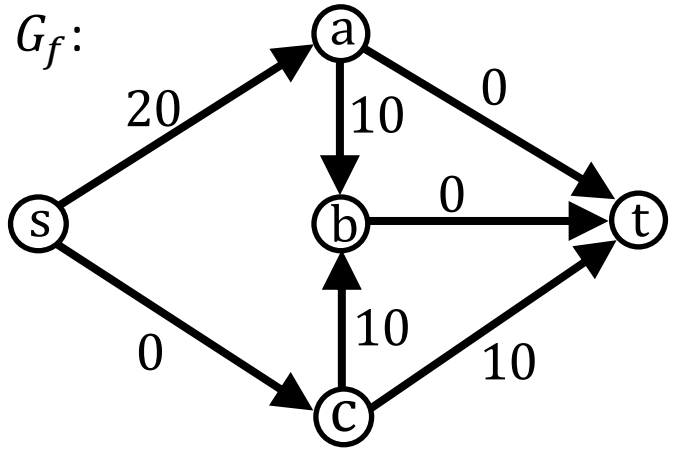

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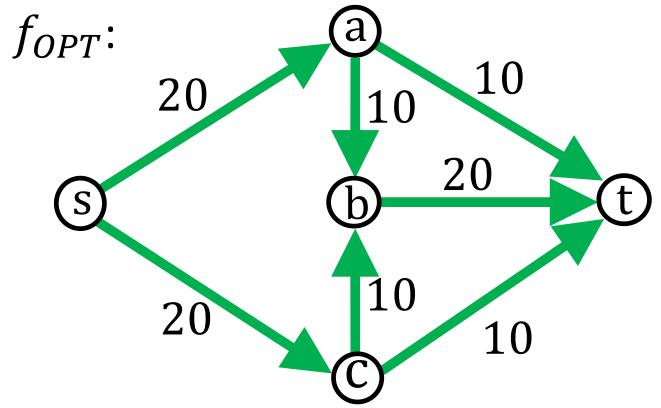

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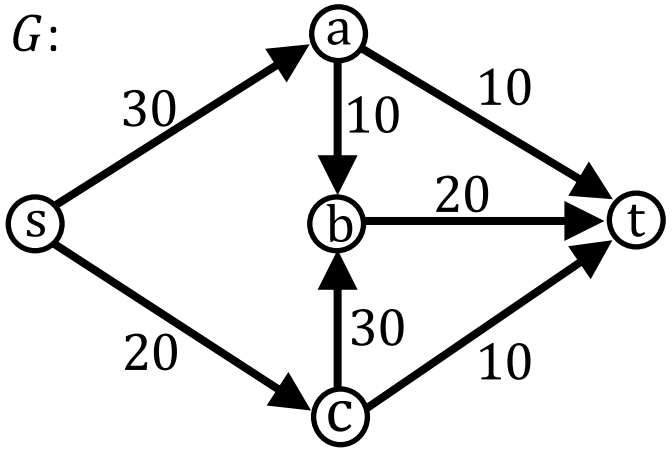
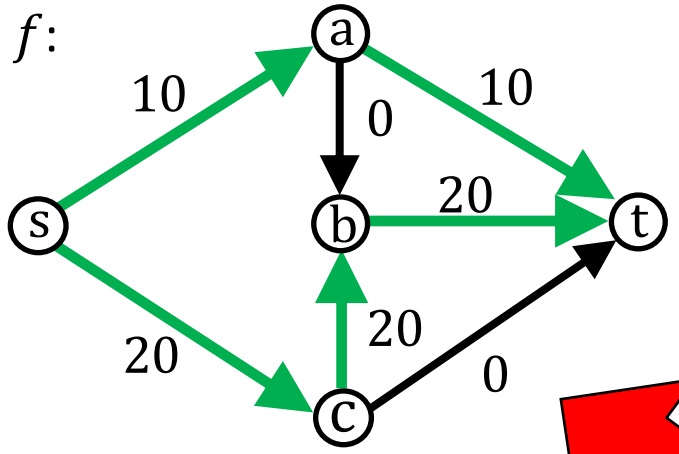
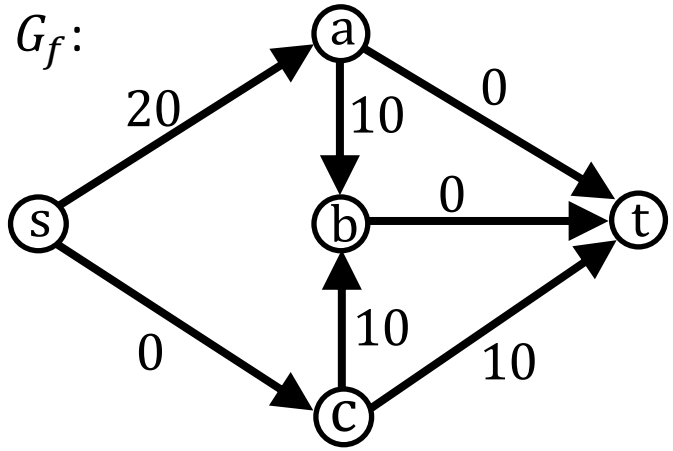
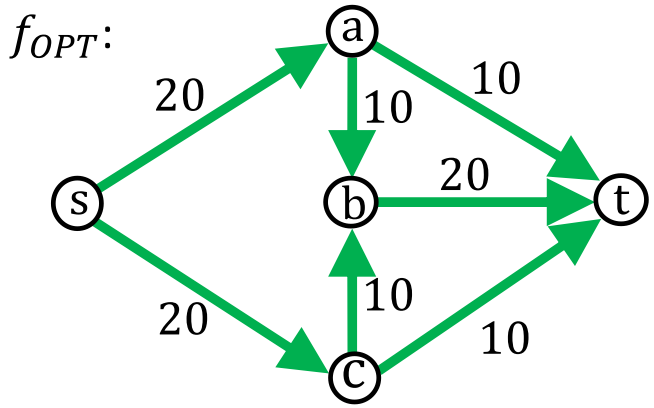






We need some way to reroute flow.



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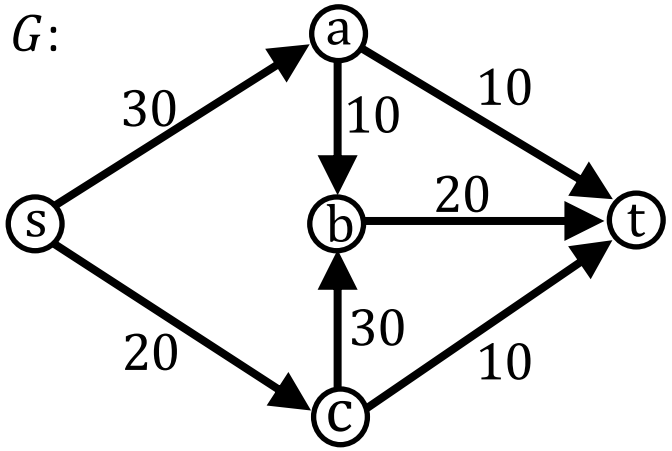

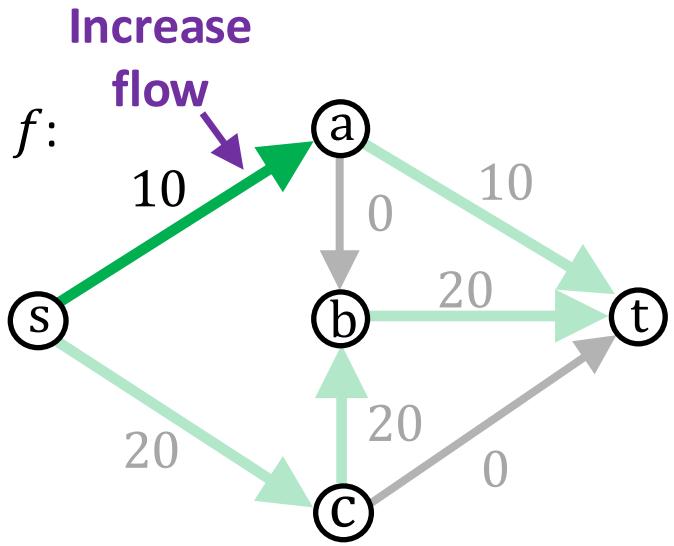

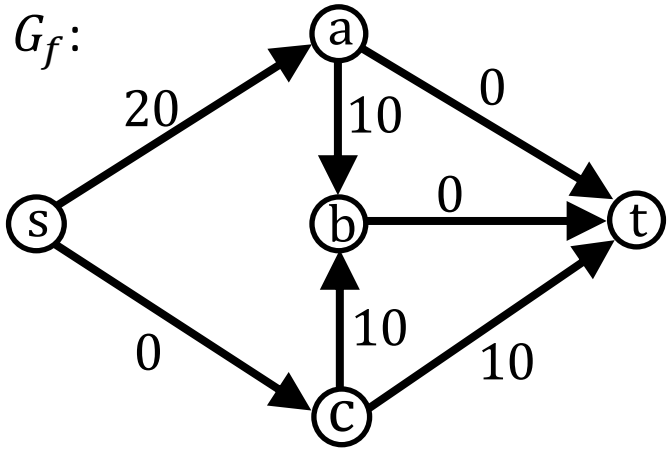

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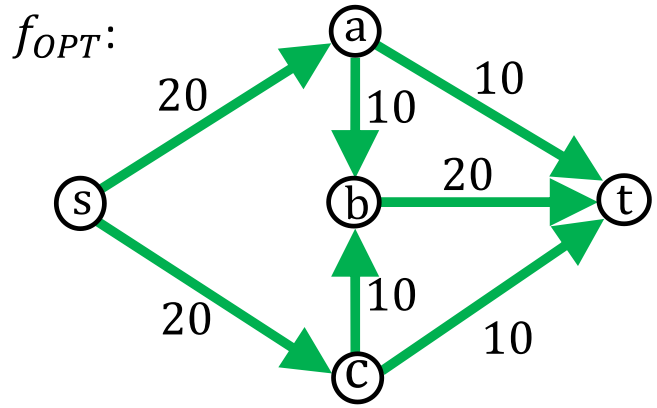

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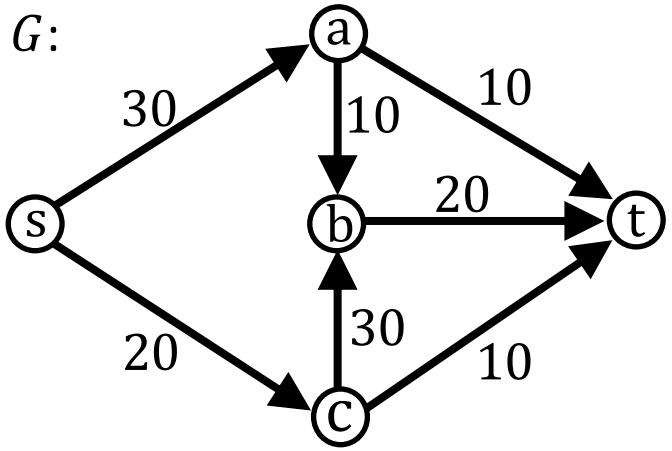

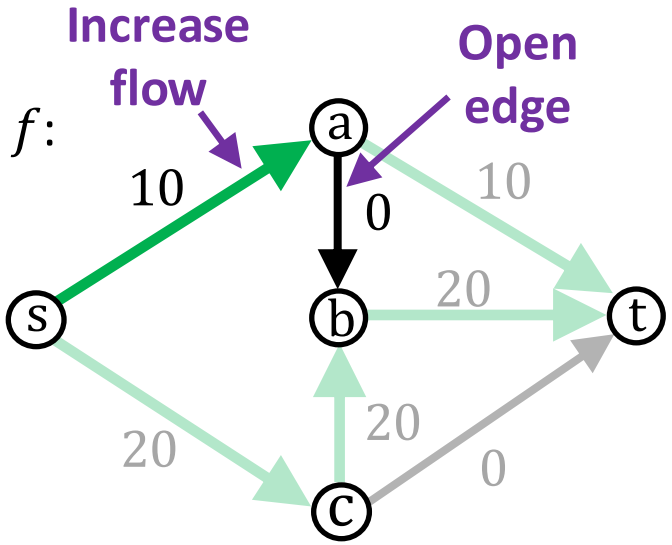

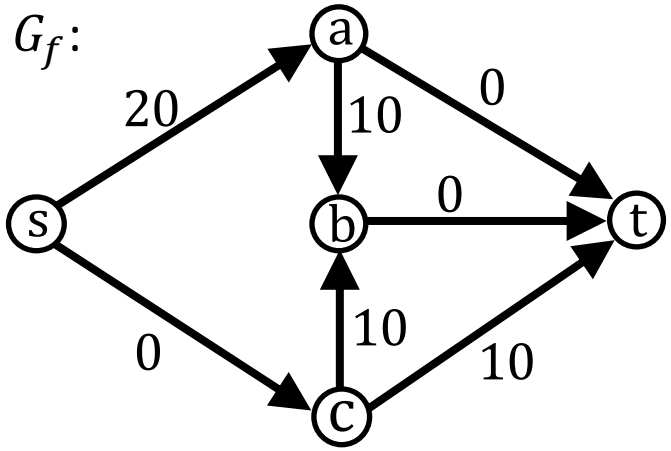

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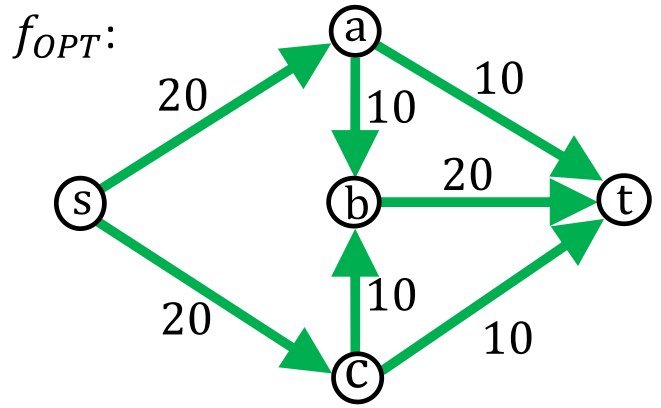

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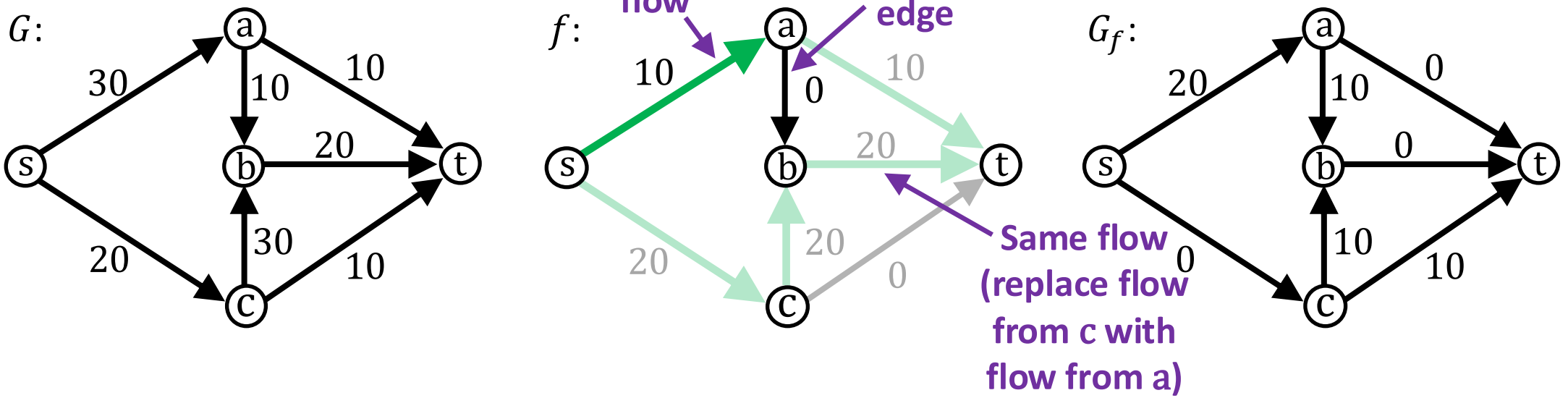

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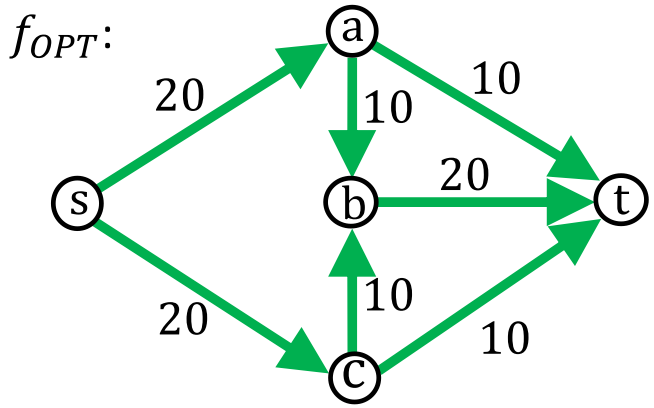
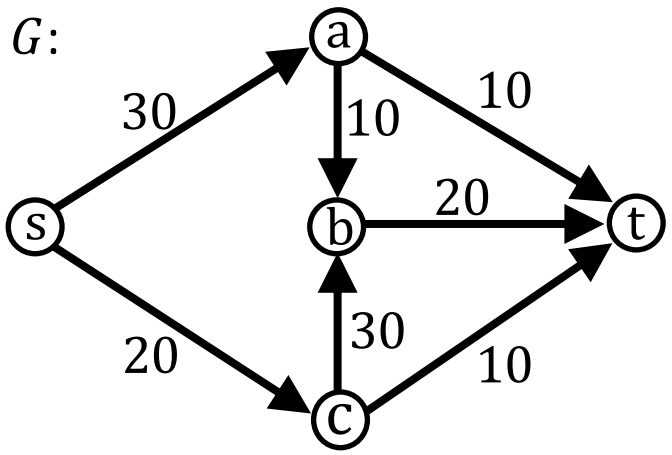
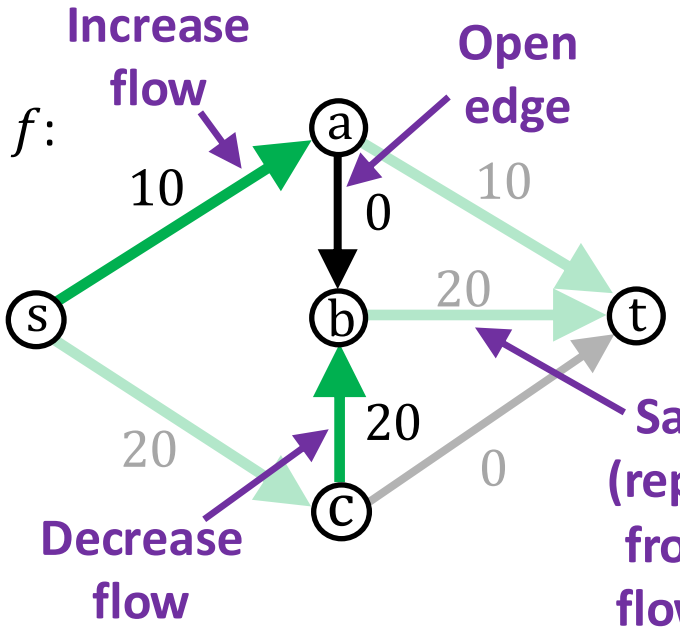
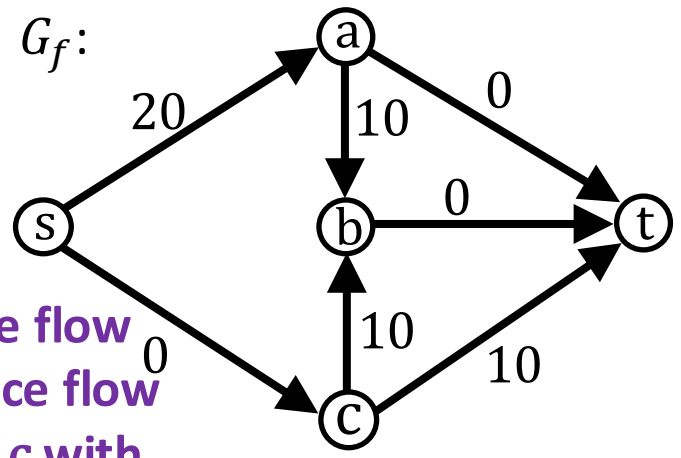


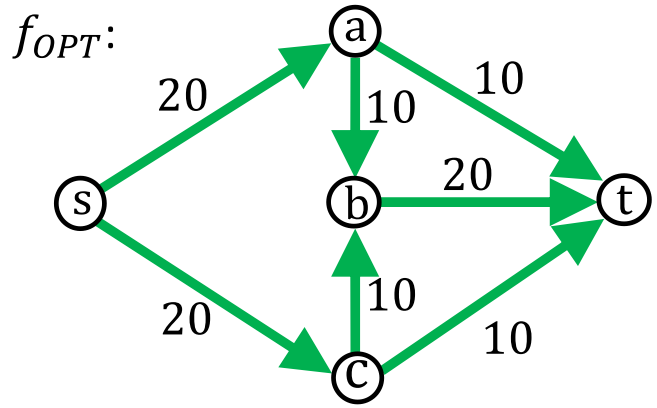
Max Flow Algorithm







We need some way to reroute flow.



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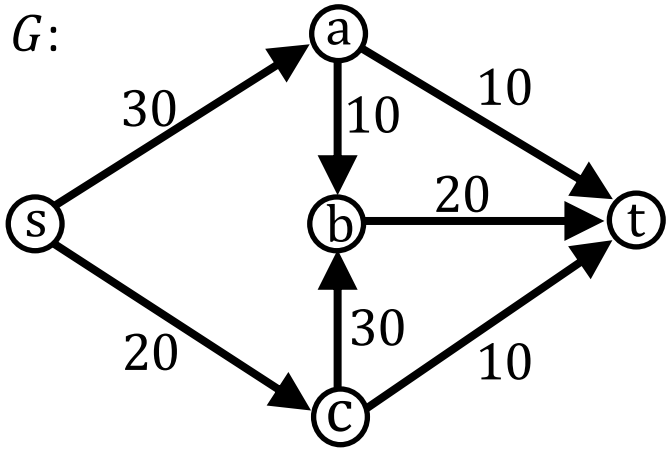

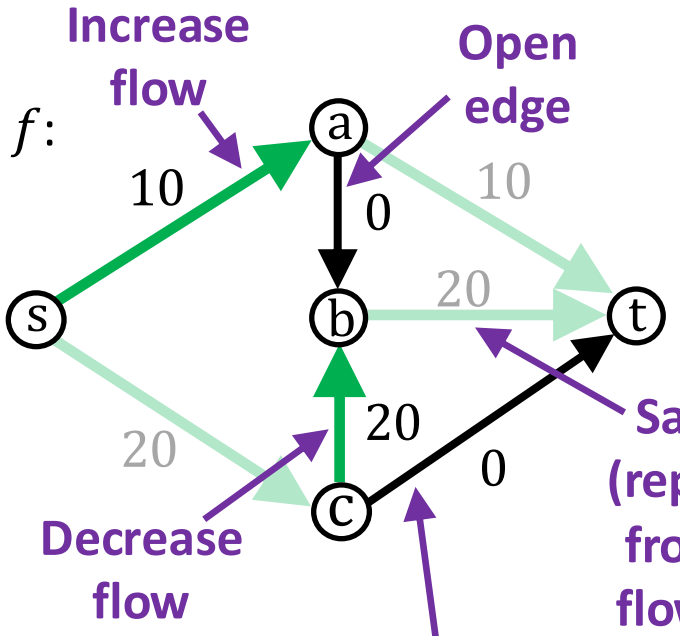

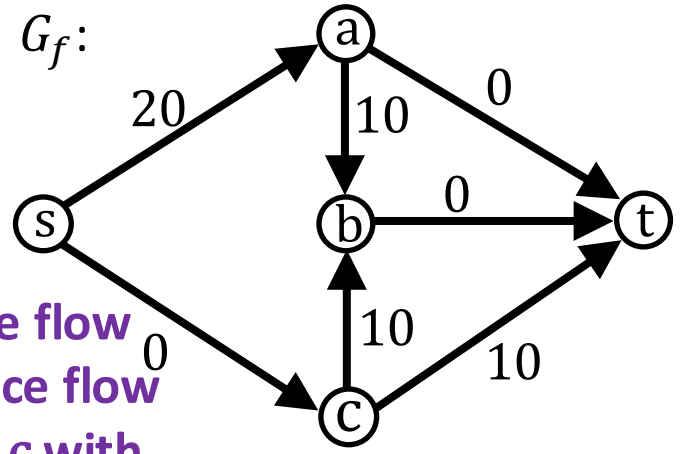

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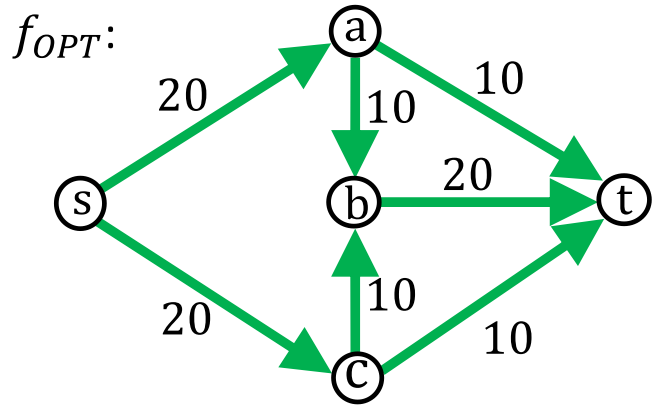

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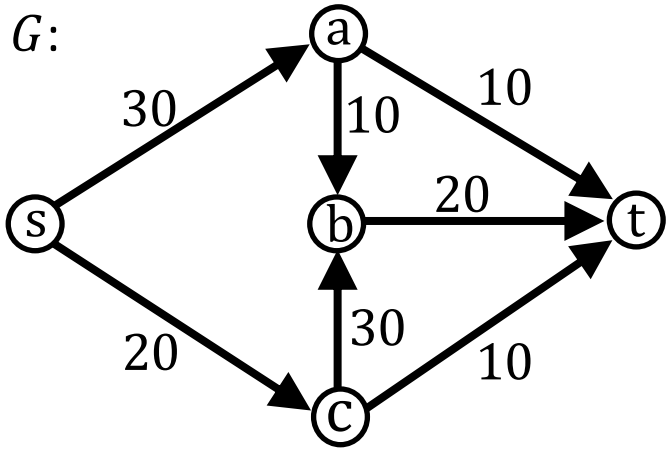

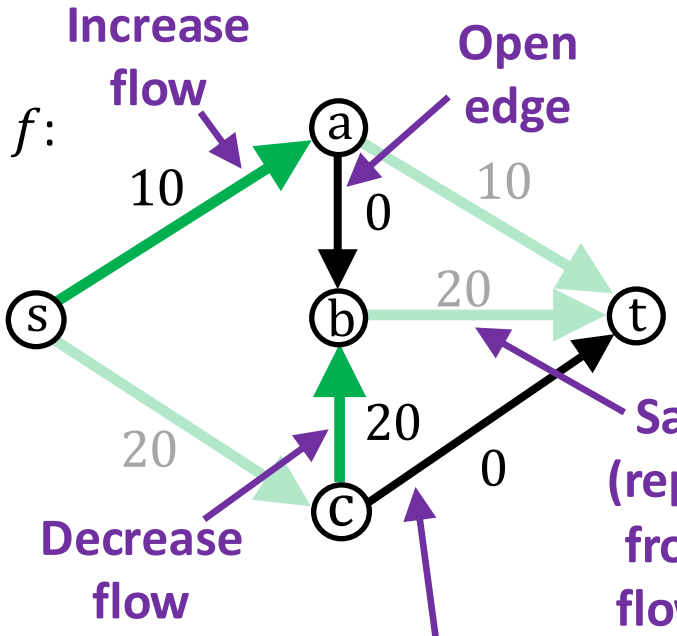

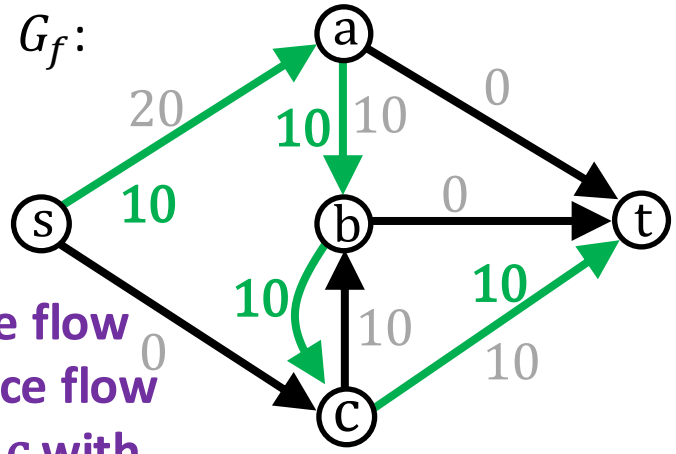

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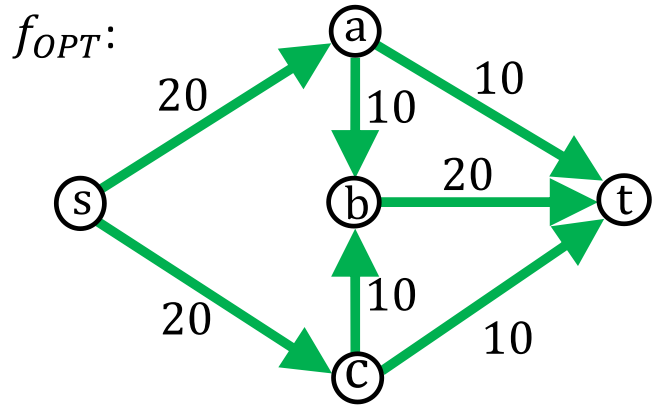

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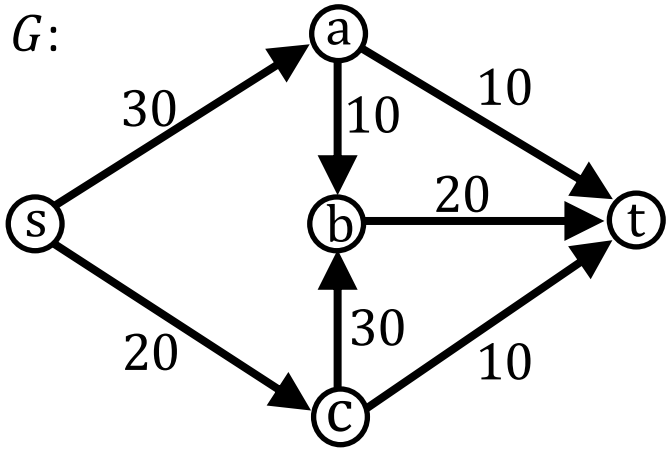

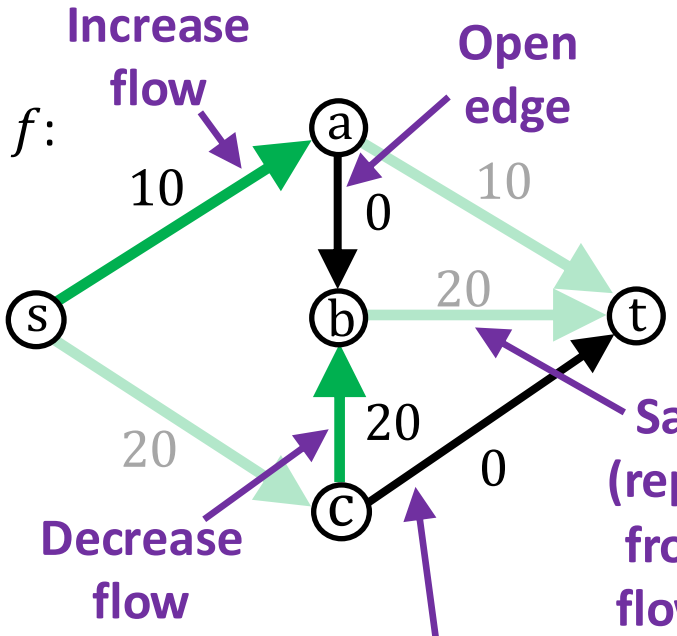

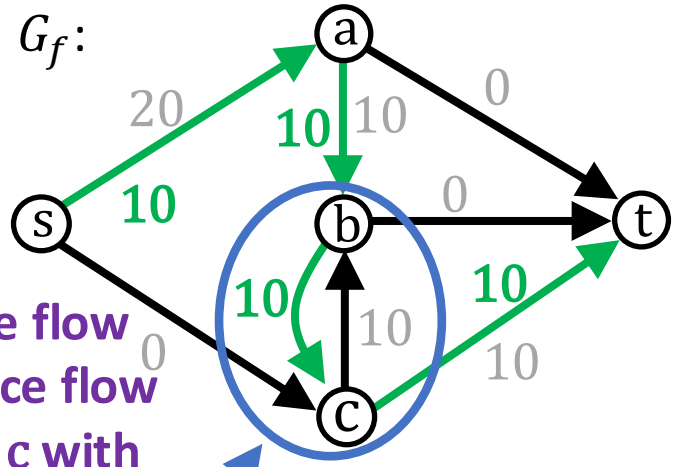

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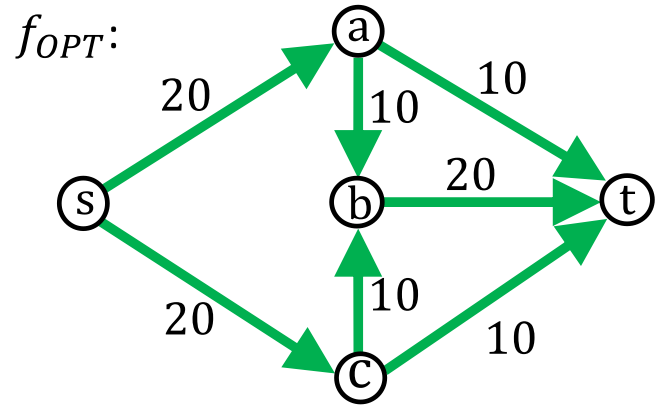
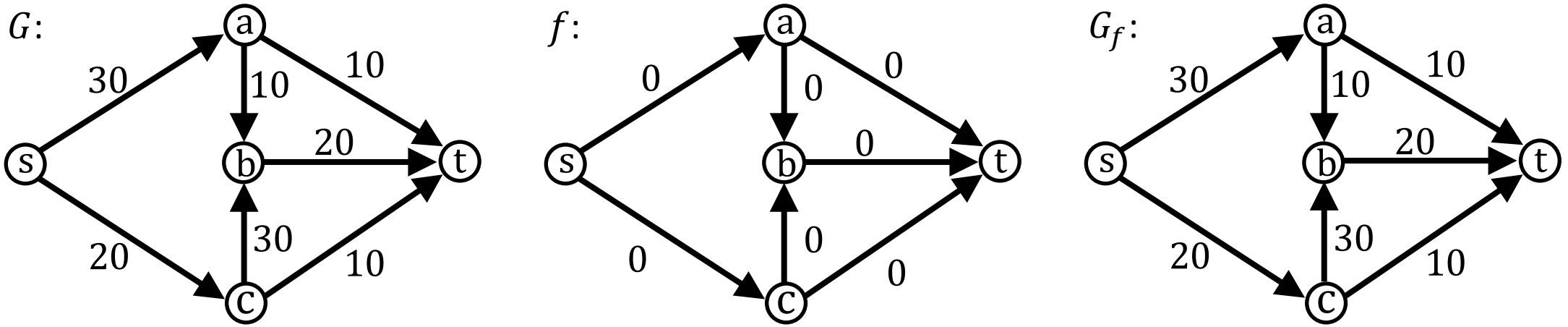


This edge does not physically exist but exists virtually as a mechanism to reroute flow that has already been pushed.

Max Flow Algorithm



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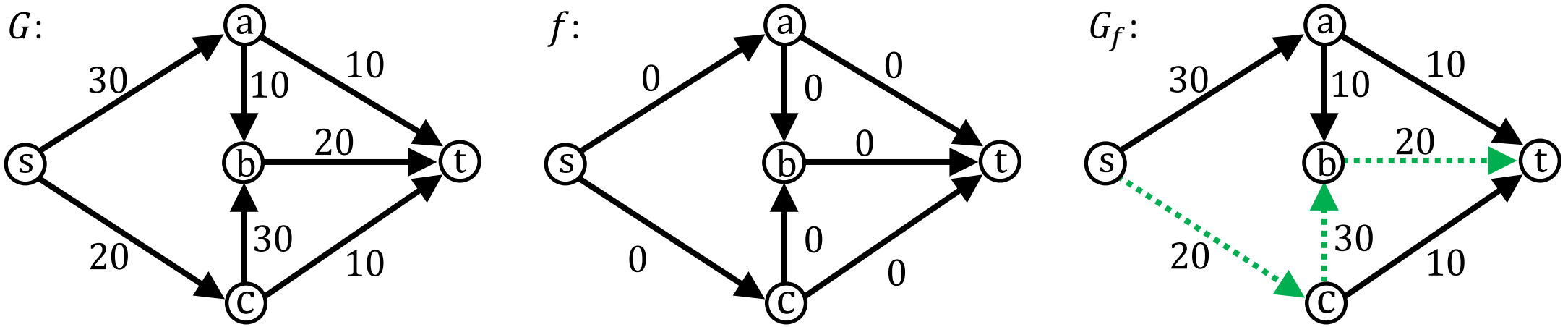

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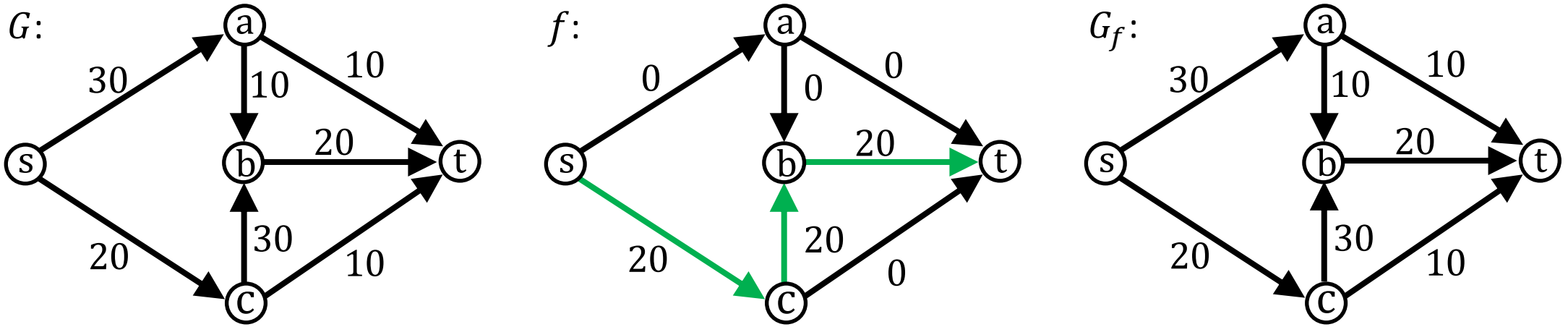

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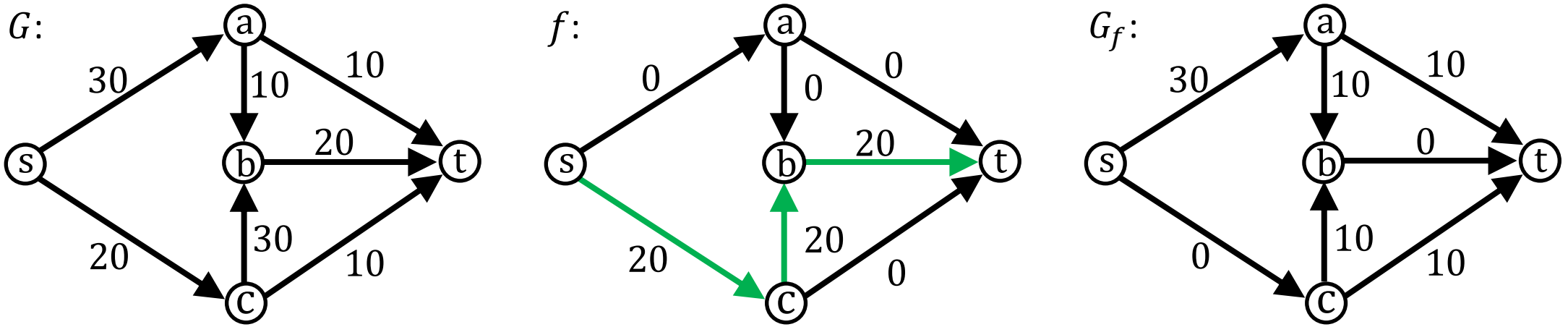

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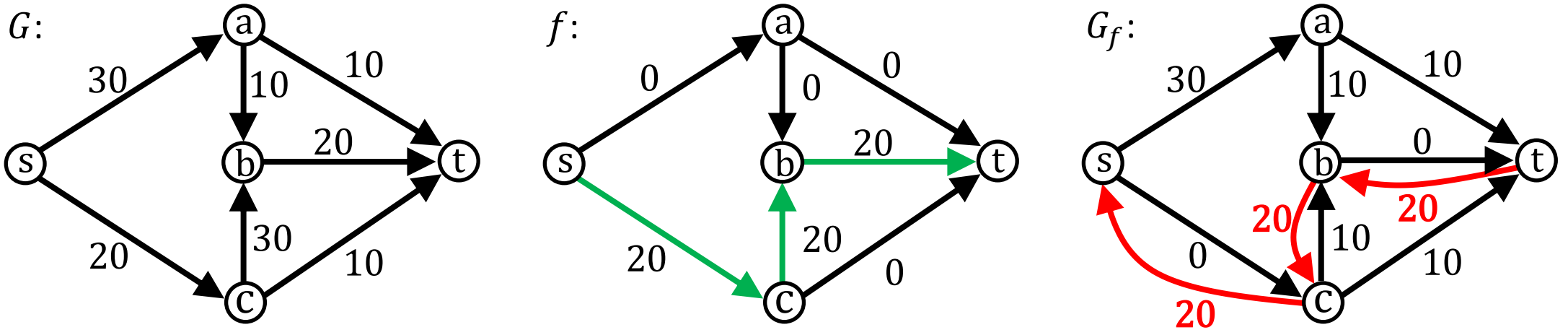

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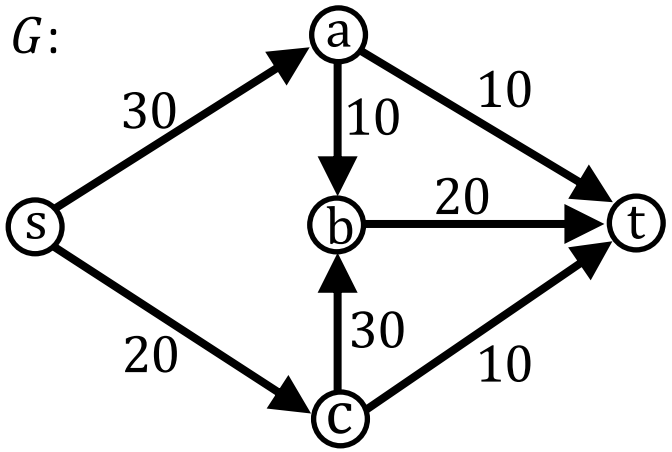

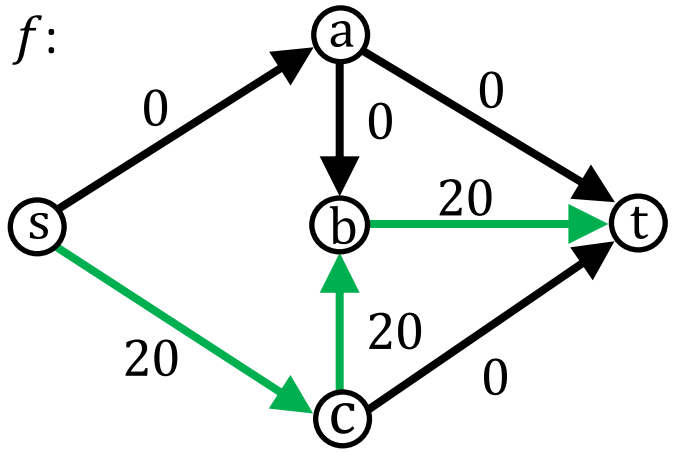

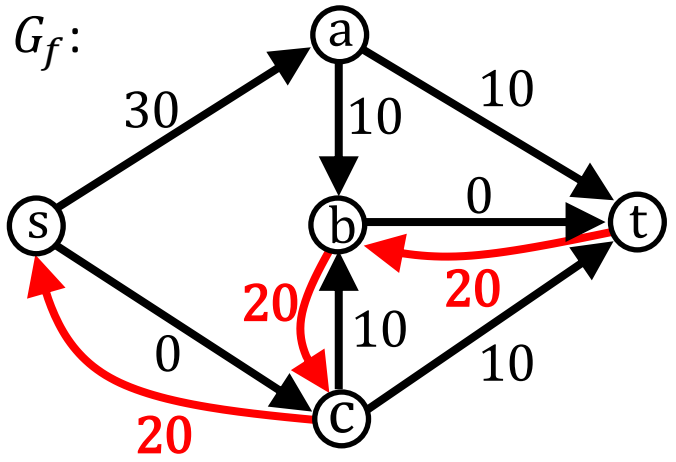

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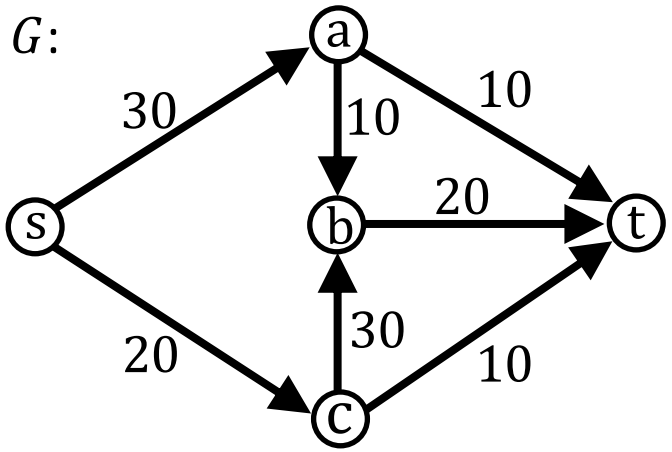
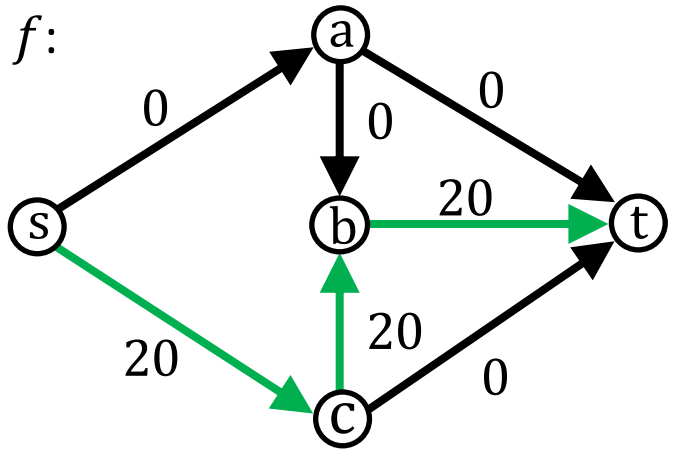
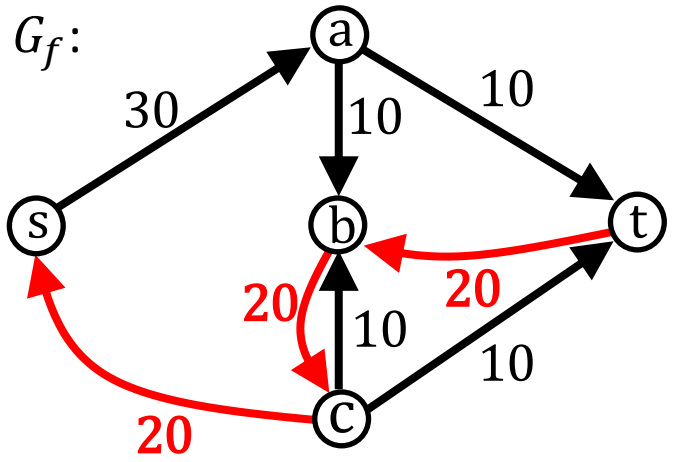
Remove edges with 0 capacity in residual graph. Can't use them anyway.

Max Flow Algorithm







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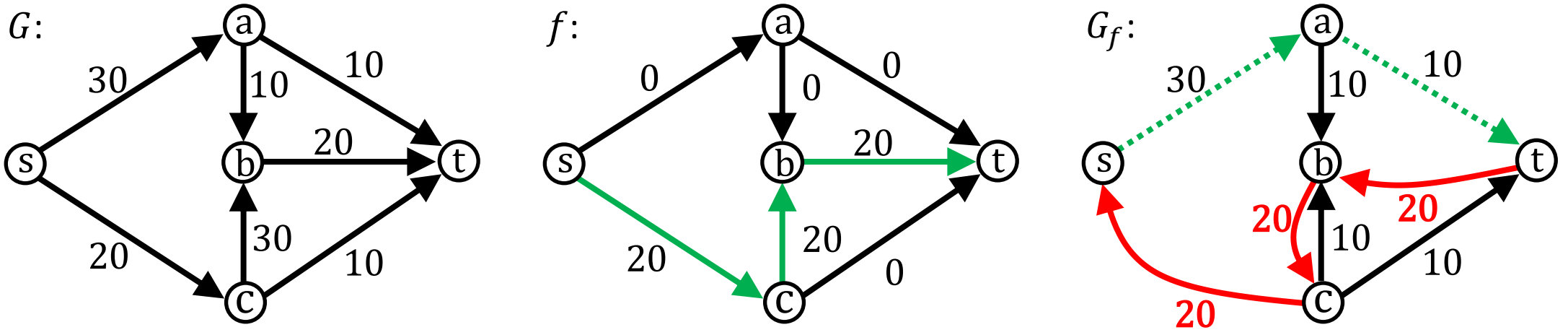

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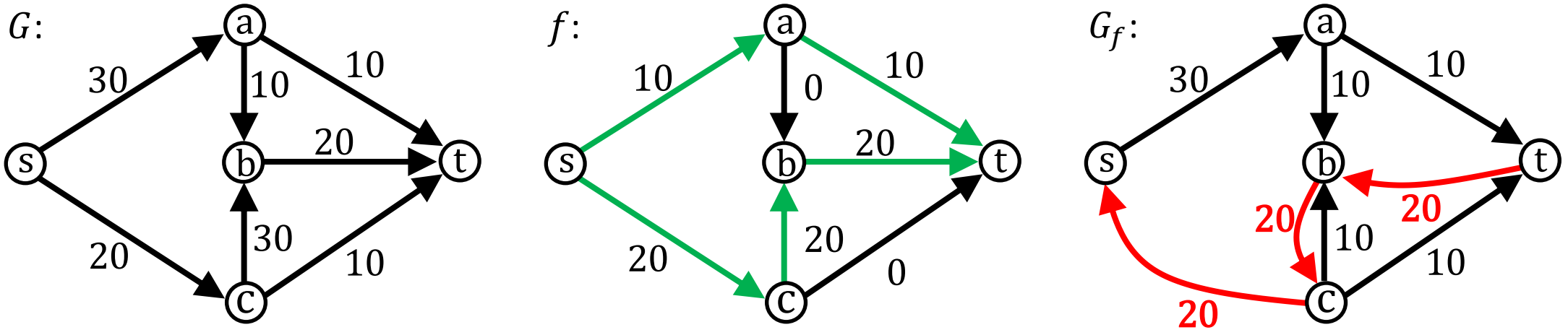

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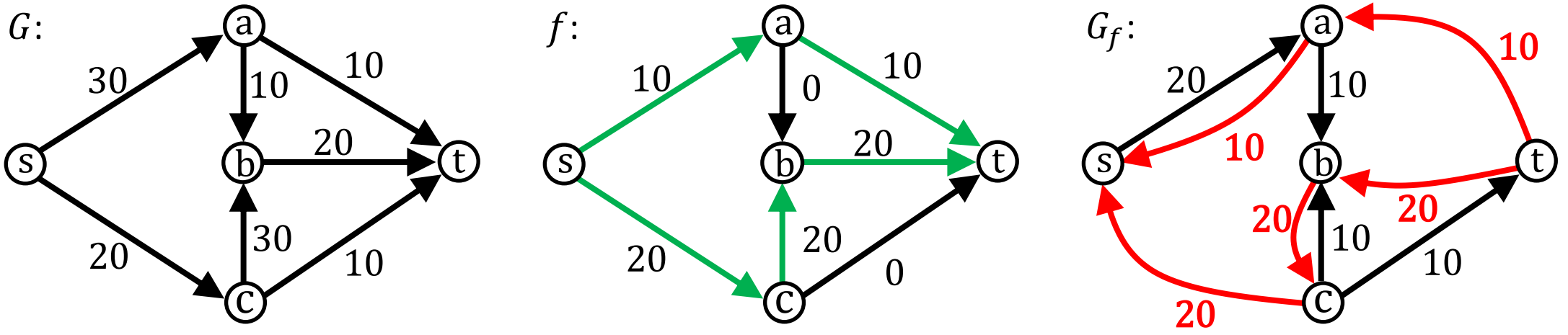

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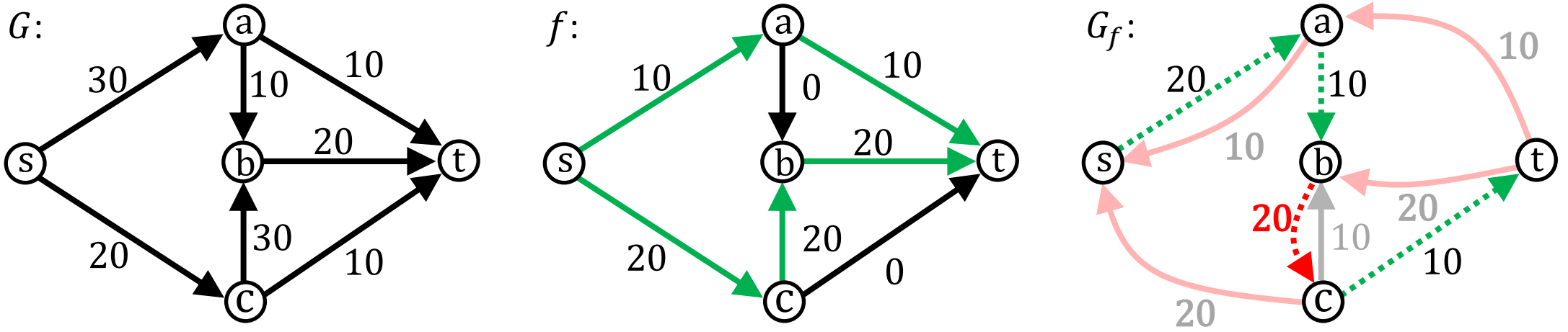

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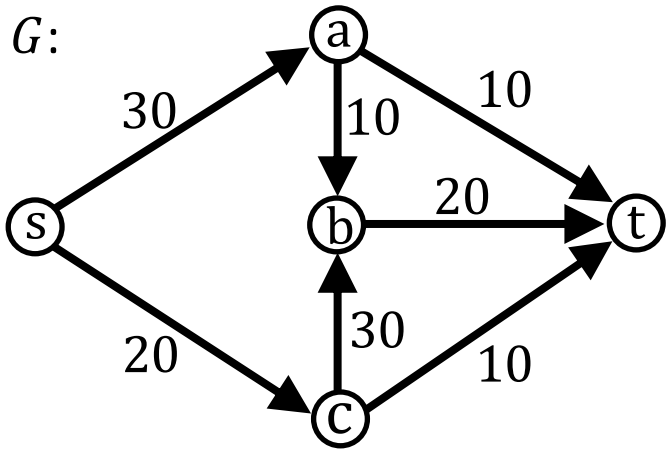

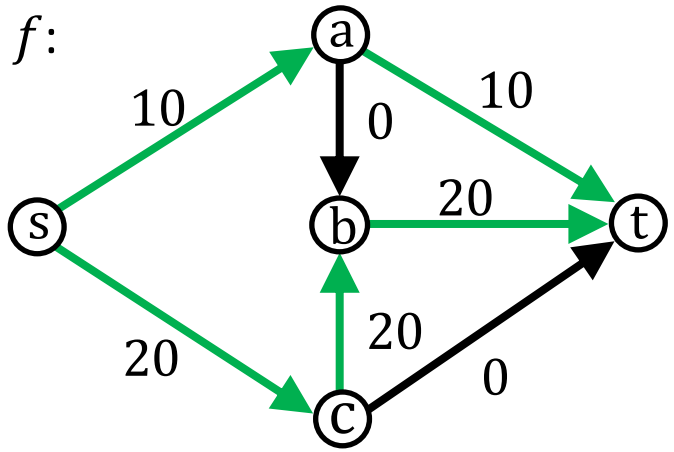

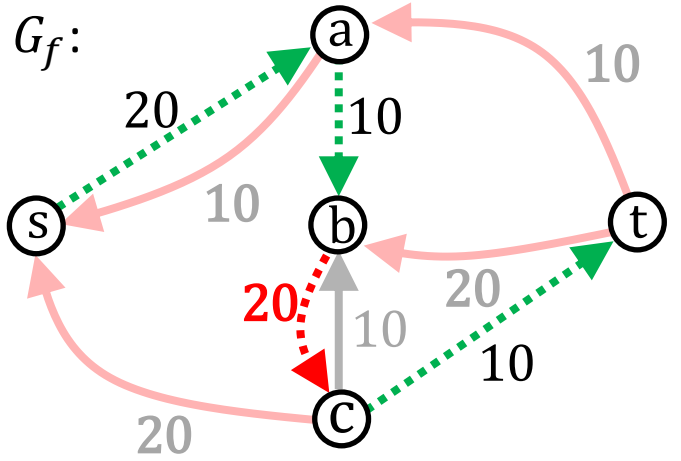

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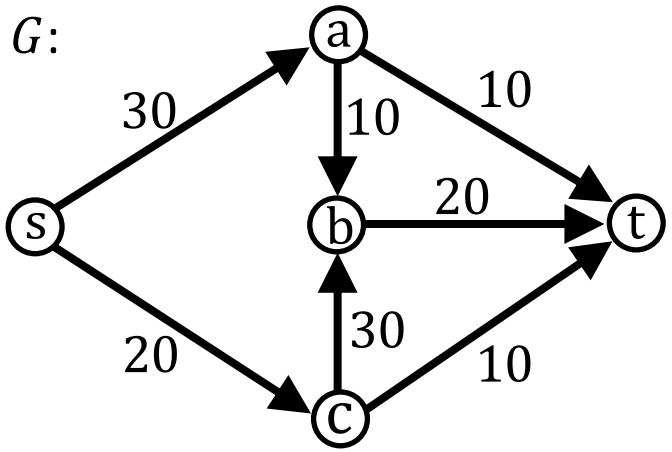

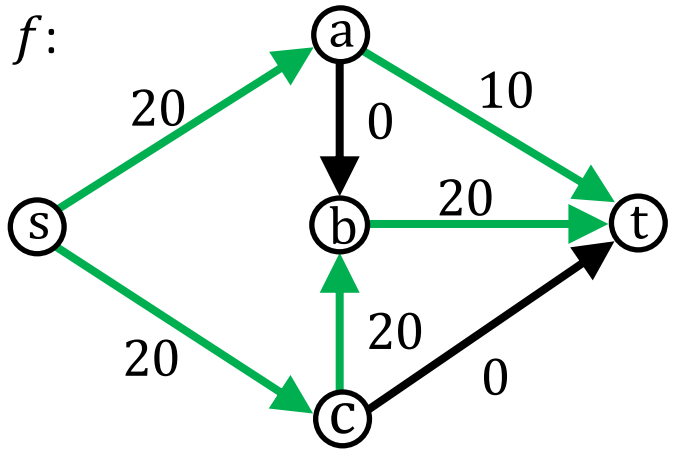

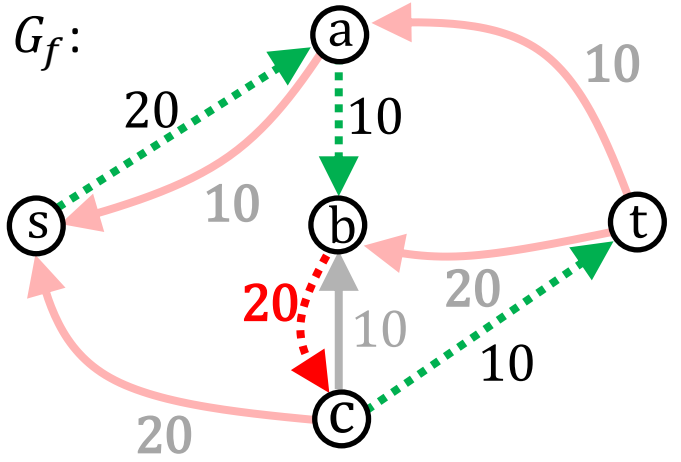

Pushing flow on a path:

1. If edge is a normal edge, increase flow.

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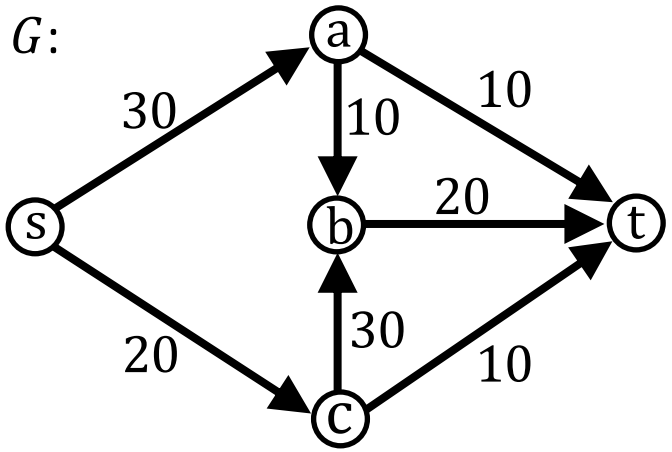

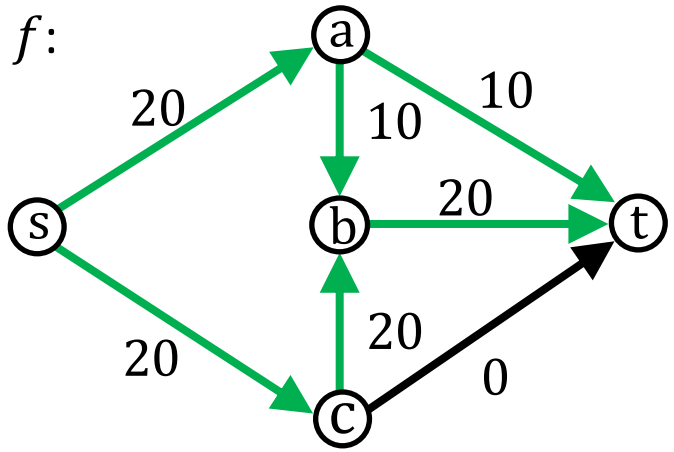

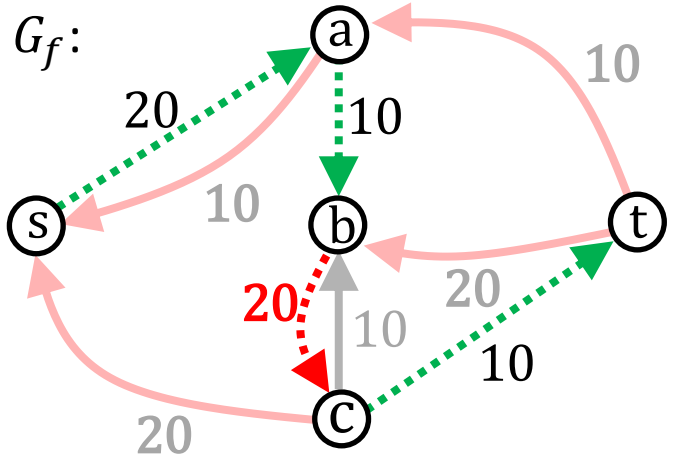

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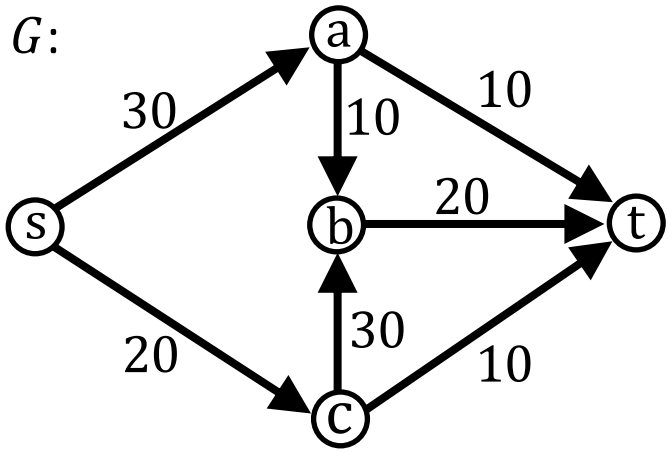

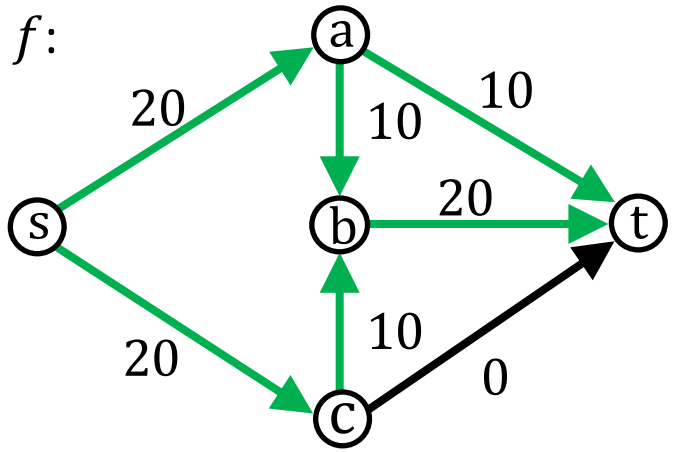

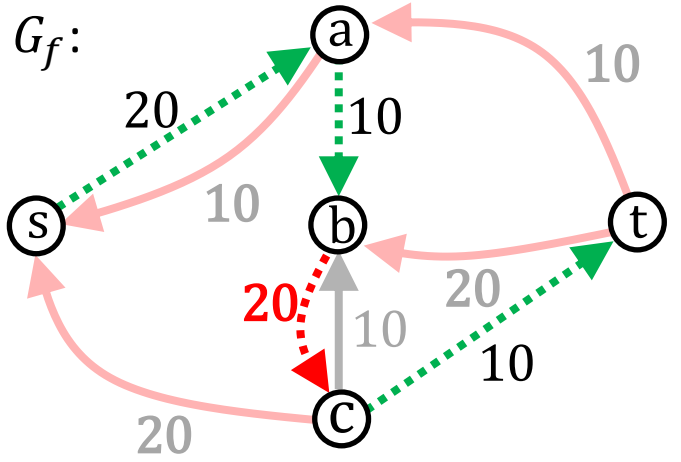

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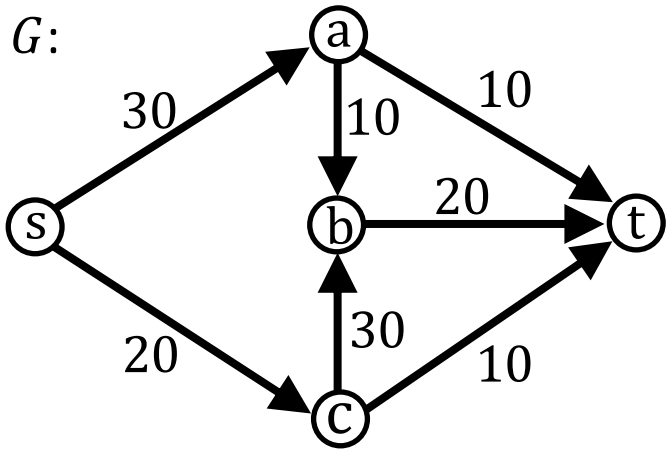

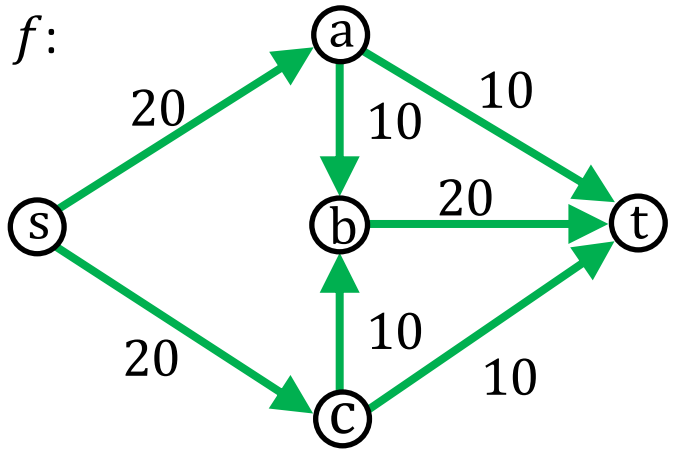

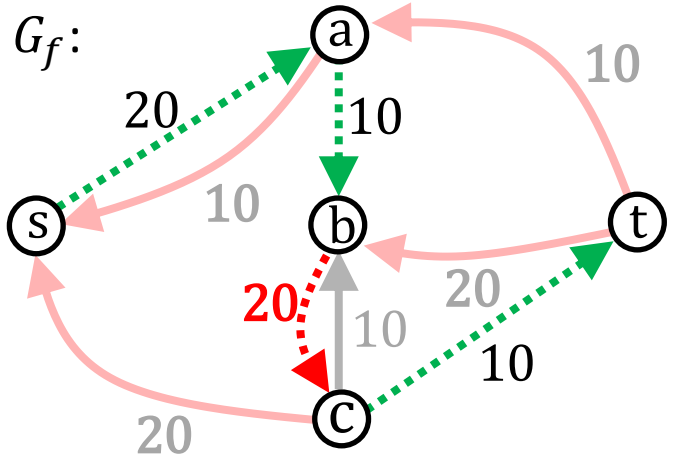

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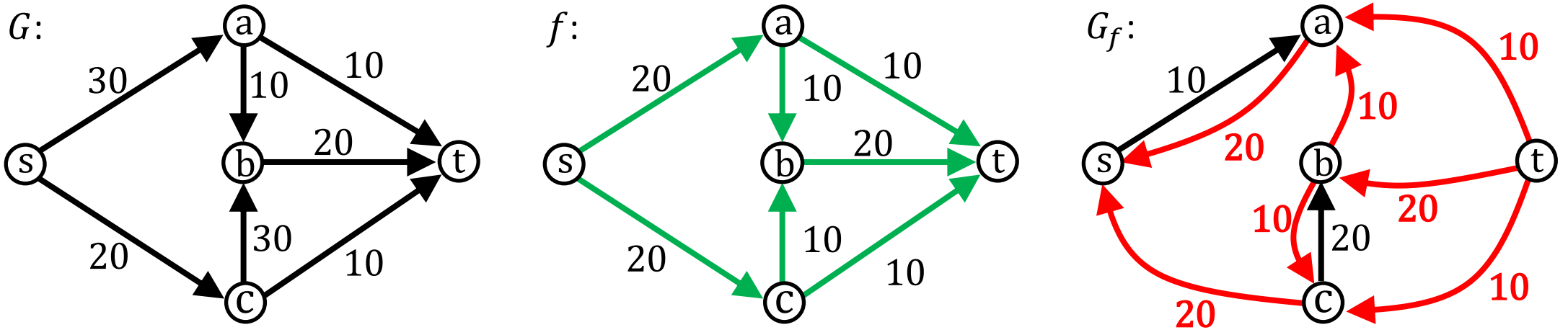

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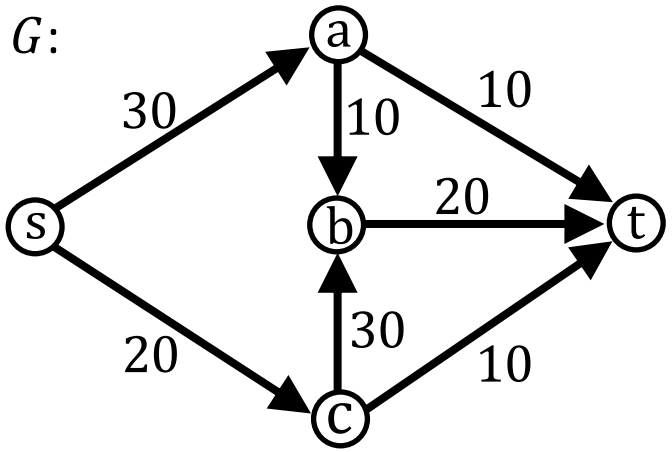
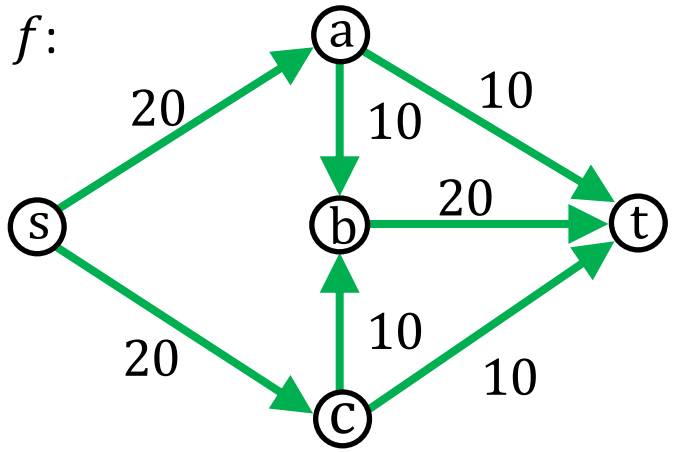
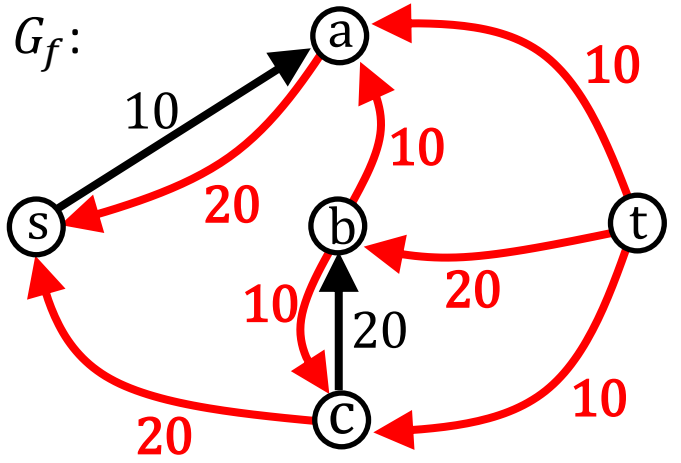


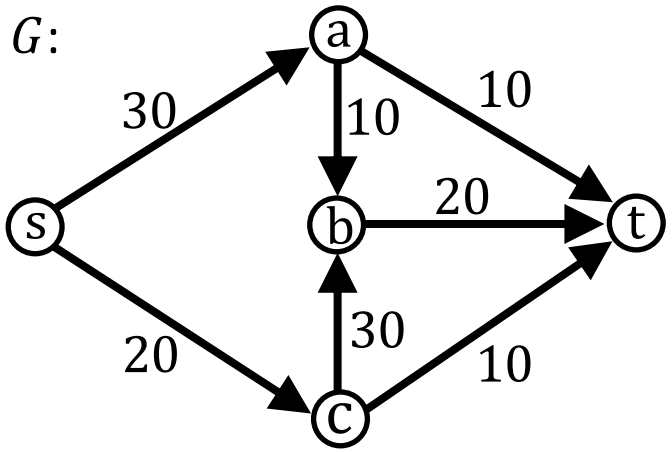


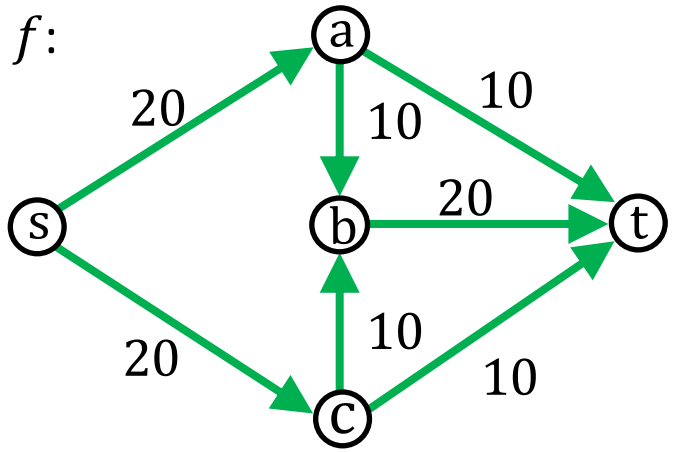


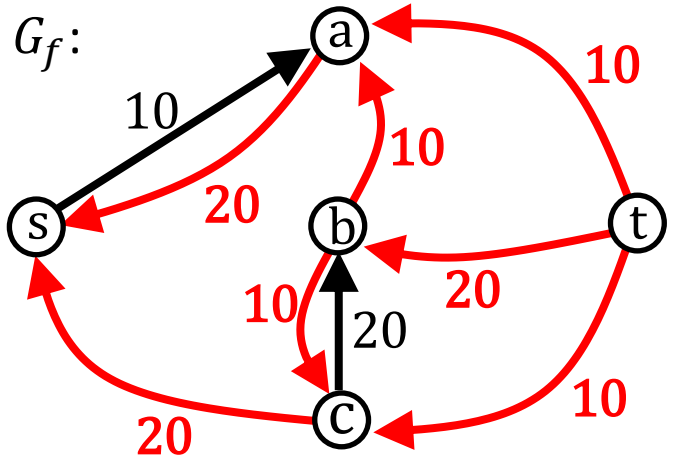
Algorithm Overview

1. Start with 0 flow and initial residual graph.
2. Select an $s - t$ path P in residual graph.
3. Push bottleneck(P) flow on P.
4. Update residual graph.
5. Repeat until no $s - t$ paths exist in residual graph.

Max Flow Algorithm







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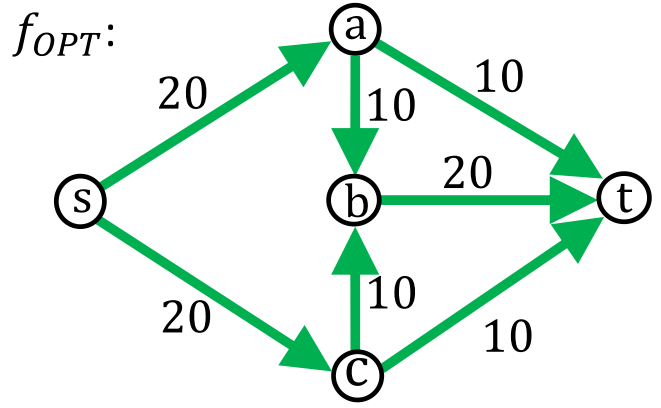
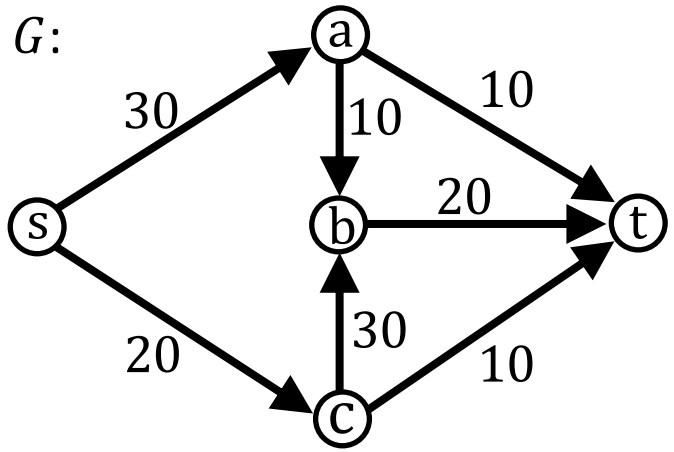
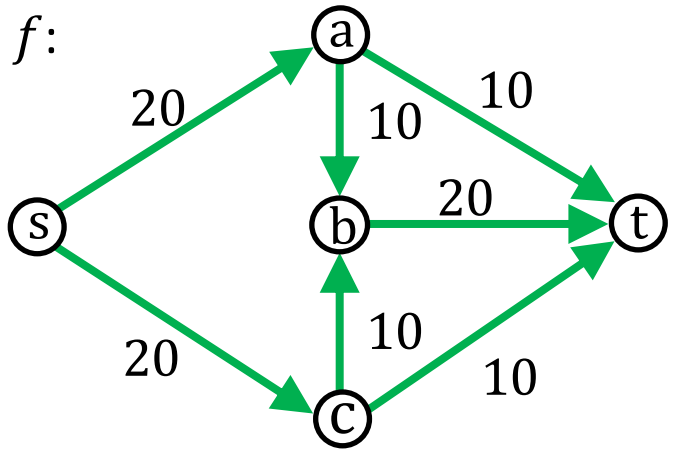
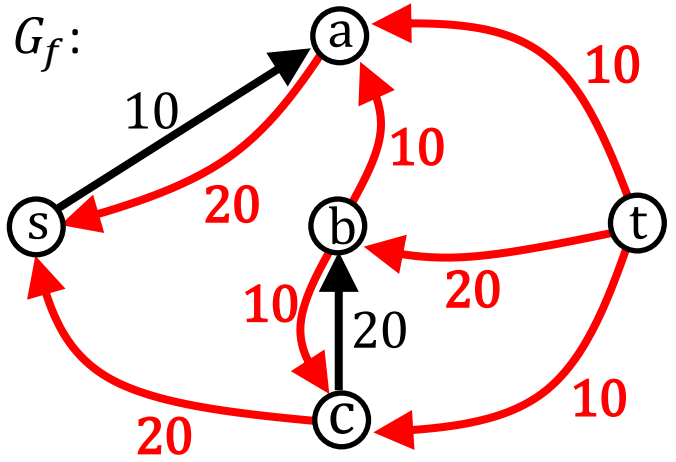


Ford-Fulkerson



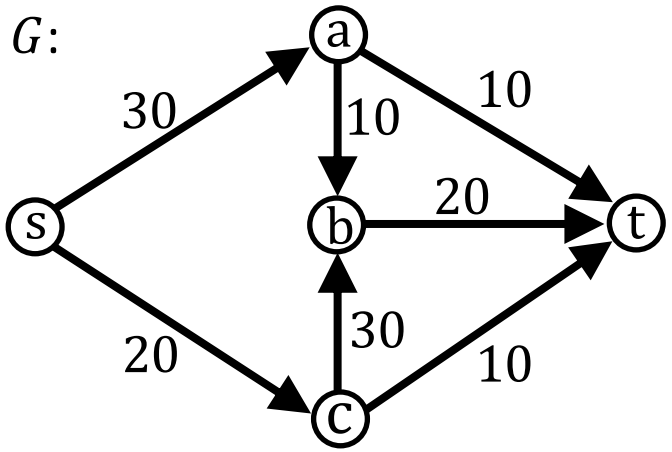


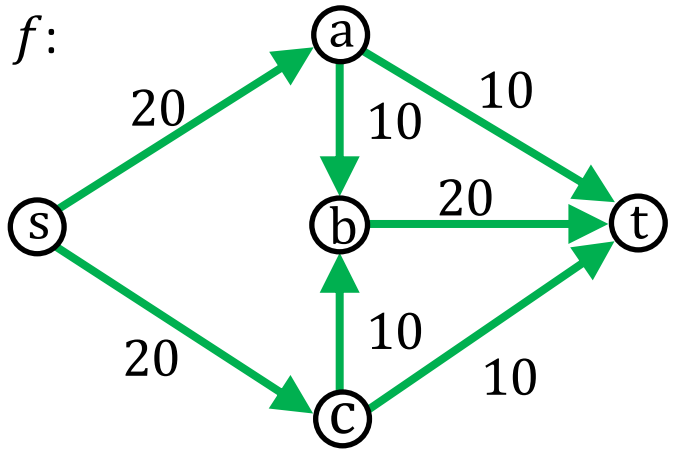


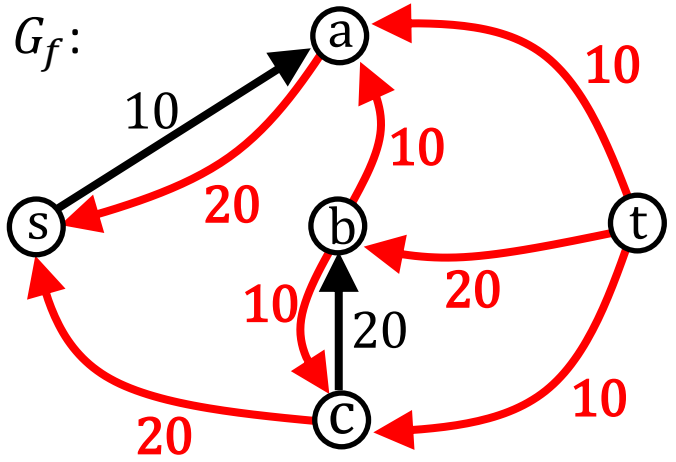

```
Max-Flow(G)
   f(e) = 0 for all e in G
   while s-t path in Gf exists
      P = simple s-t path in Gf
      f'= augment(f, P)
      f = f'
      Gf = Gf'
   return f
```

```
augment(f, P)
   b = bottleneck(P,f)
   for each edge (u, v) in P
      if (u, v) is a back edge
         f((v, u)) -= b
      else
         f((u, v)) += b
   return f
```

Ford-Fulkerson





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      f((v, u)) -= b
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```

Need to show:

1. Validity.
2. Running time.
3. Finds max flow.